Computers and Mathematics with Applications 64 (2012) 3008–3020

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On recent developments in the theory of boundary value problems for impulsive fractional differential equations[☆]

JinRong Wang[a], Yong Zhou[b,*], Michal Fečkan[c,d]

[a] Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, PR China
[b] Department of Mathematics, Xiangtan University, Xiangtan, Hunan 411105, PR China
[c] Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia
[d] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

ARTICLE INFO

Keywords:
Impulsive fractional differential equations
Boundary value problems
Solutions

ABSTRACT

This paper is motivated from some recent papers treating the boundary value problems for impulsive fractional differential equations. We first make a counterexample to show that the formula of solutions in cited papers are incorrect. Second, we establish a general framework to find the solutions for impulsive fractional boundary value problems, which will provide an effective way to deal with such problems. Third, some sufficient conditions for the existence of the solutions are established by applying fixed point methods. Meanwhile, data dependence is obtained by using a new generalized singular Gronwall inequality. Finally, three examples are given to illustrate the results.

1. Introduction

The first definition of the fractional derivative was introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus was mentioned already in 1695 by Leibniz and L'Hospital. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining much importance and attention. For more details on basic theory of fractional differential equations, one can see the monographs of Diethelm [1], Kilbas et al. [2], Miller and Ross [3], Podlubny [4] and Tarasov [5], and the Refs. [6–17].

This paper is strongly motivated from the recent research papers [18–28] treating the boundary value problems for impulsive differential equations with fractional derivative. After reading these papers carefully, one can see that the concept of piecewise continuous solutions used are not appropriate. To support our claims, we consider a simple boundary problem for impulsive fractional differential equations

☆ The first author acknowledges the support from Key Projects of Science and Technology Research in the Chinese Ministry of Education (211169), and Natural Science Foundation of Guizhou Province (2010, No. 2142). The second author was supported by National Natural Science Foundation of China (10971173). The third author acknowledges the support by Grants VEGA-MS 1/0507/11, VEGA-SAV 2/0124/10 and APVV-0134-10.
* Corresponding author.
E-mail addresses: wjr9668@126.com (J. Wang), yzhou@xtu.edu.cn (Y. Zhou), Michal.Feckan@fmph.uniba.sk (M. Fečkan).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.12.064

$$\begin{cases} {}^cD_{0,t}^q u(t) := {}^cD_t^q u(t) = h(t), & t \in J' := J \setminus \{t_1, \ldots, t_m\}, J := [0, 1], \\ \Delta u(t_k) = I_k(u(t_k^-)), \quad \Delta u'(t_k) = J_k(u(t_k^-)), & k = 1, 2, \ldots, m, \\ u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0, \end{cases} \tag{1}$$

where ${}^cD_t^q$ is the Caputo fractional derivative of order $q \in (1, 2)$ with the lower limit zero, $u_0 \in R$, $h: J \to R$ is continuous, $I_k, J_k: R \to R$ and t_k satisfy $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $u(t_k^+) = \lim_{\epsilon \to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim_{\epsilon \to 0^-} u(t_k + \epsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$.

If we use Lemma 2.2 in [18], then problem (1) is equivalent to the following integral equation

$$u(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds \\ \quad + (1-t)\left[\frac{1}{\Gamma(q)} \int_{t_k}^1 (1-s)^{q-1} h(s)ds + \frac{1}{\Gamma(q-1)} \int_{t_k}^1 (1-s)^{q-2} h(s)ds \right. \\ \quad + \sum_{0<t_k<1} \left(\frac{1}{\Gamma(q)} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} h(s)ds + I_k(u(t_k^-)) \right) \\ \quad \left. + \sum_{0<t_k<1} (2-t_k) \left(\frac{1}{\Gamma(q-1)} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-2} h(s)ds + J_k(u(t_k^-)) \right) \right], \\ \quad \text{for } t \in [0, t_1), \\ \quad \vdots \\ \quad \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} h(s)ds \\ \quad + (1-t)\left[\frac{1}{\Gamma(q)} \int_{t_k}^1 (1-s)^{q-1} h(s)ds + \frac{1}{\Gamma(q-1)} \int_{t_k}^1 (1-s)^{q-2} h(s)ds \right. \\ \quad + \sum_{0<t_k<1} \left(\frac{1}{\Gamma(q)} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} h(s)ds + I_k(u(t_k^-)) \right) \\ \quad \left. + \sum_{0<t_k<1} (2-t_k) \left(\frac{1}{\Gamma(q-1)} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-2} h(s)ds + I_k(u(t_k^-)) \right) \right] \\ \quad + \sum_{0<t_k<1} \left(\frac{1}{\Gamma(q)} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} h(s)ds + I_k(u(t_k^-)) \right) \\ \quad + \sum_{0<t_k<1} (t-t_k) \left(\frac{1}{\Gamma(q-1)} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-2} h(s)ds + J_k(u(t_k^-)) \right), \\ \quad \text{for } t \in (t_k, t_{k+1}],\ k = 1, 2, \ldots, m. \end{cases} \tag{2}$$

Then one can say that a function $u \in PC^1(J, R)$ is called a solution of problem (1) if u satisfies Eq. (2). Unfortunately, the above formula is false. The reader will find an interesting counterexample which has been made to support our claims in Section 3.

Due to the above comments, we discuss boundary value problems for impulsive differential equations with Caputo fractional derivative and seek a correct formula of the solution for such kind of problems. In the present paper, we consider the boundary value problems for the following impulsive fractional differential equations

$$\begin{cases} {}^cD_t^q u(t) = f(t, u(t)), & t \in J', q \in (1, 2), \\ \Delta u(t_k) = y_k, \quad \Delta u'(t_k) = \bar{y}_k, & k = 1, 2, \ldots, m, \\ u(0) = 0, \quad u'(1) = 0, \end{cases} \tag{3}$$

where $y_k, \bar{y}_k \in R$.

We try to seek a correct formula of solutions for problem (3). After the strict proof, we find that the formula of solutions for problem (3) should be

$$u(t)=\begin{cases}\dfrac{1}{\Gamma(q)}\displaystyle\int_0^t(t-s)^{q-1}f(s,u(s))ds\\ \qquad-\left(\dfrac{1}{\Gamma(q-1)}\displaystyle\int_0^1(1-s)^{q-2}f(s,u(s))ds+\sum_{k=1}^m\bar{y}_k\right)t, \quad \text{for } t\in[0,t_1),\\ \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t(t-s)^{q-1}f(s,u(s))ds\\ \qquad+\bar{y}_1(t-t_1)+y_1-\left(\dfrac{1}{\Gamma(q-1)}\displaystyle\int_0^1(1-s)^{q-2}f(s,u(s))ds+\sum_{k=1}^m\bar{y}_k\right)t, \quad \text{for } t\in(t_1,t_2],\\ \vdots\\ \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t(t-s)^{q-1}f(s,u(s))ds+\sum_{i=1}^k\bar{y}_i(t-t_i)+\sum_{i=1}^k y_i\\ \qquad-\left(\dfrac{1}{\Gamma(q-1)}\displaystyle\int_0^1(1-s)^{q-2}f(s,u(s))ds+\sum_{k=1}^m\bar{y}_k\right)t, \quad \text{for } t\in(t_k,t_{k+1}], k=1,2,\ldots,m.\end{cases}$$

The rest of this paper is organized as follows. In Section 2, we give some notations, recall some concepts and preparation results. In Section 3, we make a counterexample to verify that the current formula of the solutions for such problems is not correct. In Section 4, we establish a general framework to find the solutions for impulsive fractional boundary value problems, which provide an effective way to deal with such problems. In Section 5, we give main results, the first existence and uniqueness result is based on the Banach contraction principle, the second existence and data dependence results are based on Krasnoselskii's fixed point theorem and a generalized Gronwall inequality with mixed integral term (Lemma 2.10). Three examples are given in Section 6 to demonstrate the application of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

We define $PC(J,R) := \{u: J\to R: u\in C(t_k,t_{k+1}],R), k=0,\ldots,m$ and there exist $u(t_k^-)$ and $u(t_k^+)$, $k=1,\ldots,m$, with $u(t_k^-)=u(t_k)\}$ with the norm $\|u\|_{PC} := \sup\{|u(t)|: t\in J\}$. Denote $PC^1(J,R) := \{u\in PC(J,R): \dot{u}\in PC(J,R)\}$. Set $\|u\|_{PC^1} := \|u\|_{PC}+\|\dot{u}\|_{PC}$. It can be seen that endowed with the norm $\|\cdot\|_{PC^1}$, $PC^1(J,R)$ is also a Banach space.

For measurable functions $\mu: J\to R$, define the norm

$$\|\mu\|_{L^p(J)}=\begin{cases}\left(\displaystyle\int_J|\mu(t)|^pdt\right)^{\frac{1}{p}}, & 1\le p<\infty,\\ \inf\limits_{\mathrm{mes}(\bar{J})=0}\{\sup\limits_{t\in J-\bar{J}}|\mu(t)|\}, & p=\infty.\end{cases}$$

Let $L^p(J,R)$ be the Banach space of all Lebesgue measurable functions $\mu: J\to R$ with $\|\mu\|_{L^p(J)}<\infty$.

Let us recall the following known definitions. For more details, see [2].

Definition 2.1. The fractional integral of order γ with the lower limit zero for a function $f: [0,\infty)\to R$ is defined as

$$I_t^\gamma f(t)=\frac{1}{\Gamma(\gamma)}\int_0^t\frac{f(s)}{(t-s)^{1-\gamma}}ds, \quad t>0, \gamma>0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann–Liouville derivative of order γ with the lower limit zero for a function $f: [0,\infty)\to R$ can be written as

$$^LD_t^\gamma f(t)=\frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t\frac{f(s)}{(t-s)^{\gamma+1-n}}ds, \quad t>0, n-1<\gamma<n.$$

Definition 2.3. The Caputo derivative of order γ for a function $f: [0,\infty)\to R$ can be written as

$$^cD_t^\gamma f(t)={}^LD^\gamma\left[f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right], \quad t>0, n-1<\gamma<n.$$

Remark 2.4. (i) If $f(t) \in C^n[0, \infty)$, then

$$ {}^cD_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I_t^{n-\gamma} f^{(n)}(t), \quad t > 0, n-1 < \gamma < n. $$

(ii) The Caputo derivative of a constant is equal to zero.

Moreover, we need the following known results.

Lemma 2.5. *For $q > 0$, the general solution of fractional differential equation ${}^cD_t^q u(t) = 0$ is given by*

$$ u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} $$

where $c_i \in R$, $i = 0, 1, 2 \cdots n-1$ ($n = [q] + 1$) and $[q]$ denotes the integer part of the real number q.

Remark 2.6. In view of Lemma 2.5, it follows that

$$ I^q({}^cD_t^q u)(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} $$

where $c_i \in R$, $i = 0, 1, 2, \ldots, n-1$, $n = [q] + 1$.

Definition 2.7. A function $u \in PC^1(J, R)$ is said to be a solution of problem (3) if $u(t) = u_k(t)$ for $t \in (t_k, t_{k+1})$ and $u_k \in C([0, t_{k+1}], R)$ satisfies ${}^cD_t^q u_k(t) = f(t, u_k(t))$ a.e. on $(0, t_{k+1})$ with the restriction of $u_k(t)$ on $[0, t_k)$ is just $u_{k-1}(t)$ and the conditions $\Delta u(t_k) = y_k$, $\Delta u'(t_k) = \bar{y}_k$, $k = 1, 2, \ldots, m$ with $u(0) = 0$, $u'(1) = 0$.

To obtain the data dependence results of solution to problem (3), we need a new generalized Gronwall inequality with mixed integral term. Recalling a generalized Gronwall inequality which appeared in our earlier work [29].

Lemma 2.8 (*[29, Lemma 2]*)**.** *Let $u \in C(J, R)$ satisfy the following inequality:*

$$ |u(t)| \le a + b \int_0^t |u(\theta)|^{\lambda_1} d\theta + c \int_0^1 |u(\theta)|^{\lambda_2} d\theta, \quad t \in J, $$

where $\lambda_1 \in [0, 1]$, $\lambda_2 \in [0, 1)$, $a, b, c \ge 0$ are constants. Then there exists a constant $M^* > 0$ such that

$$ |u(t)| \le M^*. $$

Remark 2.9. For $\lambda_1 < 1$ we can take M^* to be the unique positive solution of $M^* = a + bM^{*\lambda_1} + cM^{*\lambda_2}$. Using the classical Gronwall inequality, for $\lambda_1 = 1$ we can take M^* to be the unique positive solution of $M^* = \left(a + cM^{*\lambda_2}\right) e^b$.

Using Lemma 2.8, we can obtain the following generalized Gronwall inequality with mixed integral term.

Lemma 2.10. *Let $u \in C(J, R)$ satisfy the following inequality:*

$$ |u(t)| \le a + b \int_0^t (t-s)^{q-1} |u(s)|^{\lambda_1} ds + c \int_0^1 (1-s)^{q-2} |u(s)|^{\lambda_2} ds, \tag{4} $$

where $q \in (1, 2)$, $a, b, c \ge 0$ are constants, $\lambda_1 \in [0, 1 - \frac{1}{p}]$, $\lambda_2 \in [0, 1 - \frac{1}{p})$, and for some $p > 1$ such that $p(q-2) + 1 > 0$. Then there exists a constant $M_* > 0$ such that

$$ |u(t)| \le M_*. $$

Proof. It follows from (4) and Hölder inequality that

$$ \begin{aligned} |u(t)| &\le a + b \left(\int_0^t (t-s)^{p(q-1)} ds \right)^{\frac{1}{p}} \left(\int_0^t |u(s)|^{\frac{\lambda_1 p}{p-1}} ds \right)^{\frac{p-1}{p}} + c \left(\int_0^1 (1-s)^{p(q-2)} ds \right)^{\frac{1}{p}} \left(\int_0^1 |u(s)|^{\frac{\lambda_2 p}{p-1}} ds \right)^{\frac{p-1}{p}} \\ &\le a + b \left[\frac{1}{p(q-1)+1} \right]^{\frac{1}{p}} \int_0^t |u(s)|^{\frac{\lambda_1 p}{p-1}} ds + c \left[\frac{1}{p(q-2)+1} \right]^{\frac{1}{p}} \int_0^1 |u(s)|^{\frac{\lambda_2 p}{p-1}} ds \\ &\le a + b \int_0^t |u(s)|^{\frac{\lambda_1 p}{p-1}} ds + c \left[\frac{1}{p(q-2)+1} \right]^{\frac{1}{p}} \int_0^1 |u(s)|^{\frac{\lambda_2 p}{p-1}} ds. \end{aligned} $$

Applying Lemma 2.8, there exists a constant $M_* > 0$ such that

$$ |u(t)| \le M_*. $$

The proof is completed. □

Remark 2.11. Constant M_* can be determined by using Remark 2.9.

Theorem 2.12 *([30, Theorem 2.1]). Let X be a Banach space and $\mathcal{W} \subset PC(J, X)$. If the following conditions are satisfied:*

(i) *$\mathcal{W}$ is uniformly bounded subset of $PC(J, X)$;*
(ii) *$\mathcal{W}$ is equicontinuous in (t_k, t_{k+1}), $k = 0, 1, 2, \ldots, m$, where $t_0 = 0$, $t_{m+1} = T$;*
(iii) *$\mathcal{W}(t) = \{u(t): u \in \mathcal{W}, t \in J \setminus \{t_1, \ldots, t_m\}\}$, $\mathcal{W}(t_k^+) = \{u(t_k^+): u \in \mathcal{W}\}$ and $\mathcal{W}(t_k^-) = \{u(t_k^-): u \in \mathcal{W}\}$ is a relatively compact subsets of X,*

then $\mathcal{W}$ is a relatively compact subset of $PC(J, X)$.

3. A counterexample

In this section, we make a counterexample to illustrate that the current formula of solutions for impulsive fractional boundary value problems is not correct.

Let us consider the following counterexample:

$$\begin{cases} {}^cD_t^{\frac{3}{2}} u(t) = t, \quad t \in (0, 1] \setminus \left\{\dfrac{1}{2}\right\}, \\ \Delta u\left(\dfrac{1}{2}\right) = 1, \qquad \Delta u'\left(\dfrac{1}{2}\right) = 1, \\ u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0. \end{cases} \tag{5}$$

As a special case of Eq. (1), the solution of Eq. (5) is given by

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_{\frac{1}{2}}^{t} (t-s)^{\frac{3}{2}-1} sds + (1-t)\left[\frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_{\frac{1}{2}}^{1} (1-s)^{\frac{3}{2}-1} sds + \frac{1}{\Gamma\left(\frac{3}{2}-1\right)} \int_{\frac{1}{2}}^{1} (1-s)^{\frac{3}{2}-2} sds \right. \\ & \left. + \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right)^{\frac{3}{2}-1} sds + 1 + \left(2 - \frac{1}{2}\right)\left(\frac{1}{\Gamma\left(\frac{3}{2}-1\right)} \int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right)^{\frac{3}{2}-2} sds + 1\right)\right] \\ & + \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right)^{\frac{3}{2}-1} sds + 1 + \left(t - \frac{1}{2}\right)\left(\frac{1}{\Gamma\left(\frac{3}{2}-1\right)} \int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right)^{\frac{3}{2}-2} sds + 1\right), \quad \text{for } t \in \left(\frac{1}{2}, 1\right]. \end{aligned}$$

Note that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma\left(\frac{5}{2}+1\right) = \frac{15\sqrt{\pi}}{8}, \qquad \Gamma\left(\frac{3}{2}+1\right) = \frac{3\sqrt{\pi}}{4}, \qquad \Gamma\left(\frac{1}{2}+1\right) = \frac{\sqrt{\pi}}{2}.$$

Meanwhile,

$$\begin{aligned} & \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_{\frac{1}{2}}^{t} (t-s)^{\frac{3}{2}-1} sds = \left(t - \frac{1}{2}\right)^{\frac{1}{2}} \frac{(2t-1)(4t+3)}{15\sqrt{\pi}}, \\ & \frac{1}{\Gamma\left(\frac{3}{2}-1\right)} \int_{\frac{1}{2}}^{t} (t-s)^{\frac{3}{2}-2} sds = \left(t - \frac{1}{2}\right)^{\frac{1}{2}} \frac{4t+1}{3\sqrt{\pi}}, \\ & \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_{\frac{1}{2}}^{1} (1-s)^{\frac{3}{2}-1} sds = \frac{7\sqrt{2}}{30\sqrt{\pi}}, \\ & \frac{1}{\Gamma\left(\frac{3}{2}-1\right)} \int_{\frac{1}{2}}^{1} (1-s)^{\frac{3}{2}-2} sds = \frac{5\sqrt{2}}{6\sqrt{\pi}}, \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right)^{\frac{3}{2}-1} sds = \frac{\sqrt{2}}{15\sqrt{\pi}}, \\ & \frac{1}{\Gamma\left(\frac{3}{2}-1\right)} \int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right)^{\frac{3}{2}-2} sds = \frac{\sqrt{2}}{3\sqrt{\pi}}. \end{aligned}$$

Therefore,

$$\begin{aligned} u(t) &= \left(t-\frac{1}{2}\right)^{\frac{1}{2}} \frac{(2t-1)(4t+3)}{15\sqrt{\pi}} \\ &\quad + (1-t)\left[\frac{7\sqrt{2}}{30\sqrt{\pi}} + \frac{5\sqrt{2}}{6\sqrt{\pi}} + 1 + \frac{\sqrt{2}}{15\sqrt{\pi}} + \left(2-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{3\sqrt{\pi}}+1\right)\right] \\ &\quad + \frac{\sqrt{2}}{15\sqrt{\pi}} + 1 + \left(t-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{3\sqrt{\pi}}+1\right) \\ &= \frac{2\sqrt{t-\frac{1}{2}}(2t-1)(4t+3) + 46\sqrt{2} + 90\sqrt{\pi} - (39\sqrt{2}+45\sqrt{\pi})t}{30\sqrt{\pi}}, \end{aligned} \tag{6}$$

for $\frac{1}{2} < t \le 1$.

On the other hand, it comes from Remark 2.6 that we can suppose that a general solution u of the first equation of (5) on $\left[0, \frac{1}{2}\right)$ can be given by

$$u(t) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^t (t-s)^{\frac{3}{2}-1} s ds + a + bt, \quad \text{for } t \in \left[0, \frac{1}{2}\right),$$

and a general solution u of the first equation of (5) on $\left(\frac{1}{2}, 1\right]$ can be given by

$$u(t) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^t (t-s)^{\frac{3}{2}-1} s ds + c + dt, \quad \text{for } t \in \left(\frac{1}{2}, 1\right].$$

Then,

$$u'(t) = \frac{1}{\Gamma\left(\frac{3}{2}-1\right)} \int_0^t (t-s)^{\frac{3}{2}-1-1} s ds + b, \quad \text{for } t \in \left[0, \frac{1}{2}\right),$$
$$u'(t) = \frac{1}{\Gamma\left(\frac{3}{2}-1\right)} \int_0^t (t-s)^{\frac{3}{2}-1-1} s ds + d, \quad \text{for } t \in \left(\frac{1}{2}, 1\right].$$

The 2nd and 3rd conditions of (5) imply the next linear system

$$-a - \frac{b}{2} + c + \frac{d}{2} = 1, \tag{7}$$
$$-b + d = 1, \tag{8}$$
$$a + b = 0, \tag{9}$$
$$\frac{28}{15\sqrt{\pi}} + c + 2d = 0, \tag{10}$$

which has a solution

$$a = \frac{5}{2} + \frac{28}{15\sqrt{\pi}}, \tag{11}$$
$$b = -\frac{5}{2} - \frac{28}{15\sqrt{\pi}}, \tag{12}$$
$$c = 3 + \frac{28}{15\sqrt{\pi}}, \tag{13}$$
$$d = -\frac{3}{2} - \frac{28}{15\sqrt{\pi}}. \tag{14}$$

So the solution of (5) is given by

$$u(t) = \begin{cases} \frac{8}{15\sqrt{\pi}} t^{\frac{5}{2}} + \frac{5}{2} + \frac{28}{15\sqrt{\pi}} - \left(\frac{5}{2} + \frac{28}{15\sqrt{\pi}}\right)t, & \text{for } 0 \le t < \frac{1}{2}, \\ \frac{8}{15\sqrt{\pi}} t^{\frac{5}{2}} + 3 + \frac{28}{15\sqrt{\pi}} - \left(\frac{3}{2} + \frac{28}{15\sqrt{\pi}}\right)t, & \text{for } \frac{1}{2} < t \le 1. \end{cases} \tag{15}$$

Consequently, u given by (6) does not satisfy (5).

4. Formula of solutions

In this section, we give a correct formula of solutions to boundary problem for impulsive fractional differential equations

$$\begin{cases} {}^cD_t^q u(t) = h(t), & t \in J', q \in (1,2), \\ \Delta u(t_k) = y_k, \quad \Delta u'(t_k) = \bar{y}_k, & k = 1, 2, \ldots, m, \\ u(0) = 0, \quad u'(1) = 0, \end{cases} \tag{16}$$

where $y_k, \bar{y}_k \in R$.

Lemma 4.1. *Let $q \in (1, 2)$ and $h: J \to R$ be continuous. A function u given by*

$$u(t) = \begin{cases} \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} h(s)ds - \left(\dfrac{1}{\Gamma(q-1)} \displaystyle\int_0^1 (1-s)^{q-2} h(s)ds + \sum_{k=1}^m \bar{y}_k \right) t, \quad \text{for } t \in [0, t_1), \\ \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} h(s)ds + \bar{y}_1(t - t_1) + y_1 - \left(\dfrac{1}{\Gamma(q-1)} \displaystyle\int_0^1 (1-s)^{q-2} h(s)ds + \sum_{k=1}^m \bar{y}_k \right) t, \\ \quad \text{for } t \in (t_1, t_2], \\ \vdots \\ \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} h(s)ds + \sum_{i=1}^k \bar{y}_i(t - t_i) + \sum_{i=1}^k y_i \\ \quad - \left(\dfrac{1}{\Gamma(q-1)} \displaystyle\int_0^1 (1-s)^{q-2} h(s)ds + \sum_{k=1}^m \bar{y}_k \right) t, \quad \text{for } t \in (t_k, t_{k+1}], k = 1, 2, \ldots, m, \end{cases} \tag{17}$$

is a unique solution of the following impulsive problem

$$\begin{cases} {}^cD_t^q u(t) = h(t), & t \in J', q \in (1,2), \\ \Delta u(t_k) = y_k, \quad \Delta u'(t_k) = \bar{y}_k, & k = 1, 2, \ldots, m, \\ u(0) = 0, \quad u'(1) = 0. \end{cases} \tag{18}$$

Proof. A general solution u of the 1th equation of (18) on each interval (t_k, t_{k+1}) $(k = 0, 1, \ldots, m)$ is given by

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds + a_k + b_k t, \quad \text{for } t \in (t_k, t_{k+1}), \tag{19}$$

where $t_0 = 0$ and $t_{m+1} = 1$.

Then, we have

$$u'(t) = \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} h(s)ds + b_k, \quad \text{for } t \in (t_k, t_{k+1}). \tag{20}$$

Applying the boundary conditions of (18), we find that

$$a_0 = 0, \qquad b_m = -\frac{1}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)ds. \tag{21}$$

Next, using the right impulsive condition of (18), we derive

$$b_k = b_{k-1} + \bar{y}_k, \tag{22}$$

which by (21) imply

$$b_j = -\frac{1}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)ds - \sum_{k=j+1}^m \bar{y}_k, \quad j = 0, 1, 2, \ldots, m-1. \tag{23}$$

Furthermore, using the left impulsive condition of (18), we derive

$$a_k + b_k t_k = a_{k-1} + b_{k-1} t_k + y_k,$$

which by (22) is equivalent to

$$a_k = a_{k-1} + (b_{k-1} - b_k)t_k + y_k = a_{k-1} + y_k - \bar{y}_k t_k,$$

so by (21) we obtain

$$a_j = \sum_{k=1}^{j} (y_k - \bar{y}_k t_k), \quad j = 1, 2, \ldots, m. \tag{24}$$

Hence for $j = 1, 2, \ldots, m$, (23) and (24) imply

$$\begin{aligned} a_j + b_j t &= \sum_{k=1}^{j} (y_k - \bar{y}_k t_k) + \left(-\frac{1}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)ds - \sum_{k=j+1}^{m} \bar{y}_k \right) t \\ &= \sum_{k=1}^{j} \bar{y}_k (t - t_k) + \sum_{k=1}^{j} y_k - \left(\frac{1}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)ds + \sum_{k=1}^{m} \bar{y}_k \right) t. \end{aligned} \tag{25}$$

Now it is clear that (19), (21) and (25) imply (17).

Conversely, assume that u satisfies (17). By a direct computation, it follows that the solution given by (17) satisfies (18). This completes the proof. □

5. Main results

This section deals with the existence and uniqueness of solutions to problem (3).

We are ready to state the first existence and uniqueness results.

Theorem 5.1. *Let $f: J \times R \to R$ be a continuous function mapping. Assume that there exists a positive constant L such that*

$$(A_1): |f(t, u) - f(t, v)| \le L|u - v|, \quad \text{for all } t \in J, u, v \in R,$$

with $L \le \frac{\Gamma(1+q)}{2(1+q)}$. Then problem (3) has a unique solution on J.

Proof. Setting $\sup_{t\in J} |f(t, 0)| = M$ and

$$B_r = \{u \in PC(J, R): \|u\|_{PC} \le r\},$$

where

$$r \ge 2 \left[\frac{1+q}{\Gamma(1+q)} M + \sum_{i=1}^{m} |\bar{y}_i| + 2 \sum_{i=1}^{m} |y_i| \right].$$

Define an operator $F: B_r \to PC(J, R)$ by

$$\begin{aligned} (Fu)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds + \sum_{i=1}^{k} \bar{y}_i (t - t_i) + \sum_{i=1}^{k} y_i \\ &\quad - \left(\frac{1}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} f(s, u(s))ds + \sum_{i=1}^{m} \bar{y}_i \right) t, \quad t \in (t_k, t_{k+1}], k = 0, 1, 2, \ldots, m. \end{aligned}$$

It is obvious that F is well defined due to $f: J \times R \to R$ and is jointly continuous and maps bounded subsets of $J \times R$ to bounded subsets of R.

Step 1. We show that $FB_r \subset B_r$.

For $u \in B_r$, $t \in J'$, we have

$$\begin{aligned} |(Fu)(t)| &= \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds + \sum_{i=1}^{k} \bar{y}_i (t - t_i) + \sum_{i=1}^{k} y_i \right. \\ &\quad \left. - \left(\frac{1}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} f(s, u(s))ds + \sum_{i=1}^{m} \bar{y}_i \right) t \right| \\ &\le \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds \right| + \left| \frac{1}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} f(s, u(s))ds \right| + \sum_{i=1}^{m} |\bar{y}_i| + 2 \sum_{i=1}^{m} |y_i| \end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,u(s))-f(s,0)|ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,0)|ds \\
&\quad + \frac{1}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}|f(s,u(s))-f(s,0)|ds + \frac{1}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}|f(s,0)|ds \\
&\quad + \sum_{i=1}^m |\bar{y}_i| + 2\sum_{i=1}^m |y_i| \\
&\le L\frac{1+q}{\Gamma(1+q)}r + M\frac{1+q}{\Gamma(1+q)} + \sum_{i=1}^m |\bar{y}_i| + 2\sum_{i=1}^m |y_i| \\
&\le r.
\end{aligned}$$

Step 2. We show that F is a contraction mapping.

For $u, v \in B_r$ and for each $t \in J'$, we obtain

$$\begin{aligned}
|(Fu)(t)-(Fv)(t)| &= \Bigg|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,u(s))ds - \frac{t}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}f(s,u(s))ds \\
&\quad - \left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,v(s))ds - \frac{t}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}f(s,v(s))ds\right]\Bigg| \\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,u(s))-f(s,v(s))|ds \\
&\quad + \frac{1}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}|f(s,u(s))-f(s,v(s))|ds \\
&\le \left(\frac{L}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds\right)\|u-v\|_{PC} + \left(\frac{L}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}ds\right)\|u-v\|_{PC} \\
&\le L\frac{1+q}{\Gamma(1+q)}\|u-v\|_{PC} \\
&\le \frac{1}{2}\|u-v\|_{PC},
\end{aligned}$$

which implies that

$$\|Fu-Fv\|_{PC} \le \frac{1}{2}\|u-v\|_{PC}.$$

Therefore F is a contraction.

Thus, the conclusion of theorem follows by the contraction mapping principle. The proof is completed. □

Our next result is based on the following well-known fixed point theorem due to Krasnoselskii.

Theorem 5.2. *Let $\mathcal{M}$ be a closed convex and nonempty subset of a Banach space X. Let $\mathcal{A}$, $\mathcal{B}$ be the operators such that*

(i) *$\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$ whenever $x, y \in \mathcal{M}$,*
(ii) *$\mathcal{A}$ is compact and continuous,*
(iii) *$\mathcal{B}$ is a contraction mapping.*

Then there exists a $z \in \mathcal{A}$ such that $z = \mathcal{A}z + \mathcal{B}z$.

Now we are ready to state and prove the following existence result.

Theorem 5.3. *Let $f: J\times R \to R$ be a continuous function mapping with $|f(t,u)| \le \mu(t)$, for all $(t,u) \in J\times R$ where $\mu \in L^{\frac{1}{\sigma}}(J,R)$ and $\sigma \in (0, q-1)$. Then problem (3) has at least one solution on J.*

Proof. Let us choose

$$r \ge \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q)\left(\frac{q-\sigma}{1-\sigma}\right)^{1-\sigma}} + \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q-1)\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}} + 2\sum_{i=1}^m |\bar{y}_i| + \sum_{i=1}^m |y_i|,$$

and denote

$$B_r = \{u \in PC(J,R): \|u\|_{PC} \le r\}.$$

We define the operators P and Q on B_r as

$$(Pu)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,u(s))ds - \left(\frac{1}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}f(s,u(s))ds\right)t,$$
$$(Qu)(t) = \sum_{i=1}^{k}\bar{y}_i(t-t_i) + \sum_{i=1}^{k} y_i - \sum_{i=1}^{m}\bar{y}_i t.$$

For any $u, v \in B_r$ and $t \in J$, using the estimation condition on f and Hölder inequality,

$$\int_0^t \left|(t-s)^{q-1}f(s,u(s))\right| ds \le \left(\int_0^t (t-s)^{\frac{q-1}{1-\sigma}} ds\right)^{1-\sigma} \left(\int_0^t (\mu(s))^{\frac{1}{\sigma}} ds\right)^{\sigma} \le \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\left(\frac{q-\sigma}{1-\sigma}\right)^{1-\sigma}},$$
$$\int_0^1 \left|(1-s)^{q-2}f(s,u(s))\right| ds \le \left(\int_0^1 (1-s)^{\frac{q-2}{1-\sigma}} ds\right)^{1-\sigma} \left(\int_0^1 (\mu(s))^{\frac{1}{\sigma}} ds\right)^{\sigma} \le \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}.$$

Therefore,

$$\|Pu+Qv\|_{PC} \le \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q)\left(\frac{q-\sigma}{1-\sigma}\right)^{1-\sigma}} + \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q-1)\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}} + 2\sum_{i=1}^{m}|\bar{y}_i| + \sum_{i=1}^{m}|y_i| \le r.$$

Thus $Pu+Qv \in B_r$. It is obvious that Q is a contraction with the constant zero. On the other hand, the continuity of f implies that the operator P is continuous. Also, P is uniformly bounded on B_r since

$$\|Pu\|_{PC} \le \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q)\left(\frac{q-\sigma}{1-\sigma}\right)^{1-\sigma}} + \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q-1)\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}} \le r.$$

Now we need to prove the compactness of the operator P.

Letting $\Omega = J \times B_r$, we can define $\sup_{(t,x)\in\Omega}|f(t,u)| = f_{\max}$, and consequently for any $t_k < \tau_2 < \tau_1 \le t_{k+1}$ we have

$$\begin{aligned}
|(Pu)(\tau_2)-(Pu)(\tau_1)| &= \left|\frac{1}{\Gamma(q)}\int_0^{\tau_2}(\tau_2-s)^{q-1}f(s,u(s))ds - \frac{\tau_2}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}f(s,u(s))ds\right.\\
&\quad \left. - \left[\frac{1}{\Gamma(q)}\int_0^{\tau_1}(\tau_1-s)^{q-1}f(s,u(s))ds - \frac{\tau_1}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}f(s,u(s))ds\right]\right|\\
&\le \left|\frac{1}{\Gamma(q)}\int_0^{\tau_2}\left[(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}\right]f(s,u(s))ds\right.\\
&\quad \left. + \frac{1}{\Gamma(q)}\int_{\tau_2}^{\tau_1}(\tau_1-s)^{q-1}f(s,u(s))ds\right| + \left|\frac{(\tau_2-\tau_1)}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}f(s,u(s))ds\right|\\
&\le f_{\max}\left[\frac{2(\tau_1-\tau_2)^q+\tau_1^q-\tau_2^q}{\Gamma(1+q)} + \frac{\tau_1-\tau_2}{\Gamma(q)}\right],
\end{aligned}$$

which tends to zero as $\tau_2 \to \tau_1$. This yields that P is equicontinuous on the interval $(t_k, t_{k+1}]$. So P is relatively compact on B_r.

Hence, by PC-type Arzela–Ascoli Theorem (see Theorem 2.12 in the case of $X = R$), P is compact on B_r. Thus all the assumptions of Theorem 5.2 are satisfied and the conclusion of Theorem 5.2 implies that problem (3) has at least one solution on J. The proof is completed. □

In order to obtain the data dependence of solutions, we revise (A_1) to the following assumption.

(A_2) There exist $L > 0$ and $\lambda \in (0, 1-\frac{1}{p})$ where $p(q-2)+1 > 0$ with $p > 1$ such that

$$|f(t,u)-f(t,v)| \le L|u-v|^{\lambda}, \quad \text{for each } t \in J, \text{ and all } u, v \in R.$$

Further, we give the following data dependence result.

Theorem 5.4. *Assume that the conditions of Theorem 5.3 and the additional condition* (A_2) *hold. Let $v(\cdot)$ be another solution of problem (3) with impulsive conditions $\Delta v(t_k) = y_k$, $\Delta v'(t_k) = \bar{y}_k$, $k = 1, 2, \ldots, m$, and boundary value conditions $v(0) = 0$, $v'(1) = 0$. Then there exists a constant $M_* > 0$ such that $\|u-v\|_{PC} \le M_*$.*

Proof. By Theorem 5.3, problem (3) has a solution $u(\cdot)$ in $PC^1(J, X)$. Keeping in mind our conditions, $v(\cdot)$ be another solution of problem (3) with impulsive conditions $\Delta v(t_k) = y_k$, $\Delta v'(t_k) = \bar{y}_k$, $k = 1, 2, \ldots, m$, and boundary value conditions $v(0) = 0$, $v'(1) = 0$. Note the condition (A_2), we obtain

$$|u(t) - v(t)| \le \frac{L}{\Gamma(q)} \int_0^t (t-s)^{q-1} |u(s) - v(s)|^{\lambda} ds + \frac{L}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} |u(s) - v(s)|^{\lambda} ds.$$

By Lemma 2.10, we obtain $\|u - v\|_{PC} \le M_*$. This completes the proof. □

Remark 5.5. Under the assumptions of Theorem 5.4, we do not obtain the uniqueness of the solutions.

Remark 5.6. By Remark 2.11 we see that M_* is the unique positive solution of

$$M_* = \frac{L}{\Gamma(q)} M_*^{\frac{\lambda p}{p-1}} + \frac{L}{\Gamma(q-1)} \left[\frac{1}{p(q-2)+1}\right]^{\frac{1}{p}} M_*^{\frac{\lambda p}{p-1}},$$

so

$$M_* = \left[\frac{L}{\Gamma(q)} + \frac{L}{\Gamma(q-1)} \left[\frac{1}{p(q-2)+1}\right]^{\frac{1}{p}}\right]^{\frac{1}{1-\frac{\lambda p}{p-1}}}.$$

6. Examples

In this section, we give three examples to illustrate the usefulness of our main results.

Example 6.1. Let us consider the first impulsive fractional boundary value problem

$$\begin{cases} {}^cD_t^{\frac{3}{2}} u(t) = \dfrac{\cos t}{(t+10)^2} \dfrac{|u(t)|}{1+|u(t)|}, & t \in [0,1] \setminus \left\{\dfrac{1}{4}\right\}, \\ \Delta u\left(\dfrac{1}{4}\right) = y_1, \quad \Delta u'\left(\dfrac{1}{4}\right) = \bar{y}_1, \\ u(0) = 0, \quad u'(1) = 0. \end{cases} \tag{26}$$

Set

$$f(t,u) = \frac{\cos t}{(t+10)^2} \frac{|u(t)|}{1+|u(t)|}, \quad (t,u) \in [0,1] \times [0,\infty).$$

Let $u, v \in [0, \infty)$ and $t \in [0, 1]$. Obviously,

$$|f(t,u) - f(t,v)| \le \frac{\cos t}{(t+10)^2} |u - v| \le \frac{1}{100} |u - v|.$$

Set $q = \frac{3}{2}$, $L = \frac{1}{100}$ and $L \le \frac{\Gamma(\frac{5}{2})}{2 \times \frac{5}{2}} = \frac{1}{5}\Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{20}$. Thus all the assumptions of Theorem 5.1 are satisfied. Hence, the impulsive fractional boundary value problem (26) has a unique solution on [0, 1].

Example 6.2. Let us consider the second impulsive fractional boundary value problem

$$\begin{cases} {}^cD_t^{\frac{3}{2}} u(t) = \dfrac{e^t}{(t+1)^2} \dfrac{|u(t)|}{1+|u(t)|}, & t \in [0,1] \setminus \left\{\dfrac{1}{4}\right\}, \\ \Delta u\left(\dfrac{1}{4}\right) = y_1, \quad \Delta u'\left(\dfrac{1}{4}\right) = \bar{y}_1, \\ u(0) = 0, \quad u'(1) = 0. \end{cases} \tag{27}$$

Set

$$f(t,u) = \frac{e^t}{(t+1)^2} \frac{|u(t)|}{1+|u(t)|}, \quad (t,u) \in [0,1] \times [0,\infty).$$

Let $u, v \in [0, \infty)$ and $t \in [0, 1]$. Obviously,

$$|f(t,u)| \le \frac{e^t}{(t+1)^2}.$$

Set $q = \frac{3}{2}$, $\sigma = \frac{1}{4}$, $L = 1$ and $\mu(t) = \frac{e^t}{(t+1)^2} \in L^4([0,1], R)$. Thus all the assumptions of Theorem 5.3 are satisfied. Hence, the impulsive fractional boundary value problem (27) has at least one solution on [0, 1].

On the other hand, for any $u, v \in R$, we have

$$|f(t,u)-f(t,v)| \le \left|\frac{|u|}{1+|u|}-\frac{|v|}{1+|v|}\right| \le |u-v| \le 2|u-v|^{\frac{1}{4}}, \quad \text{for } |u-v| \le 1,$$
$$|f(t,u)-f(t,v)| \le \left|\frac{|u|}{1+|u|}-\frac{|v|}{1+|v|}\right| \le 2 \le 2|u-v|^{\frac{1}{4}}, \quad \text{for } |u-v| \ge 1.$$

So we can use Theorem 5.4 with $\lambda = \frac{1}{4}, L = 2$ and $p = \frac{3}{2}$ to get a data dependence result of problem (27). Remark 5.6 gives $M_* = \frac{256\left(1+\sqrt[3]{2}\right)^4}{\pi^2}$. Note Theorem 5.1 is not applicable to problem (27).

Example 6.3. Now we justify that in general, assumption $|f(t,u)| \le \mu(t)$ in Theorem 5.3 is reasonable for $\mu \in L^{\frac{1}{\sigma}}(J,R)$ and not for some continuous functions μ. Let $a > 0$ and take the function

$$f(t,u) = 1 \quad \text{for } |u| \le 1,$$
$$f(t,u) = (|u|-n)\min\{t^{-a}, n+1\} + (n+1-|u|)\min\{t^{-a}, n\} \quad \text{for } |u| \in (n, n+1),$$

with any $t \in J$. Then $1 \le f(t,u) \le t^{-a}$ for any $u \in R$ and $t \in J$. Moreover, $f(t,n) = \min\{t^{-a}, n\} \to t^{-a}$ as $n \to \infty$ for any $t \in J$ fixed. So $\mu(t) = t^{-a}$ is the best function satisfying $|f(t,u)| \le \mu(t)$ for any $u \in R$ and $t \in J$. Clearly $f(t,u)$ is continuous. Next $t^{-a} \in L^{\frac{1}{\sigma}}(J,R)$ only if $0 < a < \sigma < 1$. So Theorem 5.3 is applicable if $0 < a < \sigma < q-1$.

Remark 6.4. Let $|f(t,u)| \le \mu(t)$ for all $(t,u) \in J \times R$ where $f : J \times R \to R$ is continuous, $\mu \in L^{\frac{1}{\sigma}}(J,R)$ and $\sigma \ge 1$. Set

$$f_n(t) = \max_{J\times[-n,n]} |f(t,u)|, \quad n = 1, 2, \ldots.$$

Then $\{f_n\}_{m=1}^{\infty}$ is a nondecreasing sequence of continuous functions such that $f_n(t) \le \mu(t)$ for any $t \in J$. Then $\lim_{n\to\infty} f_n(t) = \mu_*(t)$ is lower semi continuous, i.e., $\mu_*^{-1}((r,\infty))$ is open for any $r \in R$. Note that it could be $\mu_*(t) = \infty$ for some $t \in J$. Then μ_* is measurable, $\mu_*(t) \le \mu(t)$ for any $t \in J$, and so $\mu_* \in L^{\frac{1}{\sigma}}(J,R)$. This means that without loss of generality we can suppose that $\mu \in L^{\frac{1}{\sigma}}(J,R)$ is lower semi continuous on J. On the other hand, let $\mu \in L^{\frac{1}{\sigma}}(J,R)$ be nonnegative and lower semi continuous on J. Then by Theorem 10 on Page 153 in [31] there is a nondecreasing sequence of nonnegative continuous functions $\{f_n\}_{m=1}^{\infty}$ such that $\lim_{n\to\infty} f_n(t) = \mu(t)$. Then we set

$$f(t,u) = f_1(t)|u| \quad \text{for } |u| \le 1,$$
$$f(t,u) = (|u|-n)f_{n+1}(t) + (n+1-|u|)f_n(t) \quad \text{for } |u| \in (n, n+1),$$

with any $t \in J$. Then $0 \le f(t,u) \le \mu(t)$ for any $u \in R$ and $t \in J$. Moreover $f(t,n) = f_n(t) \to \mu(t)$ as $n \to \infty$ for any $t \in J$ fixed. So $\mu(t)$ is the best function satisfying $|f(t,u)| \le \mu(t)$ for any $u \in R$ and $t \in J$. Clearly $f(t,u)$ is continuous.

7. Conclusions

An essence error of the formula of solutions which appeared in the recent study on the boundary value problems for impulsive fractional differential equations are reported in this work. A correct formula of solutions for a certain boundary value problem for fractional differential equations with Caputo fractional derivative and linear impulsive perturbed conditions are presented. By applying the well known fixed point theorems, new existence and uniqueness theorems of solutions are established. More important, impulsive fractional boundary value problems can be consider as a powerful tool to deal with physics models in real words.

Our future work will be devoted to study the following interesting models (see [18,19]), which can be regarded as the more beautiful physics models in some sense.

(i) Impulsive fractional hybrid boundary value problems

$$\begin{cases} {}^cD_t^q u(t) = f(t, u(t)), & t \in J', \\ \Delta u(t_k) = I_k(u(t_k^-)), \quad \Delta u'(t_k) = J_k(u(t_k^-)), & k = 1, 2, \ldots, m, \\ u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0, \end{cases} \tag{28}$$

where $f, I_k, J_k : R \to R$ are suitable functions.

(ii) Impulsive fractional integral boundary value problems

$$\begin{cases} l^cD_t^q u(t) = f(t, u(t)), & t \in J', \\ \Delta u(t_k) = I_k(u(t_k^-)), \quad \Delta u'(t_k) = J_k(u(t_k^-)), & k = 1, 2, \ldots, m, \\ \alpha u(0) + \beta u'(0) = \int_0^1 g_1(u(s))ds, \quad \alpha u(1) + \beta u'(1) = \int_0^1 g_2(u(s))ds, \end{cases} \tag{29}$$

where $f, I_k, J_k, g_1, g_2 : R \to R$ are suitable functions and $\alpha > 0, \beta \ge 0$ are real numbers.

References

[1] K. Diethelm, The analysis of fractional differential equations, Lecture Notes in Math. (2010).
[2] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V, Amsterdam, 2006.
[3] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.
[4] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[5] V.E. Tarasov, Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media, Springer, HEP, 2010.
[6] R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta. Appl. Math. 109 (2010) 973–1033.
[7] B. Ahmad, J.J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, Comput. Math. Appl. 58 (2009) 1838–1843.
[8] Z. Bai, On positive solutions of a nonlocal fractional boundary value problem, Nonlinear Anal.: TMA 72 (2010) 916–924.
[9] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, J. Math. Anal. Appl. 338 (2008) 1340–1350.
[10] Y.-K. Chang, J.J. Nieto, Some new existence results for fractional differential inclusions with boundary conditions, Math. Comput. Modelling 49 (2009) 605–609.
[11] M. Fečkan, Y. Zhou, J. Wang, On the concept and existence of solution for impulsive fractional differential equations, Commun. Nonlinear Sci. Numer. Simul. (2011) doi:10.1016/j.cnsns.2011.11.017.
[12] J. Wang, Y. Zhou, A class of fractional evolution equations and optimal controls, Nonlinear Anal.: RWA 12 (2011) 262–272.
[13] J. Wang, Y. Zhou, Analysis of nonlinear fractional control systems in Banach spaces, Nonlinear Anal.: TMA 74 (2011) 5929–5942.
[14] J. Wang, Y. Zhou, Existence and controllability results for fractional semilinear differential inclusions, Nonlinear Anal.: RWA 12 (2011) 3642–3653.
[15] S. Zhang, Existence of positive solution for some class of nonlinear fractional differential equations, J. Math. Anal. Appl. 278 (2003) 136–148.
[16] Y. Zhou, F. Jiao, J. Li, Existence and uniqueness for fractional neutral differential equations with infinite delay, Nonlinear Anal.: TMA 71 (2009) 3249–3256.
[17] Y. Zhou, F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, Nonlinear Anal.: RWA 11 (2010) 4465–4475.
[18] B. Ahmad, S. Sivasundaram, Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations, Nonlinear Anal.: HS 3 (2009) 251–258.
[19] B. Ahmad, S. Sivasundaram, Existence of solutions for impulsive integral boundary value problems of fractional order, Nonlinear Anal.: HS 4 (2010) 134–141.
[20] B. Ahmad, G. Wang, Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order, Comput. Math. Appl. 59 (2010) 1341–1349.
[21] Y. Tian, Z. Bai, Existence results for the three-point impulsive boundary value problem involving fractional differential equations, Comput. Math. Appl. 59 (2010) 2601–2609.
[22] J. Cao, H. Chen, Some results on impulsive boundary value problem for fractional differential inclusions, Electron. J. Qual. Theory Differ. Equ. 2010 (11) (2010) 1–24.
[23] G. Wang, B. Ahmad, L. Zhang, Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order, Nonlinear Anal.: TMA 74 (2011) 792–804.
[24] G. Wang, L. Zhang, G. Song, Systems of first order impulsive functional differential equations with deviating arguments and nonlinear boundary conditions, Nonlinear Anal.: TMA 74 (2011) 974–982.
[25] G. Wang, B. Ahmad, L. Zhang, Some existence results for impulsive nonlinear fractional differential equations with mixed boundary conditions, Comput. Math. Appl. 59 (2010) 1389–1397.
[26] X. Wang, Impulsive boundary value problem for nonlinear differential equations of fractional order, Comput. Math. Appl. 62 (2011) 2383–2391.
[27] J. Cao, H. Chen, Impulsive fractional differential equations with nonlinear boundary conditions, Math. Comput. Modelling (2011) doi:10.1016/j.mcm.2011.07.037.
[28] L. Yang, H. Chen, Nonlocal boundary value problem for impulsive differential equations of fractional order, Adv. Differential Equations 2011 (2011) 16. Article ID 404917.
[29] J. Wang, X. Xiang, Y. Peng, Periodic solutions of semilinear impulsive periodic system on Banach space, Nonlinear Anal.: TMA 71 (2009) e1344–e1353.
[30] W. Wei, X. Xiang, Y. Peng, Nonlinear impulsive integro–differential equation of mixed type and optimal controls, Optimization 55 (2006) 141–156.
[31] I.P. Natanson, Theory of Functions of a Real Variable, vol. II, Frederick Ungar Publishing Co, New York, 1960.