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Essential Self-Adjointness for Semi-bounded Magnetic Schrödinger Operators on Non-compact Manifolds

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We prove essential self-adjointness for semi-bounded below magnetic Schrödinger operators on complete Riemannian manifolds with a given positive smooth measure which is fixed independently of the metric. Some singularities of the scalar potential [illegible] structed by a non-trivial smoothing procedure due to Karcher.

1. INTRODUCTION

Let (M, g) be a Riemannian manifold (i.e., M is a C^∞-manifold, (g_{jk}) is a Riemannian metric on M), $\dim M = n$. We will always assume for simplicity that M is connected. We will also assume that we are given a *positive smooth measure* $d\mu$ i.e., a measure which has a C^∞ positive density $\rho(x)$ with respect to the Lebesgue measure $dx = dx^1 \cdots dx^n$ in any local coordinates $x^1, \ldots, x^n$, so we will write $d\mu = \rho(x)\, dx$. This measure may be completely independent of the Riemannian metric, but may of course coincide with the canonical measure $d\mu_g$ induced by the metric (in this case $\rho = \sqrt{g}$ where $g = \det(g_{jk})$, so locally $d\mu_g = \sqrt{g}\, dx$).

The main purpose of this paper is to study essential self-adjointness of semi-bounded below magnetic Schrödinger operators in $L^2(M) = L^2(M, d\mu)$.

Denote $\Lambda^p_{(k)}(M)$ the set of all k-smooth (i.e., of the class C^k) complex-valued p-forms on M. We will write $\Lambda^p(M)$ instead of $\Lambda^p_{(\infty)}(M)$. A *magnetic*

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potential or *vector potential* is a real-valued 1-form $A \in \Lambda^1_{(1)}(M)$. So in local coordinates $x^1, \ldots, x^n$ it can be written as

$$A = A_j \, dx^j,$$

where $A_j = A_j(x)$ are real-valued C^1-functions of the local coordinates, and we use the standard Einstein summation convention.

The usual differential can be considered as a first order differential operator

$$d \colon C^\infty(M) \to \Lambda^1(M).$$

We will also need a deformed differential

$$d_A \colon C^\infty(M) \to \Lambda^1_{(1)}(M), \qquad u \mapsto du + iuA,$$

where $i = \sqrt{-1}$.

The Riemannian metric (g_{jk}) and the measure $d\mu$ induce an inner product in the spaces of smooth forms with compact support in a standard way. In particular, this inner product on functions has the form

$$(u, v) = \int_M u\bar{v} \, d\mu,$$

where the bar over v means the complex conjugation.

For smooth forms $\alpha = \alpha_j \, dx^j$, $\beta = \beta_k \, dx^k$ denote

$$\langle \alpha, \beta \rangle = g^{jk} \alpha_j \beta_k,$$

where (g^{jk}) is the inverse matrix to (g_{jk}). So the result $\langle \alpha, \beta \rangle$ is a scalar function on M. Then for α, β with compact support we have

$$(\alpha, \beta) = \int_M \langle \alpha, \bar{\beta} \rangle \, d\mu,$$

where

$$\bar{\beta} = \bar{\beta}_k \, dx^k.$$

Using the inner products in spaces of smooth functions and 1-forms with compact support we can define the completions of these spaces. They are Hilbert spaces which we will denote $L^2(M)$ for functions and $L^2\Lambda^1(M)$ for 1-forms. These spaces depend on the choice of the metric (g_{jk}) and the measure $d\mu$. However we will skip this dependence in the notations of the spaces for simplicity of notations. This will not lead to a confusion because both metric and measure will be fixed through the whole paper unless indicated otherwise.

The corresponding local spaces will be denoted $L^2_{loc}(M)$ and $L^2_{loc}\Lambda^1(M)$ respectively. These spaces do not depend on the metric or measure. For example $L^2_{loc}(M)$ consists of all functions u: $M \to \mathbb{C}$ such that for any local coordinates $x^1, \ldots, x^n$ defined in an open set $U \subset M$ we have $u \in L^2$ with respect to the Lebesgue measure $dx^1 \cdots dx^n$ on any compact subset in U. Similarly the space $L^p_{loc}(M)$ is well defined for any p with $1 \leqslant p \leqslant \infty$.

Formally adjoint operators to the differential operators with sufficiently smooth coefficients are well defined through the inner products above. In particular, we have an operator

$$d_A^*: \Lambda^1_{(1)}(M) \to C(M),$$

defined by the identity

$$(d_A u, \omega) = (u, d_A^* \omega), \; u \in C_c^\infty(M), \; \omega \in \Lambda^1_{(1)}(M).$$

(Here $C_c^\infty(M)$ is the set of all C^∞ functions with compact support on M.)

Therefore we can define the magnetic Laplacian Δ_A (with the potential A) by the formula

$$-\Delta_A = d_A^* d_A: C^\infty(M) \to C(M).$$

Now the main object of our study will be the *magnetic Schrödinger operator*

$$H = H_{A,V} = -\Delta_A + V, \tag{1.1}$$

where $V \in L^2_{loc}(M)$ i.e., V is a measurable locally square-integrable function which is called *electric potential* or *scalar potential*. We will always assume V to be real-valued. Then H becomes a symmetric operator in $L^2(M)$ if we consider it on the domain $C_c^\infty(M)$. In this paper we will assume that $H_{A,V}$ is semi-bounded below on $C_c^\infty(M)$ i.e., there exists a constant $C \in \mathbb{R}$ such that

$$(H_{A,V} u, u) \geqslant -C(u, u), \qquad u \in C_c^\infty(M). \tag{1.2}$$

We will impose the following local condition on V:

(H) $V = V_+ + V_-$ where $V_+ \geqslant 0$, $V_- \leqslant 0$, $V_+ \in L^2_{loc}(M)$ and $V_- \in L^p_{loc}(M)$ with $p = n/2$ if $n \geqslant 5$, $p > 2$ if $n = 4$, and $p = 2$ if $n \leqslant 3$.

Our main result is the following

THEOREM 1.1. *Let the manifold (M, g) be complete, $A \in \Lambda^1_{(1)}(M)$, V satisfies the condition (H) above, and the corresponding magnetic Schrödinger operator $H_{A,V}$ is semi-bounded below on $C_c^\infty(M)$. Then $H_{A,V}$ is essentially self-adjoint.*

Remark 1. If we assume that $V \in L^\infty_{loc}(M)$, then instead of $A \in C^1(M)$ it is sufficient to assume that $A \in \mathrm{Lip}_{loc}(M)$, i.e., A is locally Lipschitz.

Remark 2. The requirement on p in the condition (H) is almost optimal. Indeed, we must require that $V \in L^2_{loc}(M)$ if we wish $H_{A,V}$ to be defined on $C_c^\infty(M)$. This is the only requirement which is imposed for $n \leqslant 3$; the requirement $p > 2$ in case $n = 4$ is only slightly stronger. As to the requirement $p = n/2$ in case $n \geqslant 5$, it can not be replaced by $p = n/2 - \varepsilon$ with $\varepsilon > 0$. This was shown by B. Simon even in $\mathbb{R}^n$ and without magnetic field (see [66] or [54], Example 4 in Ch.X.2): the operator $-\Delta - \alpha/|x|^2$ on $C_c^\infty(\mathbb{R}^n)$ with a real parameter α is bounded from below if and only if $\alpha \leqslant (n-1)(n-3)/4 + 1/4$ and essentially self-adjoint if and only if $\alpha \leqslant (n-1)(n-3)/4 - 3/4$. However the requirement $V_- \in L^p_{loc}(M)$ can be replaced by weaker requirements formulated in less explicite terms, e.g., Stummel classes [70] and domination requirements (see e.g., [64]).

Remark 3. For the usual semi-bounded below Schrödinger operator $H = -\Delta + V(x)$ in $\mathbb{R}^3$ with a continuous potential V the essential self-adjointness was conjectured by I. M. Glazman and proved by A. Ya. Povzner ([53], Theorem 6 in Ch.I). Independently a much more general result (which includes in particular magnetic Schrödinger operators with sufficiently regular coefficients in $\mathbb{R}^n$) was obtained by E. Wienholtz [75]. The Wienholtz proof is much simpler and for the simplest case of the Schrödinger operator it is also reproduced in the book of I. M. Glazman [25]. Further improvements for operators in $\mathbb{R}^n$ and in its open subsets are due to H. Stetkær-Hansen [69], J. Walter [74], and C. Simader [64]. We will use the method of Wienholtz when we treat the case of locally bounded potentials V and the method of C. Simader [64] for more singular V.

EXAMPLE. Let us give an example which shows that the magnetic field can contribute to the fulfillment of the semi-boundedness condition (1.2) for $H_{A,V}$ so that the corresponding operator $H_{0,V}$ (with the magnetic field removed) is not essentially self-adjoint.

In this example we will take $M = \mathbb{R}^2$ with the standard flat metric, so the magnetic potential is $A = A_1\, dx^1 + A_2\, dx^2$. The magnetic field is then a

2-form $B = dA = B_{12}\, dx^1 \wedge dx^2$. Let us write B instead of B_{12} for simplicity of notation. Of course, changing the order of x^1 and x^2 would replace B by $-B$.

Using simple uncertainty principle type arguments given e.g., in [13] or [32], we can see that

$$H_{A,V} \geqslant B + V \qquad \text{and} \qquad H_{A,V} \geqslant -B + V,$$

where the inequalities are understood in the sense of quadratic forms. Assume now that $V \in L^2_{loc}(\mathbb{R}^2)$ and either $B+V$ or $-B+V$ is semi-bounded below. Then due to Theorem 1.1 the operator $H_{A,V}$ is essentially self-adjoint. This can happen in particular when $V \to -\infty$ fast enough so that $H_{0,V}$ is not essentially self-adjoint, e.g., when $V(x) = -|x|^\alpha$ with $\alpha > 2$ (see e.g., [3], Example 1.1 in Ch.3).

2. ALGEBRAIC PRELIMINARIES

We will start by considering the operator d^*, which is formally adjoint to d, so $d^*\colon \Lambda^1_{(1)}(M) \to C(M)$. This operator is related with the divergence of vector fields. Let v be a smooth vector field on M. Denote by ω_v the 1-form corresponding to v i.e., locally $\omega_v = (\omega_v)_j\, dx^j$ where

$$(\omega_v)_j = g_{jk} v^k.$$

Vice versa, for any smooth 1-form ω we will denote by v_ω the corresponding vector field, so locally $v_\omega = v^k_\omega\, \partial/\partial x_k$ where

$$v^k_\omega = g^{kj} \omega_j.$$

Then we will define the divergence of v by the formula

$$\operatorname{div} v = -d^* \omega_v. \tag{2.1}$$

Equivalently we can write

$$d^* \omega = -\operatorname{div} v_\omega. \tag{2.2}$$

A straightforward calculation shows that in local coordinates

$$\operatorname{div} v = \frac{1}{\rho} \frac{\partial}{\partial x^i} (\rho v^i), \qquad v = v^i \frac{\partial}{\partial x^i}. \tag{2.3}$$

It follows from (2.1) that $\operatorname{div} v$ (as given by (2.3)) does not depend on the choice of local coordinates but only on the metric and measure. On the other hand (2.3) implies that $\operatorname{div} v$ does not depend on the metric (even though it is not immediately seen from (2.1)).

We have the following Leibniz rule for d^* (or, equivalently, for the divergence):

$$d^*(f\omega) = f\, d^*\omega - \langle df, \omega\rangle, \qquad f \in C^1(M), \quad \omega \in \Lambda^1_{(1)}(M). \tag{2.4}$$

For the Laplacian Δ (on functions) we have

$$\Delta u = -d^*\, du = \operatorname{div}(\nabla u), \qquad u \in C^2(M),$$

where ∇u means the gradient of u associated with g, i.e., the vector field which corresponds to du and is given in local coordinates as

$$\nabla u = g^{jk} \frac{\partial u}{\partial x^j} \frac{\partial}{\partial x^k}.$$

Let us identify the magnetic potential A with the multiplication operator

$$A \colon C^\infty(M) \to \Lambda^1_{(1)}(M).$$

Then the formally adjoint operator A^* is a substitution operator of the vector field v_A into 1-forms, or in other words

$$A^*\omega = \langle A, \omega\rangle = g^{jk} A_j \omega_k. \tag{2.5}$$

This gives us a formula for d_A^*:

$$d_A^*\omega = (d^* - iA^*)\,\omega = -\operatorname{div} v_\omega - i\langle A, \omega\rangle. \tag{2.6}$$

It follows that

$$d_A^*(f\omega) = f\, d^*\omega - \langle df, \omega\rangle - if\langle A, \omega\rangle, \; f \in C^1(M), \quad \omega \in \Lambda^1_{(1)}(M). \tag{2.7}$$

The following Leibniz rules for d_A^* immediately follow:

$$d_A^*(f\omega) = f\, d_A^*\omega - \langle df, \omega\rangle,$$
$$d_A^*(f\omega) = f\, d^*\omega - \langle d_A f, \omega\rangle,$$

where f, ω are as in (2.7).

Using these formulas, we can write an explicit expression for the magnetic Laplacian $\Delta_A = -H_{A,0}$. Namely,

$$\begin{aligned}
-\Delta_A u = d_A^* \, d_A u &= (d^* - iA^*)(du + iAu) \\
&= d^* \, du - iA^* \, du + id^*(Au) + A^* Au \\
&= -\Delta u - i\langle A, du\rangle - i \operatorname{div}(u v_A) + \langle A, A\rangle \, u \\
&= -\Delta u - 2i\langle A, du\rangle + (id^* A + |A|^2) u.
\end{aligned}$$

Hence we obtain the following expression for the magnetic Schrödinger operator (1.1):

$$H_{A,V} u = -\Delta u - 2i\langle A, du\rangle + (id^* A + |A|^2)\, u + Vu. \tag{2.8}$$

On the other hand using the expressions (2.3) and (2.5) for the divergence and the operator A^* we easily obtain that in local coordinates

$$H_{A,V} u = -\frac{1}{\rho}\left(\frac{\partial}{\partial x^j} + iA_j\right)\left[\rho g^{jk}\left(\frac{\partial}{\partial x^k} + iA_k\right) u\right] + Vu, \tag{2.9}$$

or in slightly different notations

$$H_{A,V} u = \frac{1}{\rho}\,(D_j + A_j)[\rho g^{jk}(D_k + A_k)\, u] + Vu,$$

where $D_j = -i\partial/\partial x_j$.

Remark. A similar operator in $\mathbb{R}^n$ (with $\rho = 1$) was considered by T. Ikebe and T. Kato [30], K. Jörgens [34], M. S. P. Eastham, W. D. Evans and J. B. McLeod [18], A. Devinatz [17] in the space $L^2(\mathbb{R}^n, dx)$ where dx is the standard Lebesgue measure on $\mathbb{R}^n$. The general operator of the form (2.9) on manifolds was studied by H. O. Cordes [14, 15]. In this generality it includes some natural geometric situations (in particular the case $\rho = \sqrt{g}$).

3. PRELIMINARIES ON THE LIPSCHITZ ANALYSIS AND THE STOKES FORMULA ON A RIEMANNIAN MANIFOLD

Let (M, g) be a Riemannian manifold. A function $f\colon M \to \mathbb{R}$ is called *a Lipschitz function with a Lipschitz constant* L if

$$|f(x) - f(y)| \leqslant L d(x, y), \qquad x, y \in M, \tag{3.1}$$

where $d(x, y)$ means the Riemannian distance between x and y. We will denote the space of all Lipschitz functions on M by $\mathrm{Lip}(M)$. This space depends on the choice of the Riemannian metric on M. The space of all locally Lipschitz functions on M will be denoted $\mathrm{Lip}_{loc}(M)$. This space does not depend on the Riemannian metric on M.

By the well known Rademacher theorem, (3.1) implies that f is differentiable almost everywhere and

$$|df| \leqslant L \tag{3.2}$$

with the same constant L. Here $|df|$ means the length of the cotangent vector df in the metric associated with g. The corresponding partial derivatives of the first order coincide with the distributional derivatives. Vice versa if $df \in L^\infty(M)$, for the distributional differential $df = (\partial f/\partial x^j)\, dx^j$, then f can be modified on a set of measure 0 so that it becomes a Lipschitz function.

The estimate (3.2) can be also rewritten in the form

$$|\nabla f| \leqslant L \tag{3.3}$$

(again with the same constant L).

In local form (in open subsets of $\mathbb{R}^n$) these facts are discussed e.g., in the book of V. Mazya [47], Sect. 1.1. The correspondence between constants in (3.1), (3.2) and (3.3) is straightforward.

The Lipschitz vector fields, differential forms etc. are defined in an obvious way.

The formulas (2.1), (2.2), (2.3), (2.4), (2.6) apply to Lipschitz vector fields and forms instead of smooth ones.

We will also need local Sobolev spaces $W^{m,2}_{loc}$ on M for arbitrary integer m. We need these spaces for functions, vector fields and differential forms. For simplicity let us consider functions first. If $m \geqslant 0$ then the space $W^{m,2}_{loc}(M)$ consists of functions $u \in L^2_{loc}(M)$ such that their derivatives of the order $\leqslant m$ in local coordinates also belong to L^2_{loc} in these coordinates. (The functions which coincide almost everywhere are identified.) Denote also by $W^{m,2}_{comp}(M)$ the space of functions which belong to $W^{m,2}_{loc}(M)$ and have a compact support.

If $m < 0$ then $W^{m,2}_{loc}(M)$ is a dual space to $W^{-m,2}_{comp}(M)$ and it consists of all distributions which can be locally represented as sums of derivatives of order $\leqslant -m$ of functions from L^2_{loc}.

These definitions obviously extend to vector fields and differential forms.

We will need the Stokes formula, or rather the divergence formula for Lipschitz vector fields v on M in the following simplest form:

Proposition 3.1. *Let $v = v(x)$ be a vector field which is in $W^{1,2}_{comp}$ on M. Then*

$$\int_M \operatorname{div} v \, d\mu = 0.$$

The proof of the Proposition can be easily reduced to the case when v is supported in a domain of local coordinates. After that we can use mollification (regularization) of v to approximate v by smooth vector fields. A more advanced statement which does not require a compact support and includes a boundary integral, can be proved for Lipschitz vector fields ([47], Sect. 6.2).

Again using mollifiers we easily see that the formulas (2.1), (2.2), (2.3), (2.4), (2.6) apply to functions, vector fields and forms from $W^{1,2}_{loc}$ instead of smooth ones.

4. CUT-OFF FUNCTIONS

In the proofs of the main results in the next section we will need a sequence of compactly supported cut-off functions with Lipschitz gradients, such that the gradients are uniformly small. Here we will follow H. Karcher [38] to establish the existence of such functions on any complete Riemannian manifold, and they will be in fact C^∞-functions.

Proposition 4.1. *Let (M, g) be a complete Riemannian manifold. Then there exists a sequence of functions $\phi_N \colon M \to \mathbb{R}$, $N = 1, 2, \ldots,$ with the following properties:*

(a) $\phi_N \in C_c^\infty(M)$;

(b) $0 \leqslant \phi_N(x) \leqslant 1$, $x \in M$, $N = 1, 2, \ldots$;

(c) *for every compact $K \subset M$ there exists $N_0 > 0$ such that $\phi_N = 1$ on K if $N \geqslant N_0$;*

(d) $\varepsilon_N := \sup_{x \in M} |\nabla \phi_N(x)| \to 0$ *as* $\mathrm{N} \to \infty$.

Proof. Note that for any complete Riemannian manifold (M, g) it is very easy to construct a sequence of compactly supported functions $\psi_N \in \mathrm{Lip}(M)$, $N = 1, 2, \ldots,$ satisfying the conditions (b), (c), (d) above (if we substutute them for ϕ_N there). For example, we can take

$$\psi_N(x) = \chi(N^{-1} d(x, x_0)), \tag{4.1}$$

where $x_0 \in M$ is fixed, $\chi \in C_c^\infty(\mathbb{R})$, $\chi(r) = 1$ if $r \leqslant 1/3$, $\chi(r) = 0$ if $r \geqslant 2/3$, and $0 \leqslant \chi(t) \leqslant 1$ for all $r \in \mathbb{R}$. In this case clearly $|\nabla \psi_N| \leqslant C/N$.

However it is not clear how to satisfy (a), and it is even not immediately clear how to make $\nabla\phi_N \in \mathrm{Lip}(M)$. But there are many manifolds where this is easily possible, e.g., we have $\psi_N \in C^\infty(M)$ if M is $\mathbb{R}^n$ (with the flat metric), the hyperbolic space, or generally any manifold with an empty cut-locus, so that the function $x \mapsto d(x, x_0)$ is in $C^\infty(M)$ if $x \neq x_0$.

More generally, in the construction above we can replace the distance function $d(x) = d(x, x_0)$ by a *regularized distance*: a smooth function $\tilde{d}\colon M \to \mathbb{R}$ such that $\tilde{d} \geqslant 0$ and

$$C^{-1}\, d(x) - C_1 \leqslant \tilde{d}(x) \leqslant C d(x) + C_1$$

with some positive constants C, C_1. Such a function $\tilde{d} \in C^\infty(M)$ can be easily constructed on any manifold of bounded geometry (see e.g., the construction given in [60]). Subtler arguments by J. Cheeger and M. Gromov [10] (which are based on a result of U. Abresch [1] about smoothing of Riemannian metrics, I. Yomdin's theorem which is a quantitative refinement of the Sard Lemma—see [26], pp. 123–124, and some arguments from [9]), allow to construct such regularized distance on any complete Riemannian manifold with a bounded sectional curvature (without any restrictions on the injectivity radius, which are part of the usual definition of bounded geometry).

In the general case the result easily follows by use of a H. Karcher's mollifiers construction [38], applied to the family ψ_N from (4.1). Let us recall this construction.

Let us choose a point $m \in M$ and a small ball $B(m; \rho)$ centered at m with the radius $\rho > 0$, so that this ball is geodesically convex and the exponential map

$$\exp_m\colon T_m M \to M$$

restricted to the euclidean ball $D(0; \rho) \subset T_m(M)$ is a diffeomorphism of $D(0; \rho)$ onto $B(m; \rho)$. We will identify $B(m; \rho)$ with $D(0; \rho)$ via $\exp_m^{-1}$ and construct mollifiers (or mollifying kernels)

$$\Phi_\rho(m, y) = \chi\left(\frac{1}{\rho}\, d(m, y)\right)\left(\int_{B(m;\,\rho)} \chi\left(\frac{1}{\rho}\, d(m, x)\right) d_m x\right)^{-1}, \tag{4.2}$$

where χ is the same function as above, $d_m x$ is the euclidean volume in $B(m; \rho)$ (coming from $T_m M$ via the exponential map).

Choosing a compact $K \subset M$, we see that $\Phi_\rho(m, y)$ is well defined for all $m \in K$ and arbitrary $y \in M$ provided $0 < \rho < \rho_0 = \rho_0(K)$. Clearly $\Phi_\rho(\cdot, \cdot) \in C^\infty(U \times M)$ for a neighborhood U of K, $\Phi_\rho(m, y) = const$ near the diagonal $m = y$ and $\Phi_\rho(m, y) = 0$ if $d(m, y) \geqslant \rho$.

H. Karcher applied the mollifiers (4.2) to smooth maps $M \to \hat{M}$ for another Riemannian manifold $\hat{M}$. To this end he used the Riemannian center of mass on $\hat{M}$. We will only need the case $\hat{M} = \mathbb{R}$ where the construction and arguments become much simpler (but still not trivial). Taking a locally integrable function $f: M \to \mathbb{R}$, we can define the mollified functions (depending on $\rho > 0$) by

$$f_\rho(m) = \int_M f(x)\, \Phi_\rho(m, x)\, d_m x.$$

Assuming for simplicity that f has a compact support, $\operatorname{supp} f \subset K$ with a compact $K \subset M$, we see that $f_\rho \in C^\infty(M)$ if $\rho < \rho_0(K)$. It is also clear that $f_\rho = 0$ outside of the ρ-neighborhood of K.

Now let us apply this to $f = \psi_N$ taking $\rho = \rho_N$ sufficiently small, and denote the resulting mollified function by ϕ_N, i.e., $\phi_N = (\psi_N)_{\rho_N}$. Then the sequence ϕ_N, $N = 1, 2, \ldots$, satisfies the conditions (a), (b) and (c) above.

It remains to see that the functions ϕ_N satisfy (d) as well. To this end we can use Theorem 4.6 of Karcher [38]. It implies that if f is a Lipschitz function with the Lipschitz constant L i.e., (3.1) holds, and the sectional curvature varies in a finite interval $[\delta, \Delta]$ in a ρ-neighborhood of $\operatorname{supp} f$, then

$$|f_\rho(x) - f_\rho(y)| \leqslant L(1 + L^2 \cdot C(\delta, \Delta)\, \rho^2)\, d(x, y).$$

Hence for a Lipschitz function f with a compact support, we can choose ρ so small that f_ρ is Lipschitz with the Lipschitz constant $2L$. In this case we will have $|\nabla f_\rho| \leqslant 2L$ everywhere. Since the Lipschitz constant of ψ_N is $O(1/N)$, the condition (d) for ϕ_N immediately follows. ∎

5. PROOF OF THEOREM 1.1

In this section we will always write H instead of $H_{A,V}$ for simplicity of notations.

Let H_{min} and H_{max} be the minimal and maximal operators associated with the differential expression (1.1) for H in $L^2(M)$. Here H_{min} is the closure of

H in $L^2(M)$ from the initial domain $C_c^\infty(M)$, $H_{max} = H^*_{min}$ (the adjoint operator to H_{min} in $L^2(M)$). Clearly

$$\mathrm{Dom}(H_{max}) = \{u \in L^2(M) \mid Hu \in L^2(M)\},$$

where Hu is understood in the sense of distributions.

The essential self-adjointness of H means that $H_{min} = H_{max}$.

For simplicity of exposition we treat the case of a locally bounded potential first. The requirements on the vector potential A can be slightly relaxed in this case.

5.1. *Locally Bounded Scalar Potentials*

To establish the equality $H_{min} = H_{max}$ we need some information about the domain of H_{max}. We will start with a simple lemma establishing necessary local information in the simplest case $V \in L^\infty_{loc}(M)$.

LEMMA 5.1. *Assume that $A \in \mathrm{Lip}_{loc}(M)$, $V \in L^\infty_{loc}(M)$, and $u \in Dom(H_{max})$. Then $u \in W^{2,2}_{loc}(M)$.*

Proof. We will repeat an argument given in [3], Appendix 2, proof of Theorem 2.1.

Assume that $u \in \mathrm{Dom}(H_{max})$. Due to (2.8) this means that $u \in L^2(M)$ and

$$-\Delta u - 2i\langle A, du\rangle + (id^*A + |A|^2)\, u + Vu = f \in L^2(M),$$

where Δu and $\langle A, du\rangle$ are understood in the sense of distributions, so a priori $\Delta u \in W^{-2,2}_{loc}(M)$, $\langle A, du\rangle \in W^{-1,2}_{loc}(M)$. Note also that $(id^*A + |A|^2)\, u + Vu \in L^2_{loc}(M)$. It follows from the local elliptic regularity theorem applied to $-\Delta$ that $u \in W^{1,2}_{loc}(M)$.

This already implies that $\langle A, du\rangle \in L^2_{loc}(M)$. Applying the local elliptic regularity theorem again we see that $u \in W^{2,2}_{loc}(M)$. ∎

Remark. Lemma 5.1 is certainly not new, though I had difficulty to find a statement which would exactly imply it. More general equations are considered e.g., by D. Gilbarg and N. S. Trudinger ([23], Theorem 8.10), but with a stronger a priori requirement $u \in W^{1,2}$.

THEOREM 5.2. *Let us assume that the manifold (M, g) is complete, $A \in \mathrm{Lip}_{loc}(M)$, $V \in L^\infty_{loc}(M)$ and the corresponding magnetic Schrödinger operator $H_{A,V}$ is semi-bounded below on $C_c^\infty(M)$ i.e., (1.2) holds. Then $H_{A,V}$ is essentially self-adjoint.*

Proof. Note that the smoothness requirements on A, V imply that the operator $H_{A,V}$ is well defined on $C_c^\infty(M)$ and maps this space into $L^2(M)$ (see Sect. 3), as well as on $L^2(M)$ (which it maps to the space of distributions on M).

Adding $(C+1)I$ to $H_{A,V}$ we can assume that $H_{A,V} \geqslant I$ on $C_c^\infty(M)$ i.e.,

$$(H_{A,V}u, u) \geqslant (u, u), \qquad u \in C_c^\infty(M).$$

If this is true, then it is well known (see e.g., [25]) that the essential self-adjointness of $H_{A,V}$ is equivalent to the fact that the equation

$$H_{A,V}u = 0$$

has no non-trivial solutions in $L^2(M)$ (understood in the sense of distributions).

Assume that u is such a solution. First note that it is in $W^{2,2}_{loc}(M)$ due to Lemma 5.1.

Let us take a cut-off function ϕ_N on M from Proposition 4.1.

Then denoting $u_N = \phi_N u$ we see that u_N is in the domain of the minimal operator associated with $H_{A,V}$, hence

$$\|u_N\|^2 \leqslant (H_{A,V}u_N, u_N). \tag{5.1}$$

Now we will prove an identity which will be useful not only in this proof but in extending the result to singular scalar potentials.

Let us calculate $H_{A,V}(\phi u)$ for arbitrary functions u, ϕ such that $u \in W^{2,2}_{loc}$(M) and $\phi \in C^1(M)$ has a locally Lipschitz gradient. We will use the Leibniz type formulas from Sect. 2. Applying d_A^* to

$$d_A(\phi u) = \phi\, d_A u + u\, d\phi,$$

we obtain

$$d_A^*\, d_A(\phi u) = \phi\, d_A^*\, d_A u - 2\langle d\phi, d_A u\rangle + u\, d^*\, d\phi,$$

hence

$$H_{A,V}(\phi u) = \phi H_{A,V}u - 2\langle d\phi, d_A u\rangle - u\,\Delta\phi. \tag{5.2}$$

Now let us additionally assume that ϕ is real-valued and has a compact support. Then multiplying (5.2) by $\phi\bar{u}$ and integrating over M (with respect to the chosen measure $d\mu$ with a positive smooth density) we get

$$(H_{A,V}(\phi u), \phi u) = (\phi H_{A,V}u, \phi u) - \int_M [2\langle d\phi, du\rangle + 2i\langle A, d\phi\rangle\, u + u\,\Delta\phi]\, \phi\bar{u}\, d\mu.$$

Adding this formula with the complex conjugate one and dividing by 2, we see that the term with A under the integral sign cancels, so using Proposition 3.1 we obtain

$$\begin{aligned}(H_{A,V}(\phi u), \phi u) &= \operatorname{Re}(\phi H_{A,V}u, \phi u) - \int_M [\langle \phi\, d\phi, \bar{u}\, du + u\, d\bar{u}\rangle + |u|^2 \phi\, \Delta\phi]\, d\mu \\ &= \operatorname{Re}(\phi H_{A,V}u, \phi u) - \int_M [\langle \phi\, d\phi, d(|u|^2)\rangle + |u|^2 \phi\, \Delta\phi]\, d\mu \\ &= \operatorname{Re}(\phi H_{A,V}u, \phi u) - \int_M (|u|^2 d^*(\phi\, d\phi) + |u|^2 \phi\, \Delta\phi)\, d\mu.\end{aligned}$$

Since

$$d^*(\phi\, d\phi) = \phi\, d^*\, d\phi - \langle d\phi, d\phi\rangle = -\langle d\phi, d\phi\rangle - \phi\, \Delta\phi,$$

we finally obtain the desired identity

$$(H_{A,V}(\phi u), \phi u) = \operatorname{Re}(\phi H_{A,V}u, \phi u) + \int_M |d\phi|^2 |u|^2\, d\mu. \tag{5.3}$$

To use this identity in our proof assume that $H_{A,V}u = 0$. This implies

$$(H_{A,V}(\phi u), \phi u) = \int_M |d\phi|^2 |u|^2\, d\mu.$$

Now taking $\phi = \phi_N$ and applying the estimate (5.1), we obtain

$$\|\phi_N u\|^2 \leqslant \int_M |d\phi_N|^2 |u|^2\, d\mu.$$

In particular, for any compact $K \subset M$ we obtain for $N \geqslant N_0(K)$:

$$\int_K |u|^2\, d\mu \leqslant \int_M |d\phi_N|^2 |u|^2\, d\mu \leqslant \varepsilon_N \int_M |u|^2\, d\mu.$$

If now $u \in L^2(M, d\mu)$, then taking limit as $N \to \infty$, we see that $u = 0$ on K, hence $u \equiv 0$. ∎

5.2. *Singular Scalar Potentials*

Now we will consider magnetic Schrödinger operators $H_{A,V}$ on a complete Riemannian manifold (M, g), such that the conditions of Theorem 1.1 are satisfied. In particular, we will assume that $A \in \Lambda^1_{(1)}(M)$ but we will not require that V is locally bounded.

Proof of Theorem 1.1. 1. Let us choose a relatively compact coordinate neighborhood U in M with coordinates $x^1, \dots, x^n$ which are defined in a neighborhood of $\bar{U}$.

Let Δ_0 denote the flat Laplacian in these coordinates. Then due to the standard elliptic estimates the norms

$$\|\Delta_0 u\| + \|u\| \qquad \text{and} \qquad \|H_{A,0} u\| + \|u\|$$

are equivalent on functions $u \in C_c^\infty(U)$. On the other hand, if we denote the bottoms of the spectra of the Friedrichs extensions of $-\Delta_0$ and $H_{A,0}$ in U by λ_0 and λ_A, respectively, then $\lambda_0 > 0$ and also $\lambda_A > 0$ due to the diamagnetic inequality (see, e.g., [39, 68] or [46, Sect. 7.21]). It follows that

$$\|u\| \leqslant \lambda_0^{-1} \|\Delta_0 u\|, \qquad \|u\| \leqslant \lambda_A^{-1} \|H_{A,0} u\|,$$

for any $u \in C_c^\infty(U)$; hence there exists $C > 0$ such that

$$C^{-1} \|\Delta_0 u\| \leqslant \|H_{A,0} u\| \leqslant C \|\Delta_0 u\|, \qquad u \in C_c^\infty(U). \tag{5.4}$$

Now let us recall that it follows from (H) that V_- has Δ_0-bound $\varepsilon > 0$ on $C_c^\infty(U)$ for arbitrarily small ε (see Theorem X.20 and Corollary of Theorem X.21 from [54]; i.e.,

$$\|V_- u\| \leqslant \varepsilon \|\Delta_0 u\| + C_\varepsilon \|u\|, \qquad u \in C_c^\infty(U). \tag{5.5}$$

Using (5.4) we see that (5.5) is equivalent to a similar estimate with Δ_0 replaced by $H_{A,0}$:

$$\|V_- u\| \leqslant \varepsilon \|H_{A,0} u\| + C_\varepsilon \|u\|, \qquad u \in C_c^\infty(U). \tag{5.6}$$

2. We would like to extend the inequality (5.6) to functions $u \in C_c^\infty(M)$ under the condition that $V_- \in L^p_{comp}(M)$ with p as in (H) (with C_ε depending on V_-). To this end we need the following estimate:

$$\|d_A u\| \leqslant \varepsilon \|H_{A,0} u\| + C_\varepsilon \|u\|, \qquad u \in C_c^\infty(M). \tag{5.7}$$

An equivalent form of (5.7) is

$$\|d_A u\|^2 \leqslant \varepsilon \|H_{A,0} u\|^2 + \tilde{C}_\varepsilon \|u\|^2, \qquad u \in C_c^\infty(M),$$

which holds because due to the Cauchy–Schwarz inequality

$$\|d_A u\|^2 = (H_{A,0} u, u) \leqslant \|H_{A,0} u\| \cdot \|u\| \leqslant \varepsilon \|H_{A,0} u\|^2 + \frac{1}{4\varepsilon} \|u\|^2, \qquad u \in C_c^\infty(M).$$

By taking closure we see that (5.7) holds for all $u \in \mathrm{Dom}(H_{A,0})$, where the domain is understood as the domain of minimal or maximal operators (which coincide due to [62] or Theorem 5.2 above).

3. Assuming that $V_- \in L^p_{comp}(M)$, let us choose functions $\psi_1, ..., \psi_N \in C_c^\infty(M)$ such that

(a) supp $\psi_j \subset U_j$ for a relatively compact coordinate neighborhood U_j, $j = 1, ..., N$.

(b) $\sum_{j=1}^N \psi_j = 1$ in a neighborhood of supp V_-.

Using (5.6), we obtain for any $\varepsilon > 0$

$$\|V_-u\| \leqslant \sum_{j=1}^{N} \|V_-(\psi_j u)\| \leqslant \varepsilon \sum_{j=1}^{N} \|H_{A,0}(\psi_j u)\| + C_\varepsilon \|u\|. \tag{5.8}$$

Now we can use (5.2) to conclude that

$$\|H_{A,0}(\psi_j u)\| \leqslant C_1(\|H_{A,0}u\| + \|d_A u\| + \|u\|), \qquad u \in C_c^\infty(M). \tag{5.9}$$

This again holds for any $u \in \mathrm{Dom}(H_{A,0})$ due to the arguments given above in part 2 of this proof. We obtain now from (5.8) that

$$\|V_-u\| \leqslant \varepsilon \|H_{A,0}u\| + C_\varepsilon \|u\|, \qquad u \in C_c^\infty(M), \tag{5.10}$$

under the condition that $V_- \in L^p_{comp}$.

4. Define $V_-^{(N)}(x) = V_-(x)$ on supp ϕ_N, and $V_-^{(N)}(x) = 0$ otherwise. (Here ϕ_N is the function from Proposition 4.1.) Then (5.10) holds for $V_-^{(N)}$. It follows from Theorem 5.2 and from the Kato–Rellich perturbation theorem (Theorem X.12 in [54]) that the operator $H_{A,V_-^{(N)}} = H_{A,0} + V_-^{(N)}$ is essentially self-adjoint.

Now we can use the Kato inequality technique (see [39] or [54], especially Theorem X.33, and also generalization to operators on manifolds and in sections of vector bundles developed by Hess, Schrader, and Uhlenbrock [28, 29]) and the perturbation arguments from the proofs of Theorems X.28, X.29 from [54], to prove that the operator $H_N = H_{A,V_+ + V_-^{(N)}} = H_{A,0} + V_+ + V_-^{(N)}$ is essentially self-adjoint for any $N = 1, 2, ...$.

Note that the use of the Kato inequality in the last step requires that $A \in C^1$, rather than Lip_{loc} (see [39], where the Friedrich's mollifiers technique [21] is used; this technique requires the derivatives of A to be continuous).

5. In what follows we will write H instead of $H_{A,V}$. Note that for any fixed $u \in \mathrm{Dom}(H_{max})$

$$|(u, H(\phi_N f))| = |(Hu, \phi_N f)| \leqslant C\,\|f\|, \qquad f \in C_c^\infty(M). \tag{5.11}$$

Similarly to (5.2) we have

$$H(\phi_N f) = \phi_N Hf - 2\langle d\phi_N, d_A f\rangle - f\Delta\phi_N;$$

hence

$$(\phi_N u, Hf) = 2(u, \langle d\phi_N, d_A f\rangle) + (u, f\Delta\phi_N) + (u, H(\phi_N f)),$$

and using (5.11) we conclude that

$$|(\phi_N u, Hf)| \leqslant C(\|df\| + \|f\|), \qquad f \in C_c^\infty(M),$$

with the constant C depending on u, H and ϕ_N (but not on f). Since the left hand side depends only on the restriction of u to a neighborhood of supp ϕ_n, we can also write

$$|(\phi_N u, H_N f)| \leqslant C(\|df\| + \|f\|), \qquad f \in C_c^\infty(M). \tag{5.12}$$

6. Our next goal is to establish that $\mathrm{Dom}(H_{max}) \subset W_{loc}^{1,2}(M)$. It is enough to prove that (5.12) implies that $\phi_N u \in W_{loc}^{1,2}(M)$. We will repeat the arguments from [64]. Denote $v = \phi_N u$, so $v \in L^2(M)$.

By the standard domination argument we have

$$|(V_-^{(N)} f, f)| \leqslant a\,\|df\|^2 + C\,\|f\|^2, \qquad f \in C_c^\infty(M),$$

with an arbitrarily small $a > 0$ and C depending on a or, equivalently,

$$|(V_-^{(N)} f, f)| \leqslant a\,\|d_A f\|^2 + C'\,\|f\|^2, \qquad f \in C_c^\infty(M) \tag{5.13}$$

Indeed, (5.7) means the operator domination relation $V_-^{(N)} \ll H_{A,0}$ which in turn implies the same domination relation for the corresponding quadratic forms (see Theorem X.18 in [54]), i.e., (5.13) with arbitrarily small $a > 0$.

Choosing an arbitrary $\lambda > 0$, we obtain

$$\begin{aligned} ((H_N + \lambda)\, f, f) &= \|d_A f\|^2 + (V_+ f, f) + (V_-^{(N)} f, f) + \lambda\,\|f\|^2 \\ &\geqslant (1-a)\,\|d_A f\|^2 + (\lambda - C')\,\|f\|^2 \\ &\geqslant (1-a)\,\|df\|^2 + (\lambda - C'')\,\|f\|^2 \end{aligned} \tag{5.14}$$

Now let us choose here $\lambda > C''$. Taking closure, we see that the estimate (5.14) holds for all f in the domain of the closure of H_N understood as the operator with the domain $C_c^\infty(M)$. It is a standard fact that this closure coincides with $H_N^{**} = (H_N^*)^*$. However since H_N is essentially self-adjoint, we have $H_N^{**} = H_N^*$ and the domain $D_N = \mathrm{Dom}(H_N^{**})$ coincides with the domain of the corresponding maximal operator H_N^*, i.e., with the set of all $f \in L^2(M)$ such that $H_N f \in L^2(M)$, where $H_N f$ is understood in the sense of distributions. In particular, (5.14) holds for all $f \in D_N$.

Clearly, H_N is semi-bounded below. Therefore for sufficiently large $\lambda > 0$ the operator $H_N^* + \lambda\colon D_N \to L^2(M)$ is bijective. Hence for any $\phi \in C_c^\infty(M)$ supported in the domain of some local coordinates $x^1, \ldots, x^n$, and for any $j \in \{1, \ldots, n\}$ we can find $f_j \in D_N$ such that $(H_N + \lambda) f_j = \partial_j^* \phi$, where $\partial_j = \partial/\partial x^j$ and ∂_j^* means the formally adjoint operator with respect to the inner product induced by the given measure in the chosen coordinate neighborhood. It follows that for any $\varepsilon > 0$

$$(5.15) \quad |((H_N + \lambda) f_j, f_j)| = |(\partial_j^* \phi, f_j)| = |(\phi, \partial_j f_j)| \leqslant \frac{\varepsilon}{2} \|\partial_j f_j\|^2 + \frac{1}{2\varepsilon} \|\phi\|^2.$$

Combining (5.14) and (5.15) we obtain

$$\|df_j\| + \|f_j\| \leqslant C' \|\phi\|,$$

with C' independent of ϕ. Now taking $f = f_j$ in (5.12) we obtain

$$|(v, \partial_j^* \phi)| \leqslant C'' \|\phi\|.$$

This implies that $\partial_j v \in L^2_{loc}$ for all j and $v \in W^{1,2}_{loc}$ in the coordinate neighborhood. Choosing a covering of M by such coordinate neighborhoods we see that $v = \phi_N u \in W^{1,2}_{loc}(M)$. Since N was arbitrary, we see that $u \in W^{1,2}_{loc}(M)$.

7. Let us start with the identity (5.3) which was established in the case of a locally bounded V for all $u \in W^{2,2}_{loc}(M)$ and real-valued compactly supported ϕ with a Lipschitz gradient. Let us try to relax the requirement on u first, still assuming that $V \in L^\infty_{loc}(M)$. We claim that (5.3) makes sense and holds for any $u \in W^{1,2}_{loc}(M)$. Indeed, both sides of (5.3) make perfect sense for any such u if we understand the inner products as dualities between $W^{-1,2}_{loc}(M)$ and $W^{1,2}_{comp}(M)$. To prove this identity for an arbitrary $u \in W^{1,2}_{loc}(M)$ we just need to approximate u by functions from $C_c^\infty(M)$ in the $W^{1,2}$-norm in a neighborhood of supp ϕ.

This argument works also if instead of the local boundedness of V we assume that $V \in L^p_{loc}(M)$, where p is the same as in the condition (*H*). Indeed, the Sobolev inequality gives a continuous imbedding of $W^{1,2}_{loc}(M)$ into $L^q_{loc}(M)$ with $q = 2n/(n-2)$ if $n \geqslant 3$ and arbitrarily large $q < \infty$ if $n = 2$. For any $u \in W^{1,2}_{loc}(M)$ we have then $|u|^2 \in L^{q/2}_{loc}(M)$ and the last space is in a continuous duality with $L^p_{comp}(M)$ (by the usual integration) due to the Hölder inequality. Therefore in this case we can again prove the identity (5.3) for any $u \in W^{1,2}_{loc}(M)$, taking approximations by functions from $C^\infty_c(M)$.

So it remains to remove requirement $V_+ \in L^p_{loc}(M)$ for $n \geqslant 4$, replacing it by the inclusion $V_+ \in L^2_{loc}(M)$. This can be done as follows. Let us fix functions $u \in W^{1,2}_{loc}(M)$ and $\phi \in C^1_{comp}(M)$ with a locally Lipschitz gradient. Then regularize V_+, replacing it by $V^{(k)}_+(x) = V_+(x)$ if $V_+(x) \leqslant k$, and $V^{(k)}_+(x) = k$ if $V_+(x) > k$; here $k = 1, 2, \ldots$. Then the identity (5.3) holds with $V^{(k)} = V^{(k)}_+ + V_-$ instead of V because $V^{(k)} \in L^p_{loc}(M)$. But now we can take the limit as $k \to \infty$. The only terms depending on k in (5.3) will be two identical terms,

$$\int_M V^{(k)}_+ |\phi u|^2 \, d\mu,$$

on the left and right hand sides. This integral obviously has a limit (possibly $+\infty$) because the integrand converges monotonically. By the Beppo Levi theorem this limit equals

$$\int_M V_+ |\phi u|^2 \, d\mu, \tag{5.16}$$

so taking $k \to \infty$ we see that (5.3) holds for V.

If we only require that $u \in W^{1,2}_{loc}(M)$, then both sides of (5.3) can possibly be $+\infty$. If we know, however, that $u \in \mathrm{Dom}(H_{max})$ then the right hand side is finite (which in fact just means the finiteness of the integral (5.16)). Then the left hand side is finite too.

8. Using the identity (5.3) which is now established for all $u \in \mathrm{Dom}(H_{max})$, we can finish the proof of Theorem 1.1 by repeating the arguments of the proof of Theorem 5.2 which follow after this identity. ∎

6. EXAMPLES AND FURTHER COMMENTS

In this section we will provide several examples, further results and some relevant bibliographical comments (by necessity incomplete).

1. Let us comment about the *gauge invariance* for the magnetic Schrödinger operators. It is easy to see that if we replace A by $A' = A + d\phi$ with a real-valued $\phi \in C^1(M)$, such that $\nabla\phi \in \mathrm{Lip}_{loc}(M)$, then we have

$$H_{A',V} = e^{-i\phi} H_{A,\phi} e^{i\phi},$$

both for minimal and maximal operators defined by the expression $H_{A,V}$. Therefore it is clear that being essentially self-adjoint is a gauge invariant property, i.e., it does not change under any gauge transformation $A \mapsto A + d\phi$. This well known observation was extended by H. Leinfelder [42] to a very general class of operators and gauge transformations with minimal regularity conditions. He considered the case $M = \mathbb{R}^n$ (with the standard metric) but his arguments can be easily extended to the case of arbitrary Riemannian manifolds, so we will formulate the result for the general case. Let us consider a class $\mathscr{L}_2(M)$ which consists of operators $H_{A,V}$ on a Riemannian manifold (M, g) with $A \in L^4_{loc}(M)$, $d^*A \in L^2_{loc}(M)$ and $V \in L^2_{loc}(M)$. Assume further that we have two operators $H_{A,V}, H_{A',V} \in \mathscr{L}_2(M)$ and $A' = A + d\phi$ where ϕ is a distribution on M. Then the essential self-adjointness properties for A and A' are equivalent.

If M has vanishing cohomology $H^1(M, \mathbb{R})$ (e.g., if M is simply-connected) then the gauge invariance above means that the essential self-adjointness depends in fact on the magnetic field $B = dA$ (which is a 2-form or a de Rham current of degree 2) and not on the magnetic potential A itself.

2. Let us give some particular cases of Theorem 1.1.

THEOREM 6.1. *Let (M, g) be a complete Riemannian manifold. Then the magnetic Laplacian $-\Delta_A = d_A^* d_A$ is essentially self-adjoint in $L^2(M, d\mu)$ for any magnetic potential $A \in Lip_{loc}(M)$ and any positive smooth measure $d\mu$.*

Proof. Take $V \equiv 0$ and use Theorem 5.2. ∎

Theorem 6.1 generalizes the classical theorem by M. Gaffney [22] which corresponds to the case when $A = 0$ and $d\mu = d\mu_g$ is the Riemannian measure.

Note however that in fact the proof of Theorem 1.1 uses some elements of the Gaffney proof.

N. N. Ural'ceva [73] and S. A. Laptev [41] provided examples of elliptic operators in $L^2(\mathbb{R}^n, dx)$ of the form

$$\frac{\partial}{\partial x^j}\left(g^{jk}(x)\,\frac{\partial}{\partial x^k}\right)$$

(with smooth positive definite matrices (g^{jk})) which are not essentially self-adjoint due to the fact that the coefficients g^{jk} are "rapidly growing". In these examples the inverse matrix (g_{jk}) is vice versa "rapidly decaying", which implies that $\mathbb{R}^n$ with the metric (g_{jk}) is not complete.

THEOREM 6.2. *Let (M, g) be a complete Riemannian manifold with a positive smooth measure $d\mu$, $A \in \Lambda^1_{(1)}(M)$, $V \in L^2_{loc}(M)$, and $V(x) \geqslant -C$,*

$x \in M$, with a constant C. Then the magnetic Schrödinger operator $H = -\Delta_A + V(x)$ is essentially self-adjoint.

In case when $M = \mathbb{R}^n$ (with the standard metric and measure), $A = 0$ and $V \in L^\infty_{loc}(M)$, this result was established independently by T. Carleman [8] and K. Friedrichs [20], and the Carleman proof is reproduced in the book of I. M. Glazman [25], Theorem 34 in Sect. 3. The fact that in this case the requirement $V \in L^\infty_{loc}$ can be replaced by $V \in L^2_{loc}(\mathbb{R}^n)$, was established by T. Kato [39] (see also [54], Sect. X.4).

The work by T. Kato was partially motivated by the paper of B. Simon [66] who proved the essential self-adjointness under an additional restriction compared with [39]. The reader may consult Chapters X.4, X.5 in M. Reed and B. Simon [54] for more references, motivations and a review.

Though the completeness requirement looks natural in case of semi-bounded operators, sometimes it can be relaxed and incompletness may be compensated by a specific behavior of the potential (see e.g., A. G. Brusentsev [6] and also the references there).

We will mention a few more references which might be useful for the reader. Reviews of different aspects of self-adjointness can be found e.g., in [35, 37, 54, 61, 62]. Papers by M. Braverman [4], and M. Lesch [44] contain conditions of essential self-adjointness of operators on sections of vector bundles. In particular, operators considered in [44] generalize magnetic Schrödinger operators. Semi-bounded operators of higher order were studied by A. G. Brusentsev [5]. A. Iwatsuka [33] gave explicit conditions on the potentials of the magnetic Schrödinger operator in $\mathbb{R}^n$ (including interaction of electric and magnetic fields) which are sufficient for the essential self-adjointness. Different aspects of essential self-adjointness in domains in $\mathbb{R}^n$ and manifolds with boundary where behavior of the coefficients near the boundary is relevant, were studied e.g., by A. G. Brusentsev [6, 7], K. Jörgens [34], R. Mazzeo and R. McOwen [48]. Finite speed propagation is an alternative method to prove essential self-adjointness (P. Chernoff [11], A. A. Chumak [12]). I. Oleinik discovered a new method which makes the relation between classical and quantum completeness (the later means essential self-adjointness) more explicit—see [50, 51, 52, 61, 62, 4, 44]. H. Leinfelder and C. Simader [43] (see also [16]) proved the essential self-adjointness for the magnetic Schrödinger operators $H_{A,V}$ in $\mathbb{R}^n$ with $V \geqslant 0$ and with the minimal local regularity requirements on A, V.

About other conditions of essential self-adjointness for $H_{0,V}$ and $H_{A,V}$ formulated in terms of the potentials and sometimes allowing operators which are not semi-bounded below see e.g., [2, 19, 24, 27, 31, 36, 40, 45, 49, 55, 56, 57, 58, 59, 63, 65, 67, 71, 72] and references there.

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