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Monotonically monolithic spaces, Corson compacts, and D-spaces

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ABSTRACT

Monotonically monolithic spaces were recently introduced by V.V. Tkachuk, and monotonically κ-monolithic spaces by O. Alas, V.V. Tkachuk, and R. Wilson. In this note we answer some of their questions by showing that monotonically ω-monolithic compact spaces must be Corson compact, yet there is a Corson compact space which is not monotonically ω-monolithic. We obtain a characterization of monotonic monolithity that shows its close relationship to condition (G) of P. Collins and R. Roscoe. We also give an easy proof of Tkachuk's result that monotonically monolithic spaces are hereditarily D-spaces by applying a result involving nearly good relations, and finally, we generalize nearly good to nearly OK to similarly obtain L.-X. Peng's result that weakly monotonically monolithic spaces are D-spaces.

1. Introduction

All spaces are assumed to be regular and T_1. A space X is *monotonically monolithic* [14] if one can assign to each $A \subset X$ a collection $\mathcal{N}(A)$ of subsets of X such that

(1) $|\mathcal{N}(A)| \leqslant |A| + \omega$;
(2) $A \subset B \Rightarrow \mathcal{N}(A) \subset \mathcal{N}(B)$;
(3) If $\{A_\alpha\colon \alpha < \delta\}$ is an increasing collection of subsets of X, and $A = \bigcup_{\alpha<\delta} A_\alpha$, then $\mathcal{N}(A) = \bigcup_{\alpha<\delta} \mathcal{N}(A_\alpha)$;
(4) If U is open and $x \in \overline{A} \cap U$, then there is $N \in \mathcal{N}(A)$ with $x \in N \subset U$.

We call $\mathcal{N}$ a monotonically monolithic operator for X.

Further, for an infinite cardinal κ, X is said to be *monotonically κ-monolithic* [2] if $\mathcal{N}(A)$ is defined for all sets A with $|A| \leqslant \kappa$ and satisfies the above conditions.

Condition (4) may be rephrased by declaring that $\mathcal{N}(A)$ contains a network at every point of $\overline{A}$.[1] L.-X. Peng [12] called a space X *weakly monotonically monolithic* if it has an operator satisfying the above conditions but with condition (4) replaced by

(4′) If A is not closed, then $\mathcal{N}(A)$ contains a network at some point $x \in \overline{A} \setminus A$.

Tkachuk [14] proved the following results:

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[1] A collection $\mathcal{F}$ of subsets of a space X is a *network* at $x \in X$ if, given any open neighborhood U of x, there is some $F \in \mathcal{F}$ with $x \in F \subset U$.

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Theorem 1.1.

(1) *Any space with a point-countable base is monotonically monolithic*;
(2) *If X is a Lindelöf Σ-space, then $C_p(X)$ is monotonically monolithic*[2];
(3) *Monotonically monolithic spaces are hereditarily D-spaces.*[3]

This served as motivation for introducing the class of monotonically monolithic spaces, because it generalized simultaneously the results of A.V. Arhangel'skii and R. Buzyakova [1] that spaces with a point-countable base are (hereditarily) D, and our result [9] that $C_p(X)$ is hereditarily D whenever X is a Lindelöf Σ-space.

A compact space X is *Gul'ko compact* if $C_p(X)$ is a Lindelöf Σ-space. Part (2) of Theorem 1.1 implies that Gul'ko compact spaces are monotonically monolithic and hence hereditarily D. We proved [9] that Corson compacta are hereditarily D. This suggests natural questions about the relationship of monotonically monolithic spaces and Corson compacta. It was asked in [14] (resp., [2]) if monotonically monolithic (resp., monotonically ω-monolithic) compact spaces must be Corson compact. In this note, we show that the answer to this question is positive. Answering another question, we give an example of a Corson compact space which is not monotonically ω-monolithic.

Next we show that a space X is monotonically monolithic (resp., weakly monotonically monolithic) iff one can assign to each *finite* subset F of X a *countable* collection $\mathcal{N}(F)$ of subsets of X such that, for any $A \subset X$, $\bigcup_{F\in[A]^{<\omega}}\mathcal{N}(F)$ contains a network at every point of $\overline{A}$ (resp., at some point of $\overline{A}\setminus A$, if A is not closed), where $[A]^{<\omega}$ denotes the set of all finite subsets of A. It follows that condition (G) of P. Collins and R. Roscoe [4] implies monotonically monolithic. It was proven in [5] that stratifiable spaces satisfy (G). Thus stratifiable spaces are monotonically monolithic, which answers another question in [14].

As noted above, Tkachuk proved that monotonically monolithic spaces are hereditarily D-spaces; L.-X. Peng [12] proved that weakly monotonically monolithic spaces are D-spaces. Peng's result generalizes Tkachuk's result (since the class of monotonically monolithic spaces is hereditary) as well as a number of other results that say that certain classes of spaces are D-spaces. The proofs of Tkachuk and Peng are rather lengthy and involved. Here we show that Tkachuk's result follows easily from a framework for proving spaces are D-spaces that we introduced in [9], and that by a small generalization of this framework we easily obtain Peng's result as well.[4]

2. Corson compacta

A compact space X is said to be *Corson compact* if X embeds in a Σ-product of real lines, i.e., in $\{x\in\mathbb{R}^\kappa\colon |\{\alpha\colon x(\alpha)\neq 0\}|\leqslant\omega\}$ for some cardinal κ.

Consider the following game $G(H,X)$ of length ω played in a space X, where H is a closed subset of X. There are two players, O and P. In the nth round, O chooses an open superset O_n of H, and P chooses a point $p_n\in O_n$. We say O *wins* the game if $p_n\to H$ in the sense that every open superset of H contains p_n for all but finitely many $n\in\omega$. In [6], we showed that a compact Hausdorff space X is Corson compact iff O has a winning strategy in $G(\Delta, X^2)$, where Δ is the diagonal in X^2. We shall use this game characterization of Corson compacts to show that every monotonically ω-monolithic space is Corson compact. This result follows easily from the game characterization and the following lemma.

Lemma 2.1. *If X is countably compact and monotonically ω-monolithic, then O has a winning strategy in $G(H,X)$ for any closed subset H of X.*

Proof. Let the countably compact space X be monotonically ω-monolithic witnessed by the operator $\mathcal{N}$, and let H be a closed subset of X. We will prove that O has a winning strategy in $G(H,X)$. Let O's first choice be X, and let p_0 be P's response. Suppose $p_0, p_1, \ldots, p_n$ are P's plays so far. Let $A_i=\{p_j\}_{j\leqslant i}$ and let $\mathcal{N}(A_i)=\{N_{i0}, N_{i1}, N_{i2},\ldots\}$. Then let O respond in O_n, where $O_n\subset O_{n-1}$ is an open superset of H such that $\overline{O}_n\cap N_{ij}=\emptyset$ whenever $i,j\leqslant n$ and there is an open superset of H whose closure misses N_{ij}.

We claim that this strategy wins the game for O. By countable compactness, it suffices to prove that the set $A=\{p_n\}_{n\in\omega}$ has no cluster point outside of H. Suppose by way of contradiction that q is a cluster point of A, where $q\notin H$. Let U be an open neighborhood of q whose closure misses H. There is $N\in\mathcal{N}(A)$ with $q\in N\subset U$. It follows from condition (3) that $N\in\mathcal{N}(A_i)$ for some i, hence $N=N_{ij}$ for some i,j. But then $\overline{O}_n\cap N=\emptyset$ for $n\geqslant max\{i,j\}$, whence $q\notin\overline{O}_n$ and thus q cannot be a cluster point of A. □

[2] A space X is a *Lindelöf Σ-space* if it is the continuous image of closed subspace of the product of a separable metric space with a compact space, and $C_p(X)$ denotes the space of continuous real-valued functions on X with the topology of pointwise convergence.

[3] A space X is a *D-space* if whenever one assigns a neighborhood $U(x)$ of x to each $x\in X$, then there is a closed discrete set D such that $X=\bigcup\{U(x)\colon x\in D\}$.

[4] It should not be surprising that this is the case, since the proofs of Tkachuk and Peng, as well as our framework, are based fundamentally on ideas of Buzyakova in [3], where she proved that spaces $C_p(X)$ for X compact are hereditarily D-spaces.

Corollary 2.2. *If X is compact and ω-monotonically monolithic, then X is Corson compact.*

Proof. Suppose X is compact and ω-monotonically monolithic. Then so is X^2 [2], hence by the lemma O has a winning strategy in $G(\Delta, X^2)$. Thus X is Corson compact. □

But Corson compacta, while they are hereditarily D-spaces, need not be monotonically monolithic.

Example 2.3. There is a Corson compact space which is not monotonically ω-monolithic.

Proof. We exploit a Corson compact space due to Todorcevic [16,17], which was also studied in [7]. Let S be a stationary, costationary subset of ω_1, and let T' be the tree of all closed-in-ω_1 subsets of S ordered by end-extension. Every node in T' has uncountably many immediate successors. It will be handy to use a similar tree such that every node has only two immediate successors, so we stick a Cantor tree between every node of T' and its successors in T' as follows. Let $T = T' \times 2^{<\omega}$ ordered as follows. For each $t \in T'$, let $\{b_{t,\alpha}\colon \alpha \in S,\ \alpha > sup(t)\}$ be a set of branches of $2^{<\omega}$ such that, for each $s \in 2^{<\omega}$, $|\{\alpha\colon s \in b_{t,\alpha}\}| = \omega_1$. Then define $(t, s) < (t', s')$ iff

(1) $t = t'$ and $s < s'$; or
(2) $\exists \alpha > sup(t)[t \cup \{\alpha\} \leqslant_{T'} t'$ and $s \in b_{t,\alpha}]$.

In this way, the branch $b_{t,\alpha}$ gets inserted between a node t and its immediate successor $t \cup \{\alpha\}$ in T'. Also note that, for each $s, t \in T'$, we have $s \leqslant_{T'} t$ iff $(s, \emptyset) \leqslant_T (t, \emptyset)$.

Following a construction of Nyikos [11], let $\tilde{T}$ be the tree obtained from T by adding a node at the end of each branch of T. Note that this adds nodes only at limit levels. For each $t \in T$ at a successor level, let $V_t = \{s \in \tilde{T}\colon s \geqslant t\}$. Then the V_t's and their complements form a compact Hausdorff topology on $\tilde{T}$. Since each chain in $\tilde{T}$ is countable, the collection of V_t's is point-countable. It is easy to check that this collection is also T_0-separating. Thus $\tilde{T}$ with this topology is Corson compact.

We will show that T with the topology as a subspace of $\tilde{T}$ is not monotonically ω-monolithic, hence $\tilde{T}$ isn't either. It will be useful to have the following claim, which follows easily from the known fact that T' is Baire with the topology generated by the V_t's (see, e.g., Exercise H25 in Chapter VII of [10]).

Claim. *If $A = \bigcup_{n\in\omega} A_n$, where each A_n is an antichain in T, then there is a maximal antichain B in T such that for each $t \in B$ and each $a \in A$, $a \ngeqslant t$.*

Proof. If $a = (t, s) \in A$, let $C(a) = \{t \cup \{\alpha\}\colon \alpha \in S,\ \alpha > sup(t),\ s \in b_{t,\alpha}\}$. Then it is easy to check that $C_n = \bigcup_{a\in A_n} C(a)$ is an antichain in T'. Expand C_n to a maximal antichain C'_n. Let $O_n = \bigcup_{c\in C'_n} V(c)$. Then each O_n is dense open in T', so $O = \bigcap_{n\in\omega} O_n$ is too. Now let B' be a maximal antichain of elements of O. It is easy to see that $B = \{(t, \emptyset)\colon t \in B'\}$ satisfies the conclusion of the claim. □

Now suppose the operator $\mathcal{N}$ witnesses that T is monotonically ω-monolithic. For each $t \in T$, let $\mathcal{N}^*(t) = \mathcal{N}(\{s \in T\colon s \leqslant t\})$. Let A_0 be a maximal antichain of T. For each $a \in A_0$ and for each $N \in \mathcal{N}^*(a)$, let

$$m(N) = \{t \in N \cap V_a\colon \nexists s \in N(s > t)\}.$$

Since each $m(N)$ is an antichain and $\mathcal{N}^*(a)$ is countable, by the claim there is a maximal antichain $B(a)$ in V_a such that

$$\forall s \in B(a) \forall N \in \mathcal{N}^*(a) \forall u \in m(N)(s \nleqslant u).$$

Note that $A_1 = \bigcup_{a\in A_0} B(a)$ is a maximal antichain. Define A_2 from A_1 in the same way that A_1 was defined from A_0, then similarly define $A_3, A_4, \ldots$. By the claim, there is $t_0 \in T$ such that, for each $a \in \bigcup_{n\in\omega} A_n$, $t_0 \nleqslant a$. By maximality of A_n, there must be $a_n \in A_n \cap \{s\colon s < t\}$. Then $a_0 < a_1 < \cdots$. Let $t_1 \leqslant t_0$ be least such that $t_1 > a_n$ for all n. Then $t_1 \in \overline{\{a_n\}}_{n\in\omega}$.

Let t_{10} and t_{11} be the immediate successors of t, and let $U = T \setminus (V_{t_{10}} \cup V_{t_{11}})$. There is $N \in \mathcal{N}(\{a_n\}_{n\in\omega})$ with $t_1 \in N \subset U$. Then $N \in \mathcal{N}(\{a_i\}_{i<k}) \subset \mathcal{N}^*(a_k)$ for some k. Then $t_1 \in m(N) \cap V_{a_k}$, hence by the construction of A_{k+1}, we cannot have $t_1 > a_{k+1}$. So we have a contradiction. □

3. Collins and Roscoe's (G)

In [4], Collins and Roscoe introduce the following condition:

(G) For each $x \in X$, there is assigned a countable collection $\mathcal{G}(x)$ of subsets of X such that, whenever $x \in U$, U open, there is an open V with $x \in V \subset U$ such that, whenever $y \in V$, then $x \in N \subset U$ for some $N \in \mathcal{G}(y)$.

As mentioned in [8], it is straightforward to check that (G) is equivalent to the following:

(G′) For each $x \in X$, one can assign a countable collection $\mathcal{G}(x)$ of subsets of X such that, for any $A \subset X$, $\bigcup_{a\in A} \mathcal{G}(a)$ contains a network at every point of $\overline{A}$.

Indeed, $\mathcal{G}(x)$, $x \in X$, satisfies (G) iff it satisfies (G′).

Lemma 3.1. *Any X satisfying* (G) *is monotonically monolithic.*

Proof. Let $\mathcal{G} : X \to \mathcal{P}(X)$ satisfy condition (G). For $A \subset X$, let $\mathcal{N}(A) = \bigcup_{x\in A} \mathcal{G}(x)$. It is easy to check that $\mathcal{N}$ witnesses that X is monotonically monolithic. □

It is well known and easy to check that any space with a point-countable base satisfies (G). Indeed, it is a well-known open question whether or not having a point-countable base is equivalent to property (G) witnessed by an operator $\mathcal{G}$ such that $\mathcal{G}(x)$ consists of open sets.

In [5], it is proved that stratifiable spaces satisfy (G). Hence stratifiable spaces are monotonically monolithic, answering another question in [14].

We now give a characterization of monotonically monolithic which emphasizes its close relationship to (G).

Theorem 3.2. *A space X is monotonically monolithic* (*resp., weakly monotonically monolithic*) *iff one can assign to each finite subset F of X a countable collection $\mathcal{N}(F)$ of subsets of X such that, for each $A \subset X$, $\bigcup_{F\in[A]^{<\omega}} \mathcal{N}(F)$ contains a network at each point of $\overline{A}$* (*resp., at some point of $\overline{A} \setminus A$, if A is not closed*).

Proof. We give the proof for monotonically monolithic, the proof for weakly monotonically monolithic being similar.

First, suppose $\mathcal{N}(F)$ for $F \in [X]^{<\omega}$ satisfies the stated condition. Let $\mathcal{N}'(A) = \bigcup_{F\in[A]^{<\omega}} \mathcal{N}(F)$. It is easy to check that $\mathcal{N}'$ satisfies the conditions of the definition of monotonically monolithic.

For the other direction, suppose X is monotonically monolithic, witnessed by operator $\mathcal{N}$. We will show that $\bigcup_{F\in[A]^{<\omega}} \mathcal{N}(F)$ contains a network at each point of $\overline{A}$. To this end, let $x \in \overline{A} \cap U$. Then $x \in N \subset U$ for some $N \in \mathcal{N}(A)$. Let $F \subset A$ have minimal cardinality such that $N \in \mathcal{N}(F)$. We claim that F is finite. Suppose otherwise, and let $F = \{x_\alpha\colon \alpha < \kappa\}$ where $\kappa = |F|$. Now let $F_\alpha = \{x_\beta\colon \beta < \alpha\}$. Then F is the increasing union of the F_α's, so by condition (2) in the definition of monotonically monolithic, we have $N \in \mathcal{N}(F_\alpha)$ for some $\alpha < \kappa$. But $|F_\alpha| < |F|$, contradiction. □

Remark. Note that the proof shows that if operator $\mathcal{N}$ satisfies the conditions of the definition of monotonically monolithic, then for any set A, $\mathcal{N}(A) = \bigcup_{F\in[A]^{<\omega}} \mathcal{N}(F)$.

In an earlier version of this paper, we asked if (G) is equivalent to monotonically monolithic, and suggested that $C_p(X)$ for some Lindelöf Σ-space X might be a place to look for an example distinguishing the two concepts. Tkachuk [15] has since shown that $C_p(\beta D)$ does not satisfy (G) whenever D is an uncountable discrete space (but it is monotonically monolithic). However, we can show:

Theorem 3.3. *If X is Gul'ko compact, then X satisfies* (G).

Proof. Let X be Gul'ko compact. We use the following characterization of Gul'ko compact due to Sokolov [13]. A compact space X is Gul'ko compact iff X has a weakly σ-point finite T_0-separating cover $\mathcal{O} = \bigcup_{n\in\omega} \mathcal{O}_n$ by open F_σ-sets. Here, $\mathcal{O} = \bigcup_{n\in\omega} \mathcal{O}_n$ is *weakly σ-point-finite* means that, for each $x \in X$, $\mathcal{O} = \bigcup\{\mathcal{O}_n\colon ord(x, \mathcal{O}_n) < \omega\}$. Since each $O \in \mathcal{O}$ is F_σ, and X is compact Hausdorff, we can write $O = \bigcup_{n\in\omega} O_n$, where O_n is open, $\overline{O}_n \subset O$, $O_0 \subset O_1 \subset \cdots$.

For each $x \in X$, let M_x be a countable elementary submodel (of some sufficiently large $H(\kappa)$) such that $x, X, \mathcal{O}, \{\mathcal{O}_n\}_{n\in\omega}$, and the function $O \to \{O_n\}_{n\in\omega}$ are elements of M_x. Then let $\mathcal{G}(x) = \{G \in M_x\colon G \subset X\}$.

We claim that $\mathcal{G}$ witnesses (G) for X. Suppose $A \subset X$ and $p \in \overline{A}$. Let $p \in U$, U open. We need to show that there is some $a \in A$ and $n \in \mathcal{G}(a)$ with $p \in N \subset U$.

It follows easily from compactness and that $\mathcal{O}$ is T_0-separating that there are finite $\mathcal{P}_0, \mathcal{P}_1 \subset \mathcal{O}$ and natural numbers $j(O)$ for $O \in \mathcal{P}_0$ such that

$$p \in \bigcap\{O_{j(O)}\colon O \in \mathcal{P}_0\} \setminus \bigcup \mathcal{P}_1 \subset U.$$

For each $O \in \mathcal{P}_1$, by the weakly σ-point-finite property, there is $n(O) \in \omega$ such that $ord(p, \mathcal{O}_{n(O)}) < \omega$ and $O \in \mathcal{O}_{n(O)}$. There is a point $a \in A$ such that

$$a \in \bigcap \mathcal{P}_0 \cap \bigcap\bigl\{O'\colon \exists O \in \mathcal{P}_1\bigl(p \in O' \in \mathcal{O}_{n(O)}\bigr)\bigr\}.$$

Note that $\mathcal{O}$ is point-countable. Thus $\mathcal{P}_0 \cup \{O' \colon \exists O \in \mathcal{P}_1 (p \in O' \in \mathcal{O}_{n(O)})\} \subset M_a$. Then for each $O \in \mathcal{P}_1$, $\{O' \colon p \in O' \in \mathcal{O}_{n(O)}\} \in M_a$ and so

$$N(O) = X \setminus \bigcup \{O'' \in \mathcal{O}_{n(O)} \colon p \notin O''\} \in M_a.$$

Note that $p \in N(O) \subset X \setminus O$. Thus $N = \bigcap \{O_{j(O)} \colon O \in \mathcal{P}_0\} \cap \bigcap \{N(O) \colon O \in \mathcal{P}_1\}$ is in M_a and $\mathcal{G}(a)$, and

$$p \in N \subset \bigcap \{O_{j(O)} \colon O \in \mathcal{P}_0\} \setminus \bigcup \mathcal{P}_1 \subset U. \qquad \square$$

Question 3.4. *If X is compact, and satisfies* (G) *or is monotonically monolithic, must X be Gul'ko compact*?

4. *D*-spaces and nearly OK relations

Let X be a space. We say that a relation R on X (resp., from X to $[X]^{<\omega}$) is *nearly good* if $x \in \overline{A}$ implies xRy for some $y \in A$ (resp., $xR\tilde{y}$ for some $\tilde{y} \in [A]^{<\omega}$).

Given a neighborhood assignment U on X (i.e., for each $x \in X$, $U(x)$ is a neighborhood of x), call a subset Z of X *U-close* if $Z \subset U(x)$ for every $x \in Z$. In Proposition 2.4 of [9] we proved the following:

Theorem 4.1. *Let U be a neighborhood assignment on X. Suppose there is a nearly good relation R on X* (*resp., from X to $[X]^{<\omega}$*) *such that for any $y \in X$* (*resp., $F \in [X]^{<\omega}$*), *$R^{-1}(y) \setminus U(y)$* (*resp., $R^{-1}(F) \setminus \bigcup\{U(y) \colon y \in F\}$*) *is the countable union of U-close sets. Then there is a closed discrete $D \subset X$ such that $X = \bigcup_{x \in D} U(x)$.*

As we remarked immediately after this result, if U and R satisfy the conditions of the theorem, then so does their restrictions to any subspace. Hence, if for any U on X we can produce such an R, it follows that X is hereditarily D. We now show that this can be used to obtain a quick proof that monotonically monolithic spaces are hereditarily D.

Theorem 4.2. *([14]) Monotonically monolithic spaces are hereditarily D.*

Proof. Let X be monotonically monolithic witnessed by operator $\mathcal{N}$, and let U be a neighborhood assignment on X. Define a relation R from X to $X^{<\omega}$ by

$$xRF \quad \Leftrightarrow \quad \exists N \in \mathcal{N}(F) \big(x \in N \subset U(x)\big).$$

Since $\bigcup_{F \in [A]^{<\omega}} \mathcal{N}(F)$ contains a network at every $x \in \overline{A}$, it is straightforward to check that this R is nearly good.

Now let $N(x)$ denote any $N \in \mathcal{N}(F)$ such that $x \in N \subset U(x)$ (if such N exists). Let $F \in X^{<\omega}$ and $N \in \mathcal{N}(F)$. Then $X(N) = \{x \colon xRF \text{ and } N(x) = N\}$ is U-close, and $R^{-1}(F) = \bigcup_{N \in \mathcal{N}(F)} X(N)$. So $R^{-1}(F)$ is a countable union of U-close sets, hence X is hereditarily D. $\square$

To prove that weakly monotonically monolithic implies D requires a weakening of nearly good. Let us call a relation R on X (resp., from X to $[X]^{<\omega}$) *nearly OK* if A not closed implies xRy for some $x \in \overline{A} \setminus A$ and some $y \in A$ (resp., $xR\tilde{y}$ for some $x \in \overline{A} \setminus A$ and some $\tilde{y} \in [A]^{<\omega}$).

Theorem 4.1 still holds with "nearly good" replaced by "nearly OK".

Theorem 4.3. *Let U be a neighborhood assignment on X. Suppose there is a nearly OK relation R on X* (*resp., from X to $[X]^{<\omega}$*) *such that for any $y \in X$* (*resp., $F \in [X]^{<\omega}$*), *$R^{-1}(y) \setminus U(y)$* (*resp., $R^{-1}(F) \setminus \bigcup\{U(y) \colon y \in F\}$*) *is the countable union of U-close sets. Then there is a closed discrete $D \subset X$ such that $X = \bigcup_{x \in D} U(x)$.*

Proof. The argument is nearly the same as the argument for 2.0–2.4 in [9]. Instead of repeating this entire argument, we note here the few small changes that need to be made.

In Lemmas 2.0 and 2.1, and Propositions 2.2 and 2.3, replace "nearly good" with "nearly OK". The statement of Lemma 2.0 needs to be modified further to require that all limit points of D are in X'. Other than that, the statements of these four results remain valid. No change is needed in the proofs of 2.2 and 2.3, and the proof of the new 2.0 is straightforward from the definitions. The proof of 2.1 uses the same idea as the original, but one needs to be a bit more careful. In [9] only the proof of 2.1(b) is given, as 2.1(a) is similar but easier. For the new 2.1(b), first, by way of contradiction, let λ' be the least ordinal $\leqslant \lambda$ such that $E_{\lambda'} = \bigcup_{\alpha < \lambda'} D_\alpha$ is not closed discrete. Then use nearly OK to choose $x \in \overline{E'_\lambda} \setminus E'_\lambda$ and a finite $F \subset E_{\lambda'}$ such that xRF, and obtain a contradiction as in [9].

Now the new Proposition 2.4, i.e., our theorem, follows just as in [9], noting that in the proof all limit points of $D \cup E$ are in X', so the new Lemma 2.0 may be applied. $\square$

Theorem 4.4. *([12]) Weakly monotonically monolithic spaces are D-spaces.*

Proof. The proof is the same as that of Theorem 4.2, noting that the relation defined in the argument is nearly OK. □

We should mention that this does not get that weakly monotonically monolithic spaces are hereditarily D, because the weak monotonically monolithic property is not hereditary. It is also worth pointing out that the remark after Theorem 4.1 does not apply either, since the restriction of a nearly OK relation to a subspace need not be nearly OK.

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