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Discrete behavior of Seshadri constants on surfaces

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Abstract

Working over **C**, we show that, apart possibly from a unique limit point, the possible values of multi-point Seshadri constants for general points on smooth projective surfaces form a discrete set. In addition to its theoretical interest, this result is of practical value, which we demonstrate by giving significantly improved explicit lower bounds for Seshadri constants on $\mathbf{P}^2$ and new results about ample divisors on blow ups of $\mathbf{P}^2$ at general points.

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1. Introduction

The situation often arises that one has a birational morphism of smooth projective varieties $\pi : Y \to X$, where X is well understood and one wants to understand Y. For example, even if one knows precisely which divisors on X are ample, or nef, it is often a difficult problem to determine the same for Y. The problem of determining ampleness or nefness on Y is closely related to the problem of computing multi-point Seshadri constants on X.

Even in the case that X is a surface, it is quite hard to compute Seshadri constants exactly. Our approach instead is to study what values are possible. Of course, the more one knows about a surface X the more one would hope to be able to restrict what is possible. What has not been previously recognized is that easily obtained information about X already puts a lot of structure on the set of possible values of Seshadri constants: if the blown up points are general, the set of possible values is, apart possibly from a unique limit point, a discrete set. This has significant consequences for determining Seshadri constants on surfaces; one consequence, for example, is our Theorem 1.2.1, which establishes a framework for computing arbitrarily accurate lower bounds for multi-point Seshadri constants. Although we do not focus on implementing this framework here (for a detailed consideration of algorithmic concerns, see the unpublished posting [10]), we do demonstrate what our methods can achieve with results easily at hand by

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giving significant improvements to previously known lower bounds for multi-point homogeneous Seshadri constants on $\mathbf{P}^2$ (Corollary 1.2.3), and we determine ampleness for many new cases on blow ups Y of $\mathbf{P}^2$ at general points (Corollary 1.2.4).

1.1. Seshadri constants

Let X be a smooth projective variety of dimension $N > 1$, and let L be a nef divisor class (i.e., $L^r \cdot Z \geq 0$ for every effective r-cycle Z on X). Given a positive integer n and a nonzero real vector $\boldsymbol{\ell} = (l_1, \dots, l_n)$ with each $l_i \geq 0$, the multi-point Seshadri constant for $\boldsymbol{\ell}$ and points $p_1, \dots, p_n$ of X is the real number

$$\varepsilon(X, L, l_1 p_1, \dots, l_n p_n) = \inf \left\{ \frac{L \cdot C}{\sum_{i=1}^{n} l_i \operatorname{mult}_{p_i} C} \right\},$$

where the infimum is taken with respect to all curves C through at least one of the points. For the one-point and the multi-point homogeneous case (in which $l_i = 1$ for all i and which most previous work has focused on), see [6] or [17]. We also take $\varepsilon(X, L, n, \boldsymbol{\ell})$ to be defined as $\sup\{\varepsilon(X, L, l_1 p_1, \dots, l_n p_n)\}$, where the supremum is taken with respect to all choices of n distinct points p_i of X. For the homogeneous case, we write simply $\varepsilon(X, L, n)$ in place of $\varepsilon(X, L, n, (1, \dots, 1))$. Since the homogeneous case where $X = \mathbf{P}^2$ and L is the class of a line is of particular interest, we will denote $\varepsilon(\mathbf{P}^2, L, n)$ simply by $\varepsilon(n)$.

It is well known and not difficult to prove that $\varepsilon(X, L, p_1, \dots, p_n) \leq \sqrt[N]{L^N/n}$, but lower bounds are much more challenging (see [14,12,22]). It is not hard to see that $\varepsilon(X, L, n) = \varepsilon(X, L, p_1, \dots, p_n)$ for *very general* points $p_1, \dots, p_n$ (i.e., in the intersection of countably many Zariski-open and dense subsets of X^n), although some results (see [15,17]) suggest that the equality might hold in fact for *general* points (i.e., in a Zariski-open subset of X^n). When L is a big (i.e., $L^2 > 0$) and nef divisor on a surface X, our Theorem 1.2.1 gives lower bounds for $\varepsilon(X, L, n)$ which in fact hold for $\varepsilon(X, L, p_1, \dots, p_n)$ for general points p_i.

Two methods have been used to give lower bounds on $\varepsilon(X, L, n)$ for surfaces X. One involves explicit constructions of nef divisors, the other involves ruling out the existence of certain putative reduced irreducible curves of negative self-intersection (so-called L-abnormal curves). Both methods, which work also in the nonhomogeneous case, depend on looking at the surface Y obtained from X by the morphism $\pi : Y \to X$ blowing up distinct points $p_i \in X$, $1 \leq i \leq n$. If E_i is the divisor class of the exceptional curve $\pi^{-1}(p_i)$, then clearly $\varepsilon(X, L, l_1 p_1, \dots, l_n p_n)$ is the largest t such that $F_t = \pi^* L - t(l_1 E_1 + \cdots + l_n E_n)$ is nef, hence $\varepsilon(X, L, n, \boldsymbol{\ell}) \geq t$ whenever $F_t = \pi^* L - t(l_1 E_1 + \cdots + l_n E_n)$ is a nef $\mathbf{R}$-divisor class (i.e., a nef element of the divisor class group with real coefficients).

Alternatively (see Lemma 2.1.1), suppose each l_i is rational and t, $0 \leq t < \sqrt{L^2/\boldsymbol{\ell}^2}$, is rational, where $\boldsymbol{\ell}^2$ signifies the usual dot product. Then $t \leq \varepsilon(X, L, n, \boldsymbol{\ell})$ if and only if, for general points p_i there are no reduced and irreducible curves $C \subset X$ such that $F_t \cdot H < 0$ where $H = \pi^* C - h_1 E_1 - \cdots - h_n E_n$ is the class of the proper transform of C (so h_i is the multiplicity of C at p_i); note that $F_t \cdot H < 0$ is equivalent to $(L \cdot C)/(l_1 h_1 + \cdots + l_n h_n) < t$. In the homogeneous case we call such a curve C an *L-abnormal* curve (or simply *abnormal* if L is understood), following Nagata [14], who, in case $\boldsymbol{\ell} = (1, \dots, 1)$ and L is a line in $X = \mathbf{P}^2$, called any such curve C an abnormal curve (also referred to as submaximal in [2,17]). Moreover, if $\operatorname{Pic}(X)/\sim$, where $\sim$ denotes numerical equivalence, is cyclic (as is the case for $X = \mathbf{P}^2$), then for any such C we have $\varepsilon(X, L, n) = (L \cdot C)/(h_1 + \cdots + h_n)$ by Lemma 2.1.2. (For $\mathbf{P}^2$, Nagata also found all curves abnormal for each $n < 10$, showed no curve is abnormal for n when n is a square and conjectured there are no abnormal curves for $n \geq 10$.)

So, to exemplify the first method, if for some choice of distinct points p_i one finds positive integers d and t such that $d\pi^* L - t(l_1 E_1 + \cdots + l_n E_n)$ is nef, it follows that $\varepsilon(X, L, n, \boldsymbol{\ell}) \geq d/t$. This basic idea is used in [3] (for $X = \mathbf{P}^2$) and [8] (for surfaces generally) to obtain bounds of the form $\varepsilon(X, L, n) \geq (\sqrt{L^2/n})\sqrt{1 - 1/f(n)}$ where $f(n)$, for some values of n, is a quadratic function of n. Note that the bound $\varepsilon(n) \geq (1/\sqrt{n})(\sqrt{1 - 1/f(n)})$ is equivalent to the inequality $\mathcal{R}_n(L) \leq 1/f(n)$ of [3], where $\mathcal{R}_n(L)$ is what is called in [3] the *n-th remainder* of the divisor class L. Alternatively, to exemplify the second method, suppose one is given $F_t = \pi^* L - t(l_1 E_1 + \cdots + l_n E_n)$. One then constructs a set $o_n(F_t)$ of values which one somehow can show contains $(\pi^* L \cdot D)/(-(l_1 E_1 + \cdots + l_n E_n) \cdot D)$ for every effective, reduced, irreducible divisor D on Y with $F_t \cdot D < 0$, if any. (We show how to obtain a specific such set $o_n(F_t)$ after Lemma 2.1.4.) For as many values $v \in o_n(F_t)$ as possible, one attempts to show that there is no such D for which $v = (\pi^*L \cdot D)/(-(l_1E_1 + \cdots + l_nE_n) \cdot D)$. If c is the infimum of the remaining values in $o_n(F_t)$, then we conclude that F_c is nef and hence that $c \le \varepsilon(X, L, n, \boldsymbol{\ell})$. Thus the more values $v \in o_n(F_t)$ one can rule out, the better this bound becomes. For the homogeneous case, this is the basic idea used implicitly in [21,17,19,20], with the latter obtaining the bound $\varepsilon(n) \ge (1/\sqrt{n})\sqrt{1 - 1/(12n + 1)}$.

Given $\boldsymbol{\ell}$ and a big and nef L, we can, for each $c < \sqrt{L^2/\boldsymbol{\ell}^2}$, give a finite set $o_n(F_c)$ (see Theorem 2.1.5) depending only on ℓ, c, L^2 and the semigroup of L-degrees $\{C \cdot L : C$ is an effective divisor$\}$ of curves. This shows the set of possible values of $\varepsilon(X, L, n, \boldsymbol{\ell})$ is either finite or an increasing discrete sequence and, in the latter case, $\sqrt{L^2/\boldsymbol{\ell}^2}$ is its unique limit point, i.e., apart from $\sqrt{L^2/\boldsymbol{\ell}^2}$, the set of possible values of $\varepsilon(X, L, n, \boldsymbol{\ell})$ is discrete. This has a number of conceptual consequences. For example, if we write this increasing sequence as $o(n, L)_1 < o(n, L)_2 < \cdots$, and if we were to show that $o(n, L)_i < \varepsilon(X, L, n)$, then in fact it automatically follows that $o(n, L)_{i+1} \le \varepsilon(X, L, n)$. Moreover, to show $\varepsilon(X, L, n, \boldsymbol{\ell}) \ge c$ for any $c < \sqrt{L^2/\boldsymbol{\ell}^2}$, there are only finitely many values of $\varepsilon(X, L, n, \boldsymbol{\ell})$ less than c one must rule out. Moreover, carrying this calculation out will either show that $\varepsilon(X, L, n, \boldsymbol{\ell}) \ge c$, or it will compute $\varepsilon(X, L, n, \boldsymbol{\ell})$ exactly (by finding which value in $o_n(F_c)$ is the correct one).

Our general results about the existence of $o_n(F_c)$ with the structure as claimed above are stated in Theorem 2.1.5 and proved in Section 2.1. Using refinements of these results which we obtain in Section 2.2, we then prove Theorem 1.2.1 (which shows how theoretical results ruling out the existence of abnormal curves can be converted into bounds on $\varepsilon(X, L, n)$) and Corollary 1.2.3 (which gives lower bounds for $\varepsilon(n)$ that for most values of n are significantly better than what was known previously). As another application, we also obtain in Corollary 1.2.4 improved results on ample divisors on blow ups of $\mathbf{P}^2$.

1.2. Applications

Our results involve a related apparently simpler problem, that of the existence of curves with a given sequence of multiplicities $\mathbf{m} = (m_1, \ldots, m_n)$ at given points $p_1, \ldots, p_n \in X$. Let us denote by $\alpha(X, L, \mathbf{m}, p_1, \ldots, p_n)$ (respectively, $\alpha_0(X, L, \mathbf{m}, p_1, \ldots, p_n)$) the least degree $L \cdot C$ of a curve C (respectively, irreducible curve) passing with multiplicity at least m_i (respectively, exactly m_i) through each point p_i. If the points are in general position in X, we write simply $\alpha(X, L, \mathbf{m})$ and $\alpha_0(X, L, \mathbf{m})$. When focusing on the case that L is a line in $X = \mathbf{P}^2$, we will denote $\alpha(\mathbf{P}^2, L, \mathbf{m})$ and $\alpha_0(\mathbf{P}^2, L, \mathbf{m})$ simply by $\alpha(\mathbf{m})$ and $\alpha_0(\mathbf{m})$. Given an integer m, we will denote the vector $(m, \ldots, m)$ with r entries of m by $m^{[r]}$. As a consequence of our results in Section 2, we will prove the following:

Theorem 1.2.1. *Let X be a smooth projective surface, L a big and nef divisor, $n \ge 2$ an integer and $\mu \ge 1$ a real number.*

(a) *If $\alpha(X, L, m^{[n]}) \ge m\sqrt{L^2(n - 1/\mu)}$ for every integer $1 \le m < \mu$, then*

$$\varepsilon(X, L, n) > \sqrt{\frac{L^2}{n}}\sqrt{1 - \frac{1}{(n-2)\mu}}.$$

(b) *If $\alpha_0(X, L, m^{[n]}) \ge m\sqrt{L^2(n - 1/\mu)}$ for every integer $1 \le m < \mu$, and if*

$$\alpha_0((m^{[n-1]}, m + k)) \ge \frac{mn + k}{n}\sqrt{L^2(n - 1/\mu)}$$

for every integer $1 \le m < \mu/(n-1)$ and every integer k with

$$k^2 < (n/(n-1)) \min(m, m + k),$$

then

$$\varepsilon(X, L, n) \ge \sqrt{\frac{L^2}{n}}\sqrt{1 - \frac{1}{n\mu}}.$$

In order to apply the theorem, one just needs to know some values of α. Drawing on asymptotic results of Alexander and Hirschowitz, for example, it is possible to give bounds on ε for surfaces on which the Picard group is generated by a single ample divisor. In fact, the main result of [1] already implies ampleness for certain divisors (and so bounds on $\varepsilon(X, L, n)$ for some n); a suitable interpretation of Theorem 1.2.1 yields the following corollary, linking the mentioned asymptotic results to lower bounds for Seshadri constants in the form $(\sqrt{L^2/n})\sqrt{1-1/f(n)}$, analogous to what is known for $\mathbf{P}^2$.

Corollary 1.2.2. *Let X be a surface on which the Picard group is generated by a single ample divisor L, and let $\mathbf{m} : \mathbf{N} \to \mathbf{N}$ be a map such that for every $m < \mathbf{m}(n)$*

$$\dim \left| \frac{\alpha(X, L, m^{[n]})}{L^2} L \right| \geq n \frac{m(m+1)}{2} \tag{1}$$

holds. Then there is an n_0 such that for $n \geq n_0$,

$$\varepsilon(X, L, n) \geq \sqrt{\frac{L^2}{n}} \sqrt{1 - \frac{1}{(n-2)\mathbf{m}(n)}}.$$

Moreover, there exists such an $\mathbf{m}(n)$ with $\lim_{n\to\infty} \mathbf{m}(n) = \infty$ and hence $\lim_{n\to\infty} n\mathcal{R}_n(L) = 0$.

We can give much more specific bounds for $\mathbf{P}^2$. For instance, for $X = \mathbf{P}^2$ it is known that $\alpha(m^{[n]}) \geq m\sqrt{n}$ for $n \geq 10$ and $m \leq \lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 3)/2$ (see the proof of Corollary 1.2(a) of [9]), so we may apply Theorem 1.2.1(b) with $\mu = 1 + \lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 3)/2$ whenever $1 < 1 + \lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 3)/2$, so for $n \geq 16$. (Note that the hypotheses involving $k \neq 0$ are vacuous when $\mu/(n-1) < 1$.) On the other hand, results of [4] imply that $\alpha(m^{[n]}) \geq m\sqrt{n}$ for $n \geq 10$ and $m \leq 20$, so we may apply Theorem 1.2.1(b) with $\mu = 21$ and $n \geq 16$. (Here the only $k \neq 0$ allowed is for $k = m = 1$, but it is known and easy to see that a double point and general points of multiplicity 1 impose independent conditions on forms on $\mathbf{P}^2$ of degree α. Thus $(\alpha + 3/2)^2/2 > \binom{\alpha+2}{2} > 3 + (n-1) = n+2$, so for $k = m = 1$, $\alpha_0((m^{[n-1]}, m+k)) \geq \frac{mn+k}{n}\sqrt{(n-1/\mu)}$ since $(\alpha_0 + 3/2)^2 \geq (\alpha + 3/2)^2 > 2n + 4 \geq (\sqrt{n}+2)^2$ for $n \geq 16$, and $(\sqrt{n}+2)^2 \geq (\frac{n+1}{\sqrt{n}} + 3/2)^2 = (\frac{mn+k}{n}\sqrt{n} + 3/2)^2 > (\frac{mn+k}{n}\sqrt{(n-1/\mu)} + 3/2)^2$.) We thus immediately obtain an explicit bound which for most n is substantially better than what was known previously[1]:

Corollary 1.2.3. *For every $n \geq 16$,*

$$\varepsilon(n) \geq \max\left(\frac{1}{\sqrt{n}} \sqrt{1 - \frac{1}{n(1 + \lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 3)/2)}}, \frac{1}{\sqrt{n}} \sqrt{1 - \frac{1}{21\,n}} \right).$$

As a final application, again for blow ups Y of $X = \mathbf{P}^2$ where L is a line, we obtain an improved criterion for which divisor classes of the form $dL - m(E_1 + \cdots + E_n)$ are ample. If Nagata's conjecture [14] is true, it is not hard to see that $F = dL - m(E_1 + \cdots + E_n)$ is ample whenever d and m are positive integers such that $d^2 > m^2 n$, where $\pi : Y \to \mathbf{P}^2$ is given by blowing up $n \geq 10$ very general points and L is the class of a line. That F is in fact ample has been verified for $m = 1$ [23], $m = 2$ [3] and $m = 3$ [20]. Our result extends these substantially for large n (see [8], however, for an even stronger result if one merely wishes to conclude that F is nef):

Corollary 1.2.4. *Let $n \geq 16$, $t > \sqrt{n}m$, and $m > 0$ be integers and consider the divisor class $F = tL - m(E_1 + \cdots + E_n)$ on the blow up Y of $\mathbf{P}^2$ at n general points, where L is the pullback to Y of a line in $\mathbf{P}^2$. If $1 \leq m < \sqrt{\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 3)/2 + 1 - 1/n}$, then F is ample.*

We end this introduction by discussing Corollary 1.2.3 in the context of what was known previously in case $X = \mathbf{P}^2$. It is convenient for comparison to express lower bounds for Seshadri constants on $\mathbf{P}^2$ in the form $(\sqrt{1/n})\sqrt{1 - 1/f(n)}$. Note that the larger $f(n)$ is, the better is the bound. Perhaps the best previous general bound is given in [20], for which

[1] After submission of this paper, a result for $m \leq 42$ has been announced by Dumnicki [7] which, together with [5] for the $k \neq 0$ case, imply the stronger bound $\varepsilon(n) \geq (\sqrt{1/n})\sqrt{1 - 1/43\,n}$ if $n \geq 16$.

$f(n) = 12n + 1$ for all $n \geq 10$. For Corollary 1.2.3, which applies for all $n \geq 17$, $f(n)$ can be taken to be quadratic in n but always larger than $12n + 1$.

The article [3] gives bounds which for special values of n are better than those of [20], and for these special values $f(n)$ is quadratic in n. (In particular, if $n = (ai)^2 \pm 2i$ for positive integers a and i, then $f(n) = (a^2 i \pm 1)^2$, and, if $n = (ai)^2 + i$ for positive integers a and i with $ai \geq 3$, then $f(n) = (2a^2 i + 1)^2$). However, except in special cases, such as when $n - 1$ or $n \pm 2$ is a square, the bounds of Corollary 1.2.3 are better for n large enough. (To see this look at coefficients of the n^2 term in $f(n)$.)

Bounds are also given in [8]; they apply for all values of n for all surfaces and are almost always better than any bound for which $f(n)$ is linear in n (more precisely, given any constant a, let $\nu_a(n)$ be the number of integers i from 1 to n for which $f(i)$ from [8] is bigger than ai; then $\lim_{n\to\infty} \nu_a(n)/n = 1$). However, although the bounds in [8] are not hard to compute for any given value of n, there is no simple explicit formula for $f(n)$, so it is hard to make general comparisons. Nonetheless, computations in case $X = \mathbf{P}^2$ for specific values of n suggest that the bounds we obtain here for $\mathbf{P}^2$ are typically if not almost always better than those of [8].

It is worth noting that the bounds in Corollary 1.2.3 are not the best that one can obtain using our results here in conjunction with the methods of [9]. While [9] does give explicit formulas that hold in general, applying the methods of [9] for specific values of n usually gives notably better results than one can express in terms of an explicit formula. Since the simple explicit formula for $f(n)$ as given in Corollary 1.2.3 is based on an explicit but necessarily suboptimal formula from [9], one can usually get better results for specific values of n by directly applying the methods of Section 2 and [9]. (For specific examples of this, see the unpublished posting [10].)

2. Main results

In the first section we obtain results about abnormal curves in general. In the second section we sharpen and apply those results in the homogeneous case. For the rest of this paper we assume that X is a smooth projective surface.

2.1. Abnormal curves

Let $\pi : Y \to X$ be obtained by blowing up distinct points p_i on X and let $E_i = \pi^{-1}(p_i)$. Let L be a nef divisor on X. Abnormality, as we introduced it above, is related to nefness of divisors on Y of the form $\pi^* L - E_1 - \cdots - E_n$. In order more generally to study nefness of divisors of the form $\pi^* L - l_1 E_1 - \cdots - l_n E_n$, it is convenient to extend our notion of abnormality. Let F be a numerical equivalence divisor class on Y. We will then say a curve $D \subset Y$ is *F-abnormal* if D is reduced and irreducible with $F \cdot D < 0$. In case the points p_i are general, L is nef on X and $F = \pi^* L - (E_1 + \cdots + E_n)$, then a curve $C \subset X$ is L-abnormal according to our previous use of the word, if and only if its proper transform $\tilde{C}$ is F-abnormal.

For simplicity, we will by identification just write L in place of $\pi^* L$. The next lemma establishes a connection between values of s for which $F_s = L - s(l_1 E_1 + \cdots + l_n E_n)$ is nef and the occurrence of abnormal curves.

Lemma 2.1.1. *Let L be a nef divisor on X, let $\pi : Y \to X$ be obtained by blowing up n distinct points p_i on X and let $F_t = L - t(l_1 E_1 + \cdots + l_n E_n)$, where $E_i = \pi^{-1}(p_i)$ and where t and each $l_i \geq 0$ is real (such that $\boldsymbol{\ell} = (l_1, \ldots, l_n)$ is not 0).*

(a) *If F_t is nef, then $0 \leq t \leq \sqrt{L^2/\boldsymbol{\ell}^2}$.*

(b) *Let $0 \leq t \leq \sqrt{L^2/\boldsymbol{\ell}^2}$. If D is an F_t-abnormal curve on Y, then the largest s such that F_s is nef is at most $(L \cdot D)/D \cdot (l_1 E_1 + \cdots + l_n E_n)$. Moreover, any such D satisfies $D^2 < 0$.*

(c) *Let t and ℓ be rational and $0 \leq t < \sqrt{L^2/\boldsymbol{\ell}^2}$. Then the following are equivalent:*
   (i) *there exists a numerical equivalence class H which for general points p_i is the class of an F_t-abnormal curve;*
   (ii) *$\varepsilon(X, L, n, \boldsymbol{\ell}) < t$; and*
   (iii) *F_t is not nef for any choice of the points p_i.*

Proof. (a) We have $0 \leq t$ since F_t is nef and hence $tl_i = F_t \cdot E_i \geq 0$ for all i, while $t \leq \sqrt{L^2/\boldsymbol{\ell}^2}$ follows since any nef divisor has nonnegative self-intersection.

(b) If F_t is not nef, then $L^2 > 0$ (else $t = 0$ and $F_t = \pi^* L$ is nef). Since F_t is not nef, there is an F_t-*abnormal* curve D. If F_s is nef, then $L \cdot D - s(l_1 E_1 + \cdots + l_n E_n) \cdot D = F_s \cdot D \geq 0$, so $s \leq (L \cdot D)/D \cdot (l_1 E_1 + \cdots + l_n E_n)$.

To see $D^2 < 0$, note that up to numerical equivalence, we can write D as $C' - m_1 E_1 - \cdots - m_n E_n$, for some integers m_i where $C' = \pi^{-1}(\pi(D))$. Since $t \geq 0$, we have $F_t \cdot E_i \geq 0$ for all i, so D cannot by E_i for any i. Thus $\pi(D)$ is a curve, and $m_i \geq 0$ for each i. Since $L^2 > 0$, we can by the Hodge index theorem write $C = cL + B$ for some real $c \geq 0$ and some **R**-divisor B with $B \cdot L = 0$ and $B^2 \leq 0$, where $C = \pi(D)$. Thus $C^2 = c^2 L^2 + B^2 \leq (cL)^2 = (C \cdot L)^2/L^2 < (l_1 m_1 + \cdots + l_n m_n)^2/\boldsymbol{\ell}^2$, where the strict inequality follows since D is F_t-abnormal and $t \leq \sqrt{L^2/\boldsymbol{\ell}^2}$. But $(l_1 m_1 + \cdots + l_n m_n)^2/\boldsymbol{\ell}^2 \leq \sum_i m_i^2$ by Cauchy–Schwarz, so $D^2 = C^2 - \sum_i m_i^2 < 0$, as claimed.

(c) If an F_t-abnormal curve of class H exists for general sets of distinct points p_i, then since $\varepsilon(X, L, n, \boldsymbol{\ell}) = \varepsilon(X, L, n, l_1 p_1, \ldots, l_n p_n)$ on a dense set, from the definitions it follows that $\varepsilon(X, L, n, \boldsymbol{\ell}) \leq L \cdot H/(H \cdot (l_1 E_1 + \cdots + l_n E_n)) < t$. If $\varepsilon(X, L, n, \boldsymbol{\ell}) < t$, then by definition F_t is not nef for every set of points p_i. Finally, if F_t is not nef for every set of points $\mathbf{p} = (p_1, \ldots, p_n)$, then for each choice of the points $\mathbf{p}$ one can choose an F_t-abnormal $H_{\mathbf{p}}$. By Lemma 2.1.3, there are only finitely many classes of such $H_{\mathbf{p}}$ in $\mathrm{Pic}(Y)/\sim$, and each of them is effective on a Zariski-closed set. Hence one of them (say H) must be effective for all choices of the points $\mathbf{p}$ and irreducible for a general set of points $\mathbf{p}$, with $H \cdot F_t < 0$. □

We now state a lemma of particular interest, since it applies to the case of n general points on $X = \mathbf{P}^2$.

Lemma 2.1.2. *Assume the hypotheses of Lemma 2.1.1 together with the additional hypothesis that the points p_i are general points of X. If D and F_t are as in Lemma 2.1.1(b) with $1 = l_1 = \cdots = l_n$, and if every* **R***-divisor on X (up to numerical equivalence) is a real multiple of L, then the largest s such that F_s is nef is precisely $s = (L \cdot D)/D \cdot (E_1 + \cdots + E_n)$; i.e., $\varepsilon(X, L, n) = s$.*

Proof. We use the argument of Proposition 4.5 of [18]. Suppose that there is another F_t-abnormal curve D', whose class is $C'' - m'_1 E_1 - \cdots - m'_n E_n$. Since the points p_i are general, we may assume that $m_1 \geq m_2 \geq \cdots \geq m_n$ and $m'_1 \geq m'_2 \geq \cdots \geq m'_n$, and so by Chebyshev's sum inequality we have $((m_1 + \cdots + m_n)/n)((m'_1 + \cdots + m'_n)/n) \leq (m_1 m'_1 + \cdots + m_n m'_n)/n$. But C and C' are positive multiples of L, so there are positive reals c and c' such that $C = cL$ and $C' = c'L$. We have therefore that $cL^2/(m_1 + \cdots + m_n)$ and $c'L^2/(m'_1 + \cdots + m'_n)$ both are less than $\sqrt{L^2}/\sqrt{n}$, and hence

$$\frac{ncc'(L^2)^2}{\sum_i m_i m'_i} \leq \frac{cc'(L^2)^2}{\frac{\sum_i m_i}{n} \frac{\sum_i m'_i}{n}} < \frac{n^2 L^2}{n} = nL^2,$$

so $cc'(L^2) < \sum_i m_i m'_i$; i.e., $D \cdot D' < 0$. Since D and D' are integral, we must have $D = D'$. Thus every F_t-abnormal curve B gives the same value for $(L \cdot B)/B \cdot (E_1 + \cdots + E_n)$. By (b), F_s cannot be nef for any value of s bigger than $s = (L \cdot D)/D \cdot (E_1 + \cdots + E_n)$, yet for this value of s we have just shown there are no F_s-abnormal curves, so F_s is in fact nef, and hence $\varepsilon(X, L, n) = s$. □

To state the general fact used in Lemma 2.1.1(c), we define the notion of a *sufficient test system*. Let $\mathbf{p} = \{p_1, \ldots, p_n\}$ be a set of distinct points on a surface X, and let $\pi : Y_{\mathbf{p}} \to X$ be the morphism obtained by blowing up the points p_i with, as usual, $E_i = \pi^{-1}(p_i)$. Given a **Q**-divisor L on X and nonnegative rationals $m_1, \ldots, m_n$, consider a set $\{D_1, \ldots, D_k\}$ of numerical equivalence classes of divisors on X together with vectors $\mathbf{h}_1, \ldots, \mathbf{h}_k \in \mathbf{Z}^n_{\geq 0}$. We refer to $\{(D_i, \mathbf{h}_i)\}_{i=1,\ldots,k}$ as an $(L, \{m_i\})$-sufficient test system if whenever $\mathbf{p}$ is such that none of the classes $C_i = D_i - h_{i1} E_1 - \cdots - h_{in} E_n$ is (up to numerical equivalence) the class of a reduced irreducible curve, then $F = L - m_1 E_1 - \cdots - m_n E_n$ is nef. Remark that by definition a $(L, \{m_i\})$-sufficient test system is always finite.

Lemma 2.1.3. *Let L be a big and nef* **Q***-divisor on X, and let $m_1, \ldots, m_n$ be nonnegative rationals with $m_1^2 + \cdots + m_n^2 < L^2$. Then there exists an $(L, \{m_i\})$-sufficient test system $\{(D_i, \mathbf{h}_i)\}_{i=1,\ldots,k}$. Moreover, if $U \subset X^n$ is the set of all n-tuples of distinct points, then for each class $C_i = D_i - h_{i1} E_1 - \cdots - h_{in} E_n$ the subset of U such that C_i is the* *class of an effective divisor on the blowup of* $\mathbf{p} \in U$ *is Zariski-closed. (In particular, the subset of* U *such that* F *is nef on the blowup of* $\mathbf{p} \in U$ *is Zariski-open.)*

Proof. Clearly, there is an s such that sF is effective. Let $L_1, \dots, L_\rho$ be ample effective divisors which generate the group of numerical equivalence classes on X. For suitable $a_{i0}, a_{ij} \in \mathbf{N}$, with $1 \le i \le \rho$, $1 \le j \le n$, the divisor classes $A_{i0} = a_{i0}L_i - (E_1 + \cdots + E_n)$, $A_{ij} = a_{ij}L_i - (E_1 + \cdots + E_n) - E_j$ are ample and effective, and they generate the group of numerical equivalence classes, independently of the choice of the points. Let $d_{ij} = sF \cdot A_{ij}$ for all i and all $0 \le j \le n$. If C is a divisor such that both $|C|$ and $|sF - C|$ are nonempty (which is necessary in order to have an F-abnormal curve C), then $0 < C \cdot A_{ij} \le d_{ij}$. Moreover, the class of an irreducible curve meeting F negatively must be of the form $C = D - h_1E_1 - \cdots - h_nE_n$, and clearly there are only a finite number of numerical equivalence classes of such C satisfying $0 < C \cdot A_{ij} \le d_{ij}$. Let these classes be $C_i = D_i - h_{i1}E_1 - \cdots - h_{in}E_n$, $i = 1, \dots, k$; we have shown that $\{(D_i, \mathbf{h}_i)\}_{i=1,\dots,k}$ is a $(L, \{m_i\})$-sufficient test system.

The set $\{\mathcal{H}_\gamma\}_{\gamma\in\Gamma_i}$ of all components in the Hilbert scheme of curves in X numerically equivalent to D_i is indexed by some finite set Γ_i (see e.g. [13], lecture 15). Since there are only finitely many D_i, it follows that $\Gamma = \bigcup \Gamma_i$ is finite. For each $\gamma \in \Gamma$, there is a flat family $\phi_\gamma : \mathcal{D}_\gamma \subset X \times \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ whose members are the curves parameterized by $\mathcal{H}_\gamma$; every F-abnormal curve of class C_i occurs as the birational transform of a fiber of some ϕ_γ, $\gamma \in \Gamma_i$, which has multiplicity h_{ij} at a point $p_j \in X$. Now the sets of (distinct) points $(p_1, \dots, p_j) \in U$ such that there exists a fiber of ϕ_γ with multiplicity at least h_{ij} at the point p_j is Zariski-closed in U (an explicit construction of this closed set, using sheaves of principal parts, can be found e.g. in [11], Section 4). Since the subset of U such that C_i is the class of an effective divisor on the blowup of $\mathbf{p} \in U$ is the union of the finitely many closed subsets determined by the ϕ_γ, $\gamma \in \Gamma_i$, it follows that it is Zariski-closed.

Finally, the divisor F is nef if and only if none of the classes C_i is effective, and we have seen that the set of points p_i for which none of them is effective is open. □

Such general claims as in Lemma 2.1.3 regarding the existence of a finite set of test classes for F_t to be nef can be sharpened and made more explicit in the case of general blow ups, as we now show.

Given a big and nef divisor $L \subset X$ and nonnegative integers $\boldsymbol{\ell} = (l_1, \dots, l_n)$, let $F = dL - l_1E_1 - \cdots - l_nE_n$ where $d = \sqrt{\boldsymbol{\ell}^2/L^2}$, so $F^2 = 0$. For each real $\delta \ge 0$, consider the $\mathbf{R}$-divisor $F(\delta) = d'L - l_1E_1 - \cdots - l_nE_n$ where $d' = \sqrt{(\boldsymbol{\ell}^2 + \delta)/L^2}$; note that $F(\delta)^2 = \delta$. The next lemma can be seen as a sharpening and extension of Theorem 4.1 in [2] to the case of multi-point Seshadri constants:

Lemma 2.1.4. *Let* $\pi : Y \to X$ *be the blow up of general points* $p_1, \dots, p_n \in X$. *Let* F *and* $F(\delta)$ *be as in the preceding paragraph with* $\delta > 0$. *If* H *is the class of an* $F(\delta)$*-abnormal curve* $\tilde{C}$, *then* $H = \pi^*C - h_1E_1 - \cdots - h_nE_n$ *for some nonnegative integers* $h_1, \dots, h_n$ *and for some effective divisor class* C *on* X *such that:*

(a) $h_1^2 + \cdots + h_n^2 < (1 + d^2L^2/\delta)^2/\gamma$, *where* γ *is the number of nonzero coefficients* $h_1, \dots, h_n$, *and*

(b) $h_1^2 + \cdots + h_n^2 - a \le C^2 \le (C \cdot L)^2/L^2 < (l_1h_1 + \cdots + l_nh_n)^2/(d^2L^2 + \delta)$, *where* a *is the minimum positive element of* $\{h_1, \dots, h_n\}$.

Proof. The class H of $\tilde{C}$ must be of the form $H = \pi^*C - h_1E_1 - \cdots - h_nE_n$, with C effective (since $\tilde{C}$ is effective) and each h_i nonnegative (since $\tilde{C}$ is irreducible and $F(\delta) \cdot E_i \ge 0$ holds for all i).

First consider (b). By [22], Lemma 1, we have $\tilde{C}^2 \ge -a + 1$ if $a > 1$. It is easy to see that $\tilde{C}^2 \ge -1$ if $a = 1$, for suppose $\tilde{C} \cdot E_i = 1$ yet $\tilde{C}^2 < -1$. Then we would have $(\tilde{C} + E_i)^2 < 0$, hence $|\tilde{C} + E_i|$ is fixed. However, the linear system $|\tilde{C} + E_i|$ corresponds to a complete linear system on the surface Y' obtained by contracting E_i; $|\tilde{C}|$ corresponds to the subsystem vanishing at p_i. Since p_i is a general point, $|\tilde{C} + E_i|$ cannot be fixed, which contradicts $\tilde{C}^2 < -1$ when $a = 1$. Hence we may assume $\tilde{C}^2 \ge -a$, so $h_1^2 + \cdots + h_n^2 - a \le C^2$. Also, since L is big and nef, the index theorem (as in the proof of Lemma 2.1.1(b)) gives $C^2L^2 \le (C \cdot L)^2$. On the other hand, $F(\delta) \cdot \tilde{C} < 0$ gives $(C \cdot L)^2 < (l_1h_1 + \cdots + l_nh_n)^2L^2/(d^2L^2 + \delta)$.

Now consider (a). Let $h = \sqrt{h_1^2 + \cdots + h_n^2}$. From (b) we have $h^2 - a < (l_1h_1 + \cdots + l_nh_n)^2/(L^2d^2 + \delta) \le d^2L^2h^2/(d^2L^2 + \delta)$, so $h^2 < d^2L^2h^2/(d^2L^2 + \delta) + a$. But $a^2 \le h^2/\gamma$, so we have $h^2 < d^2L^2h^2/(d^2L^2 + \delta) + h/\sqrt{\gamma}$, and solving for h gives the result. □

For each $\delta > 0$, let $O_n(F(\delta))$ be the set of all numerical equivalence classes of divisors $H = \pi^*C - h_1E_1 - \cdots - h_nE_n$ where C is the class of an effective divisor on X and C and the h_i satisfy the inequalities in Lemma 2.1.4(a), (b) and (c). Then $O_n(F(\delta))$ is the set of obstructions to $F(\delta) = d'L - (l_1E_1 + \cdots + l_nE_n)$ being nef; i.e., $O_n(F(\delta))$ contains the class of every $F(\delta)$-abnormal curve (if any). In particular, $O_n(F(\delta))$ is an $(L, \{l_1, \ldots, l_n\})$-sufficient test system. Let $o_n(F(\delta))$ be the set of ratios $L \cdot C/(l_1h_1 + \cdots + l_nh_n)$ for all $H \in O_n(F(\delta))$.

Theorem 2.1.5. *Let L, $F(\delta)$, Y and X be as in Lemma 2.1.4. Then $o_n(F(\delta))$ is a finite set for each $\delta > 0$, and the union $U_n = \cup_{\delta>0} o_n(F(\delta))$ is discrete, with $t = \sqrt{L^2/\boldsymbol{\ell}^2}$ as the unique limit point (if any). Moreover, if $F(\delta)$ is not nef for some $\delta > 0$ (which is equivalent to $\varepsilon(X, L, n, \boldsymbol{\ell}) < \sqrt{L^2/\boldsymbol{\ell}^2}$), then $\varepsilon(X, L, n, \boldsymbol{\ell})$ is the maximum t such that $F_t = L - t(l_1E_1 + \cdots + l_nE_n)$ is nef and this t is an element of $o_n(F(\delta))$; i.e., $\varepsilon(X, L, n, \boldsymbol{\ell}) \in U_n$.*

Proof. Lemma 2.1.4 implies that $o_n(F(\delta))$ is finite. If $\delta' < \delta$, then every element t of $o_n(F(\delta'))$ not in $o_n(F(\delta))$ is bigger than every element of $o_n(F(\delta))$; in particular, $\sqrt{L^2/(\boldsymbol{\ell}^2 + \delta)} \le t < \sqrt{L^2/(\boldsymbol{\ell}^2 + \delta')}$, hence the only possible limit point is $t = \sqrt{L^2/\boldsymbol{\ell}^2}$. Note that $(1/c)F_c = F(\delta)$ exactly when $\delta = L^2/c^2 - \boldsymbol{\ell}^2$, so if $\delta = L^2/c^2 - \boldsymbol{\ell}^2$, then $F(\delta)$ is nef if and only if F_c is, so $F(\delta)$ not being nef for some $\delta > 0$ is by Lemma 2.1.1(c) equivalent to $\varepsilon(X, L, n, \boldsymbol{\ell}) < \sqrt{L^2/\boldsymbol{\ell}^2}$. If $F(\delta)$ is not nef, take t to be the infimum for $L \cdot C/(l_1h_1 + \cdots + l_nh_n)$ over all classes $H = C - (h_1E_1 + \cdots + h_nE_n)$ of $F(\delta)$-abnormal curves. Thus $t \in o_n(F(\delta))$ since $o_n(F(\delta))$ is finite, and $L - t(l_1E_1 + \cdots + l_nE_n)$ is nef since we have chosen t small enough to eliminate all obstruction classes. Finally, by Lemma 2.1.1(c), we also have $\varepsilon(X, L, n, \boldsymbol{\ell}) = t$. □

Observe that from Lemma 2.1.1(c) and (d) it follows that $\varepsilon(X, L, n, p_1, \ldots, p_n) = \varepsilon(X, L, n)$ for general points whenever $\varepsilon(X, L, n) < \sqrt{L^2/n}$ and the group of numerical equivalence classes has rank one. However, by Theorem 2.1.5 it now follows for all $\boldsymbol{\ell}$ and all X that $\varepsilon(X, L, n, \boldsymbol{\ell}) = \varepsilon(X, L, n, l_1p_1, \ldots, l_np_n)$ for general points whenever $\varepsilon(X, L, n, \boldsymbol{\ell}) < \sqrt{L^2/\ell^2}$. To see this, let $t = \varepsilon(X, L, n, \boldsymbol{\ell})$. By Lemma 2.1.1(c), F_t is nef for some choice of points p_i, and hence by Lemma 2.1.3 for an open set. Thus on some nonempty open set we have $\varepsilon(X, L, n, l_1p_1, \ldots, l_np_n) \ge t$. On the other hand, by the discreteness claim of Theorem 2.1.5 there exists a t' such that $t' > t$ but such that no element of $\cup_\delta o_n(F(\delta))$ is in the interval $(t, t']$. By Lemma 2.1.1(c) it follows that there is an open set for which there exists an $F_{t'}$-abnormal H. Since $F_t \cdot H \ge 0$ but $F_{t'} \cdot H < 0$, it must be that $H \cdot L/(H \cdot (l_1E_1 + \cdots + l_nE_n))$ is in the interval $[t, t')$, and hence that $t = H \cdot L/(H \cdot (l_1E_1 + \cdots + l_nE_n))$. Thus on this nonempty open set we also have $t \ge \varepsilon(X, L, n, l_1p_1, \ldots, l_np_n)$.

2.2. Applications

We now turn our attention to obtaining explicit bounds on homogeneous Seshadri constants. We begin this section by describing our conceptual basis for bounding Seshadri constants. Given general points $p_i \in X$, $1 \le i \le n$ on X and a big and nef divisor L on X, let $\pi : Y \to X$ be obtained from X by blowing up the points. Then $\varepsilon(X, L, n) \ge t$ whenever $F_t = L - t(E_1 + \cdots + E_n)$ is big and nef, by Lemma 2.1.1(c) (the case that t is real follows by taking the limit of smaller rational values).

In order to show F_t is nef for a given t for which $F_t^2 > 0$, we first consider the set $O_n(F_t)$ of test classes, which we obtained from Lemma 2.1.4. We can explicitly determine the finite set $o_n(F_t)$. If each test class is shown not to be the class of a reduced, irreducible curve (by showing, for example, that none is the class of an effective divisor), it follows that F_t is nef and hence that $\varepsilon(X, L, n) \ge t$. However, Lemma 2.1.4 applies more generally to classes $F = L - t(l_1E_1 + \cdots + l_nE_n)$. Since hereafter we will focus on $F = L - t(E_1 + \cdots + E_n)$, it behooves us to make better use of the fact that the coefficients l_i are equal. Doing so allows us to significantly sharpen Lemma 2.1.4, which we state as Corollary 2.2.2.

We need the following lemma, which generalizes a result of [18]:

Lemma 2.2.1. *Let F be an $\mathbf{R}$-divisor class on X with $F \cdot L > 0$ for some big and nef class L and with $F^2 \ge 0$. Let $C_1, \ldots, C_r$ be distinct F-abnormal curves. Then up to numerical equivalence their divisor classes $[C_1], \ldots, [C_r]$ are linearly independent in the divisor class group on X.*

Proof. If $[C_1], \ldots, [C_r]$ are dependent, we can find a nontrivial nonnegative integer combination D of some of the classes $[C_1], \ldots, [C_r]$ and another nontrivial nonnegative integer combination D' of the rest of the classes $[C_1], \ldots, [C_r]$, such that, up to numerical equivalence, $D = D'$. But $F \cdot D < 0$, so for some real number $\delta > 0$ we must have $(F + \delta L) \cdot D = 0$ with $(F + \delta L)^2 > 0$, hence by the index theorem we must have $D^2 < 0$, which contradicts $D^2 = D \cdot D' \geq 0$. □

The analysis of what F-abnormal curves can occur is especially simple when the coefficients $F \cdot E_i$ are all equal. In particular, as our next result generalizing and extending methods and results of [21,17,16] shows, they must be almost uniform, where we call a class of the form $\pi^*C - m(E_1 + \cdots + E_n)$ *uniform*, and we call a class of the form $\pi^*C - m(E_1 + \cdots + E_n) - kE_i$ *almost uniform* (called almost homogeneous in [17]).

Corollary 2.2.2. *Let L be a big and nef divisor on X. Let $\pi : Y \to X$ be the blow up of $n \geq 1$ general points $p_1, \ldots, p_n \in X$. Consider the $\mathbf{R}$-divisor class $F = (\sqrt{n/L^2})\pi^*L - E_1 - \cdots - E_n$, and let H be a divisor class on Y with $F \cdot H < 0$. If H is the class of an F-abnormal curve, then there are integers $m > 0$, k (where we require $k = 0$ if $n = 1$) and $1 \leq i \leq n$ and an effective divisor C on X such that:*

(a) $H = \pi^*C - m(E_1 + \cdots + E_n) - kE_i$;
(b) *either* $k > -m$ *and* $k^2 < (n/(n-1)) \min(m, m+k)$, *or* $m = -k = 1$;
(c) $(m^2n + 2mk + \max(k^2 - m, k^2 - (m+k), 0))L^2 \leq C^2L^2 \leq (C \cdot L)^2 < (m^2n + 2mk + k^2/n)L^2$ *when* $k^2 > 0$, *but* $(m^2n - m)L^2 \leq C^2L^2 \leq (C \cdot L)^2 < (m^2n)L^2$ *when* $k = 0$; *and*
(d) $C \cdot (C + K_X) - (m+k)^2 - (n-1)m^2 + mn + k \geq -2$.

Proof. The case $n = 1$ (and so $k = 0$) is easy to treat along the same lines as below; we leave it to the reader. Thus we assume $n \geq 2$.

(a) In [17], Corollary 2.8, this result is proved for surfaces of Picard number 1. We adjust their argument to prove the result for arbitrary Picard numbers. Because the points are general and F is uniform, permuting the coefficients m_i of the class $H = \pi^*C - m_1E_1 + \cdots + m_nE_n$ of an F-abnormal curve gives another such class. Since all such permutations are in the subspace of the span of $\pi^*C, E_1, \ldots, E_n$ orthogonal to $F - (F \cdot H)/(C \cdot L)\pi^*L$, it follows from Lemma 2.2.1 that there are at most n such curves. But it is not hard to check that there are always more than n permutations unless at most one of the coefficients is different from the rest. Thus H is of the form $H = \pi^*C - m(E_1 + \cdots + E_n) - kE_i$ with $1 \leq i \leq n$, which gives (a).

Since $H \cdot F(0) = H \cdot F < 0$, it follows that for $\delta > 0$ small enough, $H \cdot F(\delta) < 0$. For the proof of (b) and (c), fix a $\delta > 0$ such that H is the class of a $F(\delta)$-abnormal curve.

Consider (b). Since H is the class of a reduced irreducible curve with $C \cdot L > 0$, we must have $H \cdot E_i \geq 0$ for all i, hence $-m \leq k$. If $k = -m$, then Lemma 2.1.4(b) says $(m^2(n-1) - m) < (m(n-1))^2/n$, which simplifies to $m^2(n-1) < mn$, and hence $m = -k = 1$. Now, again by Lemma 2.1.4(b) with $a = \min(m, m+k)$, we have $(m^2n + 2mk + k^2 - a)n < (mn + k)^2$, which simplifies to give $k^2 < (n/(n-1))$ (a).

Likewise, (c) follows from Lemma 2.1.4(b) in the case that $k = 0$, as does $(m^2n + 2mk + \max(k^2 - m, k^2 - (m + k)))L^2 \leq (C \cdot L)^2 < (m^2n + 2mk + k^2/n)L^2$ when $k \neq 0$. If $k \neq 0$, then $\pi^*C - m(E_1 + \cdots + E_n) - kE_1$ and $\pi^*C - m(E_1 + \cdots + E_n) - kE_n$ are classes of distinct irreducible curves, so their intersection is nonnegative, hence $m^2n + 2mk \leq C^2$, and $(m^2n + 2mk + \max(k^2 - m, k^2 - (m+k), 0))L^2 \leq (C \cdot L)^2$ as claimed.

Finally, we prove (d). A reduced, irreducible curve must have a nonnegative genus g, hence by adjunction we must have $H^2 + K_Y \cdot H = 2g(H) - 2 \geq -2$, which is (d). □

It may be interesting to note that item (d) above is implied by (b) and (c) if $X = \mathbf{P}^2$ and the number of points is $n \geq 11$. The proof of this implication follows from a straightforward but somewhat lengthy computation that we leave to the interested reader to carry through.

It may also be of interest that Corollary 2.2.2 takes the following very simple form if $m < n$. Since we will not use the following result we omit a proof.

Corollary 2.2.3. *Let $\pi : Y \to X$ be the blow up of n general points $p_1, \ldots, p_n \in X$. Let L be a big and nef divisor on X and let $F = (\sqrt{n/L^2})\pi^*L - E_1 - \cdots - E_n$. Assume $H = \pi^*C - (m+k)E_1 - mE_2 - \cdots - mE_n$ is the class of an almost uniform F-abnormal curve H with $n > m > 0$. Then $-\sqrt{m} \leq k \leq \sqrt{m}$. Moreover, if $k \neq 0$, then also $C^2 = 2mk + m^2n$ (and so $H^2 = -k^2$) and $m\sqrt{n} - 1 < \sqrt{C^2} \leq C \cdot L/\sqrt{L^2} < m\sqrt{n} + 1$.*

Remark 2.2.4. We note that if the Néron–Severi group of X is generated by a single ample divisor L with $L^2 = r^2$ a square, when moreover Corollary 2.2.3 applies, there is for each m at most one $k \neq 0$ and one t for which an abnormal curve $[H] = t\pi^*L - (m+k)E_1 - mE_2 - \cdots - mE_n$ could exist. Indeed, $t^2r^2 = 2mk + m^2n$ implies that t^2r^2 has the same parity as m^2n, and only one integer tr in the range $m\sqrt{n} - 1 < tr < m\sqrt{n} + 1$ has this property.

The next corollary is just a refined version of Corollary 2.2.2. Note that

$$\sqrt{\frac{L^2}{n}}\sqrt{1 - \frac{1}{\mu n}} = \sqrt{\frac{L^2}{n+\delta}}$$

is equivalent to $\delta = (\mu - 1/n)^{-1}$. We will denote an almost uniform class of the form $\pi^*C - m(E_1 + \cdots + E_n) - kE_i$ by $H(C, m, k)$, with n being understood.

Corollary 2.2.5. *Let L be a big and nef divisor on X. Let $\pi : Y \to X$ be the blow up of $n > 1$ general points $p_1, \ldots, p_n \in X$. Let $\mu \geq 1$ be real and consider the $\mathbf{R}$-divisor class $F(\delta) = \sqrt{(n+\delta)/L^2}L - (E_1 + \cdots + E_n)$, where $\delta = (\mu - 1/n)^{-1}$. Then any $F(\delta)$-abnormal class is of the form $H(C, m, k)$, where C, m and k are as in Corollary 2.2.2 and where $0 < m < \mu$ and either $k = 0$ or $m(n-1) < \mu$.*

Proof. Let H be an $F(\delta)$-abnormal class. Then $H = H(C, m, k)$, where C, m and k satisfy the criteria of Corollary 2.2.2. First, say $k = 0$; then $m^2n - m \leq (C \cdot L)^2/L^2$, while $F(\delta) \cdot C < 0$ implies $(C \cdot L)\sqrt{(n+\delta)/L^2} < mn$, hence $m^2n - m < m^2n^2/(n+\delta)$ or $(1/n)(1 - 1/(mn)) < 1/(n+\delta)$. This simplifies to $m - 1/n < 1/\delta = \mu - 1/n$, or $m < \mu$. Now assume $k \neq 0$. This time we have $(C \cdot L)\sqrt{(n+\delta)/L^2} < mn + k$ and $m^2n + 2mk + \max(k^2 - m, k^2 - (m+k), 0) \leq (C \cdot L)^2/L^2$, hence $(m^2n + 2mk)/(mn+k)^2 \leq (C \cdot L)^2/((mn+k)^2L^2)$. Note that $(1/n)(1 - 1/(mn(n-1))) \leq (m^2n + 2mk)/(mn+k)^2$ is the same as $1 - 1/(mn(n-1)) \leq (m^2n^2 + 2mkn)/(mn+k)^2 = 1 - k^2/(mn+k)^2$ or $mn(n-1)k^2 \leq (mn+k)^2$. This holds when $k > 0$ because in this case $k^2 < mn/(n-1)$. It also holds when $k < 0$, because now $k^2 < (m+k)n/(n-1)$ or $mn(n-1)k^2 < (m+k)mn^2$, but $(m+k)mn^2 \leq (mn+k)^2$ holds since it simplifies to $kmn(n-2) < k^2$, but k is negative. So, putting everything together, we have

$$\frac{1}{n}\left(1 - \frac{1}{mn(n-1)}\right) \leq \frac{m^2n + 2mk}{(mn+k)^2} \leq \frac{(C \cdot L)^2}{(mn+k)^2L^2} < \frac{1}{n+\delta}.$$

But $(1/n)(1 - 1/(mn(n-1))) < 1/(n+\delta)$ simplifies to $m(n-1) - 1/n < 1/\delta = \mu - 1/n$, or $m(n-1) < \mu$. $\square$

We can now prove Theorem 1.2.1, Corollaries 1.2.2 and 1.2.4:

Proof of Theorem 1.2.1. Let us prove part (b) of Theorem 1.2.1 first. Since $\sqrt{\frac{L^2}{(n+\delta)}} = \sqrt{\frac{L^2}{n}}\sqrt{1 - \frac{1}{\mu n}}$, the statement that $\varepsilon(X, L, n)$ is at least as big as $\sqrt{\frac{L^2}{n}}\sqrt{1 - \frac{1}{\mu n}}$ follows if $F(\delta) = \sqrt{(n+\delta)/L^2}L - (E_1 + \cdots + E_n)$ is nef. If $F(\delta)$ were not nef, then there would exist an $F(\delta)$-abnormal class $H = H(C, m, k)$, hence $0 > F(\delta) \cdot H$, so $(nm+k)/\sqrt{L^2/(n+\delta)} > L \cdot C \geq \alpha_0((m^{[n-1]}, m+k))$. But our hypotheses on α_0, together with Corollaries 2.2.2 and 2.2.5, guarantee that this cannot happen.

Now consider (a). For every integer $1 \leq m < \mu$, assume that

$$\alpha(m^{[n]}) \geq m\sqrt{L^2(n - 1/\mu)} > m\sqrt{L^2(n - 1/(\mu(1 - 2/(n+1))))}.$$

Then, whenever $1 \leq m < \mu' = \mu(1 - 2/(n+1))$, we claim that $\alpha_0(m^{[n]}) \geq m\sqrt{L^2(n - 1/\mu')}$, and whenever $1 \leq m < \mu'/(n-1)$, $k^2 < (n/(n-1))\min(m, m+k)$, we claim that $\alpha_0((m^{[n-1]}, m+k)) \geq ((mn+k)/n)\sqrt{L^2(n - 1/\mu')}$. Part (b) will then imply that

$$\varepsilon(X, L, n) \geq \sqrt{L^2/n}\sqrt{1 - 1/(n\mu')} > \sqrt{L^2/n}\sqrt{1 - 1/((n-2)\mu)},$$

as wanted.

The first claim is immediate, for $m < \mu' < \mu$, so

$$\alpha_0(m^{[n]}) \geq \alpha(m^{[n]}) \geq m\sqrt{L^2\left(n - \frac{1}{\mu}\right)} > m\sqrt{L^2\left(n - \frac{1}{\mu'}\right)}.$$

For the second claim, given a reduced and irreducible curve $C = C_n$ with multiplicity m at general points $p_1, \dots, p_{n-1}$, multiplicity $m+k$ at p_n and $C \cdot L = \alpha_0((m^{[n-1]}, m+k))$, consider curves $C_1, \dots, C_{n-1}$ such that C_i has multiplicity $m+k$ at p_i and multiplicity m at the other points (which exist because the points are general). Then $D = C_1 + \cdots + C_n$ is a (reducible) curve with multiplicity $nm + k$ at each of the points. But $k^2 < (n/(n-1)) \min(m, m+k)$ implies that $k \leq m$ (since otherwise $k^2 \geq (m+1)^2 > 2m \geq nm/(n-1) \geq (n/(n-1)) \min(m, m+k)$, but this contradicts Corollary 2.2.2(b)). So if $m < \mu'/(n-1)$, then $nm + k \leq (n+1)m < (n+1)\mu'/(n-1) = \mu$, and

$$\alpha_0((m^{[n-1]}, m+k)) \geq \frac{1}{n}\alpha((nm+k)^{[n]}) \geq \frac{nm+k}{n}\sqrt{L^2\left(n - \frac{1}{\mu'}\right)} = \frac{nm+k}{\sqrt{n}}\sqrt{L^2\left(1 - \frac{1}{n\mu'}\right)},$$

as claimed. □

Proof of Corollary 1.2.2. Note that $\alpha(X, L, m^{[n]})/L^2$ is an integer which increases with n. Thus the Riemann–Roch formula together with ampleness of L gives that

$$\dim \left| \frac{\alpha(X, L, m^{[n]})}{L^2} L \right| = \frac{\alpha(X, L, m^{[n]})(\alpha(X, L, m^{[n]}) - L \cdot K)}{2L^2} + p_a,$$

where K denotes the canonical class and p_a the arithmetic genus of the surface X, provided that $\alpha(X, L, m^{[n]})$ is large enough, which certainly holds (independent of $m \geq 1$) for n large enough. Thus, in order to apply Theorem 1.2.1, it will be enough to prove for n large enough that

$$\frac{\alpha(\alpha - L \cdot K)}{2L^2} \geq n\frac{m(m+1)}{2} - p_a$$

implies $\alpha^2 \geq m^2 L^2 n$. If $L \cdot K \geq 0$ this is clear, so assume $(L \cdot K)/L^2 = -\beta < 0$. Then, in order to have $\alpha^2 < m^2 L^2 n$ it would be necessary that $\beta\alpha/2 > nm/2 - p_a$ or $\alpha > nm/\beta - c$ with $c = 2p_a/\beta$ independent of n and m. But then $\alpha^2 > (nm/\beta - c)^2 \geq m^2 L^2 n(n/(\beta^2 L^2) - 2c/(m\beta L^2)) \geq m^2 L^2 n(n/(\beta^2 L^2) - 2c/(\beta L^2))$, and for n large enough this is bigger than $m^2 L^2 n$, as desired. So it suffices to pick n_0 large enough, then for $n \geq n_0$ we obtain the claimed lower bound on $\varepsilon(X, L, n)$.

We now verify that such an $\mathbf{m}(n)$ exists. Indeed, thanks to [1], a map $\mathbf{n} : \mathbf{N} \to \mathbf{N}$ exists such that for $n > \mathbf{n}(m)$ the inequality (1) holds. Among such maps we may clearly choose one which is increasing. So, defining $\mathbf{m} : \mathbf{N} \to \mathbf{N}$ as $\mathbf{m}(n) = \min\{m | \mathbf{n}(m) > n\}$, we have for every $m < \mathbf{m}(n)$ that (1) holds. Moreover, $\mathbf{m}$ is nondecreasing and unbounded since $\mathbf{n}$ is increasing, hence $0 = \lim_{n\to\infty} 1/\mathbf{m}(n) = \lim_{n\to\infty} n\mathcal{R}_n(L)$. (Although [1] does not give an explicit $\mathbf{n}$, we have been informed by the authors that one may take $\mathbf{n}(m) \simeq \exp(\exp(m))$, in which case $\mathbf{m}(n) \simeq \log(\log(n))$.) □

Proof of Corollary 1.2.4. Let $\delta = (\lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 3)/2 + 1 - 1/n)^{-1}$. Then by Corollary 1.2.3 and the discussion immediately before Corollary 2.2.5 we have $\sqrt{n+\delta} \geq \varepsilon(n)^{-1}$. By hypothesis, $m < 1/\sqrt{\delta}$, so $1/m^2 > \delta$ so $\sqrt{n + 1/m^2} > \sqrt{n+\delta}$. Now $sL - E_1 - \cdots - E_n$ is nef by Lemma 2.1.1(c), for every rational s such that $s \geq \sqrt{n+\delta}$, hence $F(\delta)$ is itself nef. Of course, $F \cdot E_i > 0$ for all i. For any other reduced irreducible curve C it is enough to show $C \cdot F(\delta) < C \cdot F$, since $0 \leq C \cdot F(\delta)$, and $C \cdot F(\delta) < C \cdot F$ will follow if $t/m > \sqrt{n+\delta}$. But $t^2 \geq m^2 n + 1$, so $t/m \geq \sqrt{n + 1/m^2} > \sqrt{n+\delta}$, as needed. □

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