Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Discrete Mathematics 308 (2008) 4616–4624

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

On the existence and on the number of (k, l)-kernels in the lexicographic product of graphs

Waldemar Szumny, Iwona Włoch, Andrzej Włoch*

Technical University of Rzeszów, Department of Mathematics, ul W.Pola 2, 35-359 Rzeszów, Poland

Received 6 March 2006; received in revised form 14 August 2007; accepted 16 August 2007
Available online 27 September 2007

Abstract

In [G. Hopkins, W. Staton, Some identities arising from the Fibonacci numbers of certain graphs, Fibonacci Quart. 22 (1984) 225–228.] and [I. Włoch, Generalized Fibonacci polynomial of graphs, Ars Combinatoria 68 (2003) 49–55] the total number of k-independent sets in the generalized lexicographic product of graphs was given. In this paper we study (k, l)-kernels (i.e. k-independent sets being l-dominating, simultaneously) in this product and we generalize some results from [A. Włoch, I. Włoch, The total number of maximal k-independent sets in the generalized lexicographic product of graphs, Ars Combinatoria 75 (2005) 163–170]. We give the necessary and sufficient conditions for the existence of (k, l)-kernels in it. Moreover, we construct formulas which calculate the number of all (k, l)-kernels, k-independent sets and l-dominating sets in the lexicographic product of graphs for all parameters k, l. The result concerning the total number of independent sets generalizes the Fibonacci polynomial of graphs. Also for special graphs we give some recurrence formulas.

MSC: 05C69

Keywords: Counting; (k, l)-kernel; Efficient dominating set; Lexicographic product

1. Introduction

For general concepts we refer the reader to [2,10]. By a graph G we mean a finite, undirected, connected, simple graph. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. By a P_n we mean a graph with the vertex set $V(P_n)=\{t_1, \dots, t_n\}$ and the edge set $E(P_n)=\{\{t_i, t_{i+1}\}; i=1, \dots, n-1\}, n\geqslant 2$. Moreover, P_1 is the graph that consists of only one vertex. Let K_x denote the complete graph on x vertices, $x\geqslant 1$. Let G be a graph on $V(G)=\{t_1, \dots, t_n\}, n\geqslant 2$, and $h_n=(H_i)_{i\in\{1,\dots,n\}}$ be a sequence of vertex disjoint graphs on $V(H_i)=\{(t_i, y_1), \dots, (t_i, y_x)\}, x\geqslant 1$. By the generalized lexicographic product of G and $h_n=(H_i)_{i\in\{1,\dots,n\}}$ we mean the graph $G[h_n]$ such that $V(G[h_n])=\bigcup_{i=1}^n V(H_i)$ and $E(G[h_n])=\{\{(t_i, y_p), (t_j, y_q)\}; (t_i=t_j \text{and} \{(t_i, y_p), (t_i, y_q)\}\in E(H_i)) \text{or} \{t_i, t_j\}\in E(G)\}$. By H_i^c, $i=1, \dots, n$ we will denote the copy of the graph H_i in $G[h_n]$. If $H_i=H$ for $i=1, \dots, n$, then $G[h_n]=G[H]$, where $G[H]$ is the lexicographic product of two graphs. By $d_G(x, y)$ we denote the length of the shortest path joining vertices x and y in G.

* Corresponding author.
E-mail addresses: wszumny@prz.rzeszow.pl (W. Szumny), iwloch@prz.rzeszow.pl (I. Włoch), awloch@prz.rzeszow.pl (A. Włoch).

0012-365X/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2007.08.078

In [12] it has been proved:

Theorem 1 (*Włoch and Włoch [12]*). *Let $(t_i, y_p), (t_j, y_q) \in V(G[h_n])$. Then*

$$d_{G[h_n]}((t_i, y_p), (t_j, y_q)) = \begin{cases} d_G(t_i, t_j) & \text{for } i \neq j, \\ 1 & \text{for } i = j \text{ and } d_{H_i}(y_p, y_q) = 1, \\ 2 & \text{otherwise.} \end{cases}$$

Let $k \geqslant 2$, $l \geqslant 1$ be integers. We say that $J \subset V(G)$ is a (k, l)-kernel of a graph G if:

(1) for each $t_i, t_j \in J$, $d_G(t_i, t_j) \geqslant k$,
(2) for each $t_s \notin J$ there exists $t_i \in J$ such that $d_G(t_s, t_i) \leqslant l$.

From the definition of (k, l)-kernel it follows that if J is a (k, l)-kernel of G, then J is also a (k_0, l_0)-kernel of G where $k_0 \leqslant k$ and $l_0 \geqslant l$. If the set J satisfies condition in (1) or in (2), then we shall call it a k-independent set of G or an l-dominating set of G, respectively. We notice that 2-independent set is an independent set and 1-dominating set is a dominating set of G. In addition a subset containing only one vertex and the empty set also are k-independent sets. The set $V(G)$ is an l-dominating set of G. If an l-dominating, $l \geqslant 1$, set of G has exactly one vertex, then we shall call this vertex an l-dominating vertex of G. Moreover the l-dominating vertex of G also is a (k, l)-kernel of G, for $k \geqslant 2$.

From the definitions of k-independent set, l-dominating set and by Theorem 1 it follows:

Proposition 1. *Let $k \geqslant 2$, $n \geqslant 2$ be integers. A subset $S^* \subset V(G[h_n])$ is a k-independent set of $G[h_n]$ if and only if there exists a k-independent set $S \subset V(G)$, such that $S^* = \bigcup_{i \in \mathscr{I}} S_i$, where $\mathscr{I} = \{i, t_i \in S\}$, $S_i \subset V(H_i^c)$ and*

(a) *for $k = 2$, S_i is an independent set of H_i^c,*
(b) *for $k \geqslant 3$, S_i contains exactly one vertex from $V(H_i^c)$*

for every $i \in \mathscr{I}$.

Proposition 2. *Let $l \geqslant 1$, $n \geqslant 2$ be integers. A subset $Q^* \subseteq V(G[h_n])$ is an l-dominating set of $G[h_n]$ if and only if there exists an l-dominating set $Q \subseteq V(G)$, such that $Q^* = \bigcup_{i \in \mathscr{I}} Q_i$, where $\mathscr{I} = \{i, t_i \in Q\}$, $Q_i \subseteq V(H_i^c)$ and*

(a) *for $l = 1$, Q_i is a dominating set of H_i^c if for each $j \in \mathscr{I}$ and $i \neq j$, $\{t_i, t_j\} \notin E(G)$ or Q_i is a nonempty subset of $V(H_i^c)$ otherwise,*
(b) *for $l \geqslant 2$, Q_i is a nonempty subset of $V(H_i^c)$,*

for every $i \in \mathscr{I}$.

The concept of (k, l)-kernels was introduced by Kwaśnik in [5]. A (2, 1)-kernel is a kernel in Berge's sense. A (3, 1)-kernel is named as efficient dominating set and it was studied in [1]. The $(k, k-1)$-kernels, $k \geqslant 2$, were considered in [3,5,13]. In [5] it has been proved:

Theorem 2 (*Kwaśnik [5]*). *Let $k \geqslant 2$, $l \geqslant k-1$ be integers. Then every maximal (with respect to set inclusion) k-independent set of G is a (k, l)-kernel of G.*

The graph G has not always a (k, l)-kernel, for $k \geqslant 3$ and $l \geqslant 1$.

Theorem 3 (*Kwaśnik [5]*). *Let $k \geqslant 2$, $l \geqslant 1$ be integers. If the set J is a (k, l)-kernel of G and $|J| \geqslant 2$, then $l \geqslant \frac{k-1}{2}$.*

It is not easy to find a general rule when a graph G has a (k, l)-kernel. In fact there are some difficulties in finding a complete characterization of graphs having a (k, l)-kernel for $l < k-1$. For special case of k, l or for special classes of graphs see [1,5,12,13]. The main objectives of this paper are to study (k,l)-kernels in $G[h_n]$ and next counting (k,l)-kernels, k-independent sets and l-dominating sets of this product. In [8] Prodinger and Tichy gave impetus to the study of the number of independent sets of a graph and the literature includes many papers dealing with the theory of counting of independent sets in graphs, see for instance [7,9]. The problem of counting of independent sets of a graph is $\mathcal{NP}$-complete. In the chemical literature the number of independent sets of a graph is referred to as the Merrifield–Simmons index. This index is one of the most popular topological indices in chemistry. Results concerning counting independent sets in graphs may have potential use in the combinatorial chemistry.

2. The existence of (k,l)-kernels in $G[h_n]$

In this section we give necessary and sufficient conditions for the existence of (k,l)-kernel in $G[h_n]$. By Theorem 2 for $k\geqslant 2$ and $l\geqslant k-1$ every maximal k-independent set of $G[h_n]$ is a (k,l)-kernel of $G[h_n]$.

Theorem 4. *Let $k\geqslant 4$, $2\leqslant l\leqslant k-2$, $n\geqslant 2$ be integers. Then $G[h_n]$ has a (k,l)-kernel if and only if G has a (k,l)-kernel.*

Proof. Assume that $G[h_n]$ has a (k,l)-kernel, say J. From Theorem 1 and by Proposition 1(b) it follows that at most one vertex from H_i^c, $i=1,\dots,n$, can belong to the set J. Using the definition of the graph $G[h_n]$ immediately follows that the set $J_1=\{t_i\in V(G);\ J\cap V(H_i^c)\neq\emptyset\}$ is a (k,l)-kernel of the graph G. Suppose that G has a (k,l)-kernel J' and let $J'=\{t_i : i\in\mathcal{I}\}$, where $\mathcal{I}\subset\{1,\dots,n\}$ and $|\mathcal{I}|=p$, $p\geqslant 1$. We shall show that for an arbitrary sequence of graphs $H_1,\dots,H_n$ the graph $G[h_n]$ has a (k,l)-kernel. From the definition of the graph $G[h_n]$ and by Proposition 1(b) we deduce that to obtain a (k,l)-kernel of $G[h_n]$ we have to choose exactly one of the x vertices in each of the p-copies H_i^c, $i\in\mathcal{I}$. Such chosen subset J^* of the $V(G[h_n])$ is k-independent. We shall show that J^* is l-dominating. Let $(t_i,y_j)\notin J^*$. If $t_i\notin J'$, then $d_{G[h_n]}((t_i,y_j),J^*)=d_G(t_i,J')\leqslant l$. In case $t_i\in J'$ by Theorem 1 holds $d_{G[h_n]}((t_i,y_j),J^*)\leqslant 2$. Consequently J^* is a (k,l)-kernel of $G[h_n]$. Thus the theorem is proved. □

Theorem 5. *Let $k\geqslant 3$, $n\geqslant 2$ be integers. Then $G[h_n]$ has a $(k,1)$-kernel if and only if*:

(a) *for $k\geqslant 4$ there exists a dominating vertex t_i of G, $1\leqslant i\leqslant n$, such that H_i has a dominating vertex*,
(b) *for $k=3$ there exists a $(3,1)$-kernel $J=\{t_i;\ i\in\mathcal{I}\}$, $\mathcal{I}\subset\{1,\dots,n\}$ of G such that H_i has a dominating vertex, for every $i\in\mathcal{I}$.*

Proof. (a) Assume that $G[h_n]$ has a $(k,1)$-kernel, for $k\geqslant 4$. By Theorem 3 it follows that the $(k,1)$-kernel J of the graph $G[h_n]$ has exactly one vertex. Let $J=\{(t_i,y_j)\}$, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant x$, be a $(k,1)$-kernel of the graph $G[h_n]$. Then by Proposition 2(a) immediately follows that t_i is a dominating vertex of G and y_j is a dominating vertex of the graph H_i^c. Assume now that there exists a dominating vertex t_i of the graph G such that H_i has a dominating vertex. Consequently $\{t_i\}$ is a $(k,1)$-kernel of G. Let y_j be a dominating vertex of H_i. Then by the definition of the lexicographic product we obtain that (t_i,y_j) is a dominating vertex of $G[h_n]$. So, $\{(t_i,y_j)\}$ is a $(k,1)$-kernel of $G[h_n]$.

(b) Assume that $G[h_n]$ has a $(3,1)$-kernel, say J. Then by fact that $(3,1)$-kernel is 3-independent, Proposition 1(b) implies that at most one vertex from each copy of the graph H_i, $i=1,\dots,n$ can belong to the set J. So, for each $(t_i,y_p)\in J$ by Proposition 2(a) the vertex (t_i,y_p) is a dominating vertex of the graph H_i^c. Consequently, y_p is a dominating vertex of H_i. Moreover using the definition of the graph $G[h_n]$ it immediately follows that $J_1=\{t_i\in V(G);\ J\cap V(H_i^c)\neq\emptyset\}$ is a $(3,1)$-kernel of the graph G. Hence there exists a $(3,1)$-kernel of G such that if $t_i\in J_1$, then H_i has a dominating vertex.

Let us now suppose that there exists $(3,1)$-kernel of G, say $J'=\{t_i : i\in\mathcal{I}\}$, where $\mathcal{I}\subset\{1,\dots,n\}$ and $|\mathcal{I}|=p$, $p\geqslant 1$, such that if $t_i\in J'$, then H_i has a dominating vertex. We shall show that $G[h_n]$ has a $(3,1)$-kernel. Because H_i has a dominating vertex, so H_i^c in $G[h_n]$ also has a dominating vertex. By Proposition 1(b) and by the definition of the graph $G[h_n]$ to obtain a $(3,1)$-kernel of $G[h_n]$ we have to choose a dominating vertex in H_i^c, for each $i\in\mathcal{I}$. Evidently such chosen subset J of the $V(G[h_n])$ is a $(3,1)$-kernel of $G[h_n]$. Thus the theorem is proved. □

Corollary 1. *If $H_i=K_x$, $i=1,\dots,n$, then $G[K_x]$ has a $(3,1)$-kernel if and only if G has a $(3,1)$-kernel.*

3. The number of all (k, l)-kernels of $G[h_n]$

Let $r_G^{k,l}(n, p)$ denote the number of all p-element, $p \geqslant 1$, (k, l)-kernels of the graph G on n, $n \geqslant 2$, vertices. If $R^{k,l}(G)$ denotes the total number of (k, l)-kernels of the graph G, then it is clear that $R^{k,l}(G) = \sum_{p \geqslant 1} r_G^{k,l}(n, p)$. For $k = 2$ and $l = 1$ we put $r_G^{2,1}(n, p) = r_G(n, p)$ and $R^{2,1}(G) = R(G)$.

Theorem 6. *Let $k \geqslant 3$, $l \geqslant 2$, $n \geqslant 2$, $x \geqslant 1$. Then for an arbitrary graph G on n vertices and for an arbitrary sequence $h_n = (H_i)_{i \in \{1,\ldots,n\}}$ of vertex disjoint graphs on x vertices $R^{k,l}(G[h_n]) = \sum_{p \geqslant 1} r_G^{k,l}(n, p)x^p$.*

Proof. From the definition of the graph $G[h_n]$ and by Theorems 1, 2 and 4 we deduce that to obtain a p-element, $p \geqslant 1$, (k, l)-kernel of $G[h_n]$ first we have to choose a p-element (k, l)-kernel of the graph G. Evidently we can do it in $r_G^{k,l}(n, p)$ ways. Because $k \geqslant 3$ and $l \geqslant 2$ by Propositions 1(b) and 2(b) to obtain a (k, l)-kernel of $G[h_n]$ we have to choose one of the x vertices in each of the p chosen copies of H_i, $i = 1, \ldots, p$. Each of these vertices can be chosen in x ways, so we have $r_G^{k,l}(n, p)x^p$ p-element (k, l)-kernels of $G[h_n]$. Hence $R^{k,l}(G[h_n]) = \sum_{p \geqslant 1} r_G^{k,l}(n, p)x^p$. $\square$

If $l = k - 1$, $k \geqslant 3$, then we obtain result from [13]:

Theorem 7 (*Włoch and Włoch [13]*). *Let $k \geqslant 3$, $n \geqslant 2$, $x \geqslant 1$. Then for an arbitrary graph G on n vertices and for an arbitrary sequence $h_n = (H_i)_{i \in \{1,\ldots,n\}}$ of vertex disjoint graphs on x vertices $R^{k,k-1}(G[h_n]) = \sum_{p \geqslant 1} r_G^{k,k-1}(n, p)x^p$.*

Theorem 8. *Let $G[h_n]$ be a lexicographic product of graph G on n vertices, $n \geqslant 2$, and of a sequence $h_n = (H_i)_{i \in \{1,\ldots,n\}}$ of vertex disjoint graphs on x vertices, $x \geqslant 1$. Let $\mathscr{J} = \{J_1, \ldots, J_j\}$, $j \geqslant 1$, be a family of all $(3, 1)$-kernels of G such that if $t_i \in J_r$, $1 \leqslant r \leqslant j$, then H_i has a dominating vertex. Let $\mathscr{J} \ni J_r = \{t_i; i \in \mathscr{I}_r\}$ where $\mathscr{I}_r \subset \{1, \ldots, n\}$. If $d(H_i)$ is the number of dominating vertices of H_i, then $R^{3,1}(G[h_n]) = \sum_{r=1}^{j} \prod_{i \in \mathscr{I}_r} d(H_i)$.*

Proof. To obtain a $(3, 1)$-kernel of $G[h_n]$ by Theorem 5(b) we deduce that first we have to choose a $(3, 1)$-kernel from family $\mathscr{J}$. Let $J_r \in \mathscr{J}$ and $J_r = \{t_i, i \in \mathscr{I}_r\}$, where $\mathscr{I}_r \subset \{1, \ldots, n\}$. Next by Propositions 1(b) and 2(a) in each of H_i^c, $i \in \mathscr{I}_r$ we have to choose a dominating vertex of H_i. Evidently we can do it in $d(H_i)$ ways. Hence from fundamental combinatorial statements we have that $R^{3,1}(G[h_n]) = \sum_{r=1}^{j} \prod_{i \in \mathscr{I}_r} d(H_i)$. Thus the theorem is proved. $\square$

Corollary 2. *If $H_i = K_x$, $i = 1, \ldots, n$, then $R^{3,1}(G[K_x]) = \sum_{p \geqslant 1} r_G^{3,1}(n, p)x^p$.*

Using the same methods we can prove:

Theorem 9. *Let $k \geqslant 4$, $n \geqslant 2$ be integers. Let $G[h_n]$ be a lexicographic product of graph G on n vertices and of a sequence $h_n = (H_i)_{i \in \{1,\ldots,n\}}$ of vertex disjoint graphs on x vertices, $x \geqslant 1$. Let $J = \{t_i; i \in \mathscr{I}\}$, $\mathscr{I} \subset \{1, \ldots, n\}$, be the set of dominating vertices of G such that H_i has a dominating vertex. If $d(H_i)$ is the number of dominating vertices of H_i, then $R^{k,1}(G[h_n]) = \sum_{i \in \mathscr{I}} d(H_i)$.*

Theorem 10. *Let $G[h_n]$ be a lexicographic product of graph G on n vertices, $n \geqslant 2$, and of a sequence $h_n = (H_i)_{i \in \{1,\ldots,n\}}$ of vertex disjoint graphs on x vertices, $x \geqslant 1$. Let $\mathscr{J} = \{J_1, \ldots, J_j\}$, $j \geqslant 1$, be a family of all kernels of G and let $\mathscr{J} \ni J_r = \{t_i; i \in \mathscr{I}_r\}$ where $\mathscr{I}_r \subset \{1, \ldots, n\}$. Then $R(G[h_n]) = \sum_{r=1}^{j} \prod_{i \in \mathscr{I}_r} R(H_i)$.*

Theorem 11. *Let $l \geqslant 2$, $n \geqslant 2$, $x \geqslant 1$ be integers. Let $G[h_n]$ be a lexicographic product of graph G on n vertices and of a sequence $h_n = (H_i)_{i \in \{1,\ldots,n\}}$ of vertex disjoint graphs on x vertices. Let $\mathscr{J} = \{J_1, \ldots, J_j\}$, $j \geqslant 1$, be a family of all $(2, l)$-kernels of G and let $\mathscr{J} \ni J_r = \{t_i; i \in \mathscr{I}_r\}$ where $\mathscr{I}_r \subset \{1, \ldots, n\}$. If $F^*(H_i)$ is the number of nonempty independent sets of H_i, then $R^{2,l}(G[h_n]) = \sum_{r=1}^{j} \prod_{i \in \mathscr{I}_r} F^*(H_i)$.*

Let $k \geqslant 3$, $n \geqslant 1$ be integers. For the graph P_n on $V(P_n) = \{t_1, \ldots, t_n\}$ we use the following notation:
$\widehat{r}_{P_n}^{k,k-2}(n, p)$—the number of all p-element $(k, k-2)$-kernels of the graph P_n containing the vertex t_n.
$\widetilde{r}_{P_n}^{k,k-2}(n, p)$—the number of all p-element $(k, k-2)$-kernels of the graph P_n not containing the vertex t_n.

Theorem 12. *Let $k\geqslant 3$, $n\geqslant 2$, $p\geqslant 1$ be integers. If $n>p(2k-3)$, then $r_{P_n}^{k,k-2}(n,p)=0$.*

Proof. It is obvious that to construct a p-element $(k,k-2)$-kernel of the graph P_n we need at most $p(2k-3)$ vertices. In otherwise if $n>p(2k-3)$ then for an arbitrary p-element k-independent set S of P_n there exists $t_i\in V(P_n)$ such that $d_{P_n}(t_i,S)\geqslant k-1$, a contradiction. $\square$

Theorem 13. *Let $k\geqslant 3$, $n\geqslant 2$, $p\geqslant 1$. Then the number $r_{P_n}^{k,k-2}(n,p)$ satisfies following recurrence relations*:

$$r_{P_n}^{k,k-2}(n,1)=0,\quad n>2k-3,$$

$$r_{P_n}^{k,k-2}(n,1)=n,\quad n\leqslant k-1,$$

$$r_{P_n}^{k,k-2}(2k-2-i,1)=i,\quad i=1,\ldots,k-2,$$

$$\widehat{r}_{P_n}^{k,k-2}(n,1)=0\quad \textit{if } n\geqslant k,$$

$$\widehat{r}_{P_n}^{k,k-2}(n,1)=1\quad \textit{if } n\leqslant k-1,$$

for $p\geqslant 2$, $r_{P_n}^{k,k-2}(n,p)=0$ if $n>p(2k-3)$ and for $n\leqslant p(2k-3)$

$$r_{P_n}^{k,k-2}(n,p)=\widehat{r}_{P_n}^{k,k-2}(n,p)+\widetilde{r}_{P_n}^{k,k-2}(n,p),$$

$$\widetilde{r}_{P_n}^{k,k-2}(n,p)=\sum_{i=1}^{k-2}\widehat{r}_{P_{n-i}}^{k,k-2}(n-i,p),$$

$$\widehat{r}_{P_n}^{k,k-2}(n,p)=r_{P_{n-k}}^{k,k-2}(n-k,p-1)-\widehat{r}_{P_{n-2k+2}}^{k,k-2}(n-2k+2,p-1).$$

Proof. Assume that $p=1$. If $n>2k-3$, then by Theorem 12, $r_{P_n}^{k,k-2}(n,1)=0$. If $n\leqslant k-1$, then every vertex of $V(P_n)$ is a $(k,k-2)$-kernel of P_n, so $r_{P_n}^{k,k-2}(n,1)=n$ in this case. If $n=2k-2-i$ for $i=1,\ldots,k-2$, then by simple observation we have that $r_{P_n}^{k,k-2}(2k-2-i,1)=i$, $i=1,\ldots,k-2$. Moreover, it is clear that $\widehat{r}_{P_n}^{k,k-2}(n,1)=0$ if $n\geqslant k$ and $\widehat{r}_{P_n}^{k,k-2}(n,1)=1$ if $n\leqslant k-1$. Let now $p\geqslant 2$. If $n>p(2k-3)$, then by Theorem 12, $r_{P_n}^{k,k-2}(n,p)=0$. So let $n\leqslant p(2k-3)$. Assume that $\mathscr{F}_1$ be the family of all p-element $(k,k-2)$-kernels of P_n not containing the vertex t_n, hence $|\mathscr{F}_1|=\widetilde{r}_{P_n}^{k,k-2}(n,p)$. Let $\mathscr{F}_2$ be the family of all p-element $(k,k-2)$-kernels of P_n containing the vertex t_n, so $|\mathscr{F}_2|=\widehat{r}_{P_n}^{k,k-2}(n,p)$.Then it is clear, that $r_{P_n}^{k,k-2}(n,p)=|\mathscr{F}_1|+|\mathscr{F}_2|=\widetilde{r}_{P_n}^{k,k-2}(n,p)+\widehat{r}_{P_n}^{k,k-2}(n,p)$. We have to calculate the numbers $\widetilde{r}_{P_n}^{k,k-2}(n,p)$ and $\widehat{r}_{P_n}^{k,k-2}(n,p)$.

Let $S\in\mathscr{F}_1$. Then $t_n\notin S$ and exactly one of the vertices $t_{n-1},\ldots,t_{n-(k-2)}$ belongs to S. Consequently, $|\mathscr{F}_1|=\sum_{i\geqslant 1}^{k-2}\widehat{r}_{P_{n-i}}^{k,k-2}(n-i,p)$. Assume now that $S^*\in\mathscr{F}_2$, so $t_n\in S^*$. This means that $t_{n-i}\notin S^*$, $i=1,\ldots,k-1$ and $S^*=S'\cup\{t_n\}$ where S' is a $(p-1)$-element $(k,k-2)$-kernel of P_{n-k} and S' is not a $(p-1)$-element $(k,k-2)$-kernel of graph P_{n-2k+2} containing the vertex t_{n-2k+2}. Because there exist exactly $r_{P_{n-k}}^{k,k-2}(n-k,p-1)-\widehat{r}_{P_{n-2k+2}}^{k,k-2}(n-2k+2,p-1)$ sets S', hence we obtain that $\widehat{r}_{P_n}^{k,k-2}(n,p)=r_{P_{n-k}}^{k,k-2}(n-k,p-1)-\widehat{r}_{P_{n-2k+2}}^{k,k-2}(n-2k+2,p-1)$. Thus the theorem is proved. $\square$

From Theorems 6, 13 and Corollary 2 we obtain:

Corollary 3. *Let $n\geqslant 2$, $x\geqslant 1$ be integers. Then $R^{3,1}(P_n[K_x])=\sum_{p\geqslant 1}r_{P_n}^{3,1}(n,p)x^p$.*

Corollary 4. *Let $k\geqslant 4$, $n\geqslant 2$, $x\geqslant 1$ be integers. Then for an arbitrary sequence $h_n=(H_i)_{i\in\{1,\ldots,n\}}$ of vertex disjoint graphs on x vertices, $x\geqslant 1$, $R^{k,k-2}(P_n[h_n])=\sum_{p\geqslant 1}r_{P_n}^{k,k-2}(n,p)x^p$.*

4. The total number of k-independent sets of $G[h_n]$

In [4] the total number of independent sets of $G[K_x]$ was given using the concept of the Fibonacci polynomial of graphs. More general results for the number of all k-independent sets, $k\geqslant 3$, of $G[h_n]$ were obtained in [11]. In this section we give the total number of independent sets of $G[h_n]$. This result generalizes the Fibonacci polynomial of graph.

By $F^k(G)$ we denote the number of all k-independent sets of G (named as the generalized Fibonacci number of a graph) and we put $F^2(G)=F(G)$. Moreover, let $f_G^k(n,p)$ be the number of all p-element, $p\geqslant 0$, k-independent sets of a graph G on n vertices and also we put $f_G^2(n,p)=f_G(n,p)$. Consequently $F^k(G)=\sum_{p\geqslant 0} f_G^k(n,p)$. The coefficients $f_{P_n}(n,p)$ and $f_{P_n}^k(n,p)$ are equal to the Fibonacci numbers and the generalized Fibonacci numbers, respectively, see [8,6]. For k-independent sets it has been proved:

Theorem 14 (*Hopkins and Staton [4]*). *For an arbitrary graph G, on n vertices, $n\geqslant 2$, $F(G[K_x])=\sum_{p\geqslant 0} f_G(n,p)x^p$*

Theorem 15 (*Włoch [11]*). *Let $k\geqslant 3$, $x\geqslant 1$ be integers. Then for an arbitrary graph G on n, $n\geqslant 2$, vertices and for an arbitrary sequence $h_n=(H_i)_{i\in\{1,\dots,n\}}$ of vertex disjoint graphs on x vertices, $x\geqslant 1$, $F^k(G[h_n])=\sum_{p\geqslant 0} f_G^k(n,p)x^p$.*

The polynomials appearing in above Theorems are the Fibonacci polynomial of a graph and the generalized Fibonacci polynomial of a graph, respectively. For the graph P_n it has been proved

Theorem 16 (*Hopkins and Staton [4]*). *Let $n\geqslant 2$, $x\geqslant 1$ be integers. Then $F(P_n[K_x])=\sum_{p\geqslant 0}\binom{n-p+1}{p}x^p$.*

Theorem 17 (*Włoch [11]*). *Let $k\geqslant 3$, $n\geqslant 2$ be integers. Then for an arbitrary sequence $h_n=(H_i)_{i\in\{1,\dots,n\}}$ of vertex disjoint graphs on x vertices, $x\geqslant 1$, $F(P_n[h_n])=\sum_{p\geqslant 0}\binom{n-p-(p-1)(k-2)+1}{p}x^p$.*

Theorem 18. *Let $G[h_n]$ be a lexicographic product of graph G on n vertices, $n\geqslant 2$, and of a sequence $h_n=(H_i)_{i\in\{1,\dots,n\}}$ of vertex disjoint graphs on x vertices, $x\geqslant 1$. Let $\mathscr{S}=\{S_1,\dots,S_j\}$, $j\geqslant 1$ be a family of all nonempty independent sets of G and let $\mathscr{S}\ni S_r=\{t_i;\, i\in\mathscr{I}_r\}$ and $\mathscr{I}_r\subset\{1,\dots,n\}$. Then $F(G[h_n])=1+\sum_{r=1}^{j}\prod_{i\in\mathscr{I}_r}(F(H_i)-1)$.*

Proof. The definition of $G[h_n]$ implies that to obtain an independent set of $G[h_n]$ first we have to choose an independent set of G. Let $\mathscr{S}=\{S_1,\dots,S_j\}$, $j\geqslant 1$, be the family of all nonempty independent sets of G. Assume that $\mathscr{S}\ni S_r=\{t_i;\, i\in\mathscr{I}_r\}$ and $\mathscr{I}_r\subset\{1,\dots,n\}$. Next by Proposition 1(a) in each of the H_i^c, $i\in\mathscr{I}_r$, we have to choose a nonempty independent set of H_i^c. Evidently we can do it in $F(H_i)-1$ ways. Hence from fundamental combinatorial statements we have $\sum_{r=1}^{j}\prod_{i\in\mathscr{I}_r}(F(H_i)-1)$ independent sets of $G[h_n]$ having at least one vertex. Moreover, the empty set also is an independent set of $G[h_n]$. Consequently $F(G[h_n])=1+\sum_{r=1}^{j}\prod_{i\in\mathscr{I}_r}(F(H_i)-1)$. Thus the theorem is proved. $\square$

If $H_i=K_x$ for $i=1,\dots,n$, then we obtain Theorem 14.

5. The total number of l-dominating sets of $G[h_n]$

By $T^l(G)$ we denote the number of all l-dominating sets of G and we put $T^1(G)=T(G)$. Moreover, by $t_G^l(n,p)$ we denote the number of all p-element, $1\leqslant p\leqslant n$, l-dominating sets of a graph G on n vertices and also we put $t_G^1(n,p)=t_G(n,p)$. Consequently $T^l(G)=\sum_{p\geqslant 1}^{n} t_G^l(n,p)$. In this section we determine the number $T^l(G[h_n])$, $l\geqslant 1$, where $h_n=(H_i)_{i\in\{1,\dots,n\}}$ is an arbitrary sequence of vertex disjoint graphs on x vertices, $x\geqslant 1$.

Theorem 19. *Let $l\geqslant 2$, $n\geqslant 2$, $x\geqslant 1$ be integers. Then for an arbitrary graph G on n vertices and for an arbitrary sequence $h_n=(H_i)_{i\in\{1,\dots,n\}}$ of vertex disjoint graphs on x vertices $T^l(G[h_n])=\sum_{p\geqslant 1}^{n} t_G^l(n,p)(2^x-1)^p$.*

Proof. From the definition of the graph $G[h_n]$ and by Theorem 1 we deduce that to obtain an arbitrary l-dominating set of $G[h_n]$, first we have to choose an l-dominating set in the graph G. Assume that the chosen l-dominating set has

p-element, $1 \leqslant p \leqslant n$. So, we can choose it in $t^l_G(n, p)$ ways. Next by Proposition 2(b) we have to choose an arbitrary, nonempty subset in each of the p chosen copies of H_i. Because an arbitrary nonempty subset of H^c_i can be chosen in $(2^x - 1)$ ways, so we have $t^l_G(n, p)(2^x - 1)^p$ such l-dominating sets. Hence $T^l(G[h_n]) = \sum_{p \geqslant 1}^n t^l_G(n, p)(2^x - 1)^p$. Thus the theorem is proved. $\square$

Theorem 20. *Let $G[h_n]$ be a lexicographic product of graph G on n vertices, $n \geqslant 2$, and a sequence $h_n = (H_i)_{i \in \{1,\dots,n\}}$ of vertex disjoint graphs on x vertices, $x \geqslant 1$. Let $\mathcal{Q} = \{Q_1, \dots, Q_j\}$, $j \geqslant 1$ be a family of all dominating sets of G and let $\mathcal{Q} \ni Q_r = \{t_i; i \in \mathcal{I}_r\}$ and $\mathcal{I}_r \subseteq \{1, \dots, n\}$. Then $T(G[h_n]) = \sum_{r=1}^j \prod_{i \in \mathcal{I}_r} \widehat{f}(H_i)$, where*

$$\widehat{f}(H_i) = \begin{cases} T(H_i) & \textit{if for each } j \in \mathcal{I}_r \textit{ and } j \neq i \{t_i, t_j\} \notin E(G), \\ 2^x - 1 & \textit{otherwise.} \end{cases}$$

Proof. By Theorem 1 we have that to obtain a dominating set of $G[h_n]$ first we have to choose a dominating set of G. Let $\mathcal{Q} = \{Q_1, \dots, Q_j\}$, $j \geqslant 1$ be a family of all dominating sets of G and let $\mathcal{Q} \ni Q_r = \{t_i; i \in \mathcal{I}_r\}$ and $\mathcal{I}_r \subseteq \{1, \dots, n\}$. Next by Proposition 1(a) in each H^c_i, $i \in \mathcal{I}_r$, we have to choose a dominating set of H^c_i if for each $i \neq j \in \mathcal{I}_r$ holds $\{t_i, t_j\} \notin E(G)$ or if otherwise we have to choose an arbitrary nonempty subset of H^c_i. Consequently we can do it in $T(H_i)$ ways or in $2^x - 1$ ways, respectively. By above considerations we put

$$\widehat{f}(H_i) = \begin{cases} T(H_i) & \text{if for each } j \in \mathcal{I}_r \text{ and } j \neq i \{t_i, t_j\} \notin E(G), \\ 2^x - 1 & \text{otherwise.} \end{cases}$$

So by fundamental combinatorial statements $T(G[h_n]) = \sum_{r=1}^j \prod_{i \in \mathcal{I}_r} \widehat{f}(H_i)$. Thus the theorem is proved. $\square$

Corollary 5. *For an arbitrary graph G on n vertices, $n \geqslant 2$ holds $T(G[K_x]) = \sum_{p \geqslant 1}^n t_G(n, p)(2^x - 1)^p$.*

Let $l \geqslant 1$, $n \geqslant 2$ be integers. For the graph P_n on $V(P_n) = \{t_1, \dots, t_n\}$ we use the following notation. Let $\widehat{t}^l_{P_n}(n, p)$ be the number of all p-element l-dominating sets of the graph P_n, containing the vertex t_n. Consequently, we put $\widehat{T}^l(P_n) = \sum_{p \geqslant 1} \widehat{t}^l_{P_n}(n, p)$.

Theorem 21. *Let $l \geqslant 1$, $n \geqslant 1$, $p \geqslant 1$ be integers. Then $t^l_{P_n}(n, p) = 0$ if $n < p$ or $n > (2l + 1)p$ and for $p \leqslant n \leqslant (2l + 1)p$ the number $t^l_{P_n}(n, p)$ satisfies the following recurrence relations*:

$$t^l_{P_n}(n, p) = \sum_{i=0}^{l} \widehat{t}^l_{P_{n-i}}(n - i, p) \quad \textit{for } n \geqslant l + 1,$$

$$\widehat{t}^l_{P_n}(n, p) = t^l_{P_{n-1}}(n - 1, p - 1) + \sum_{k=l+2}^{2l+1} \widehat{t}^l_{P_{n-k}}(n - k, p - 1) \quad \textit{for } n \geqslant p \geqslant 2$$

with the initial conditions

$$t^l_{P_n}(n, p) = \binom{n}{p} \quad \textit{for } p \leqslant n \leqslant l,$$

$$\widehat{t}^l_{P_n}(n, p) = 0 \quad \textit{for } n < p,$$

$$\widehat{t}^l_{P_n}(n, 1) = 1 \;\; \textit{if } n \leqslant l + 1 \quad \textit{and} \quad \widehat{t}^l_{P_n}(n, 1) = 0 \;\; \textit{if } n \geqslant l + 2.$$

Proof. If $n < p$ or $n > (2l + 1)p$, then $t^l_{P_n}(n, p) = 0$, because there does not exist a p-element l-dominating set in this case. If $p \leqslant n \leqslant l$, then an arbitrary p-element subset of $V(P_n)$ is an l-dominating set of P_n, so $t^l_{P_n}(n, p) = \binom{n}{p}$. Assume that $l + 1 \leqslant n \leqslant p(2l + 1)$. Let $\mathcal{F}$ be the family of all p-element l-dominating sets of graph P_n. Hence $|\mathcal{F}| = t^l_{P_n}(n, p)$. Let $S^* \in \mathcal{F}$. It is clear that at least one vertex from vertices $t_n, t_{n-1}, \dots, t_{n-l}$ belongs to the set S^*. Consequently

$t^l_{P_n}(n,p)=\sum_{i=0}^{l}\widehat{t}^{\,l}_{P_{n-i}}(n-i,p)$. Next we have to calculate the number $\widehat{t}^{\,l}_{P_n}(n,p)$, for $n\geqslant 1$. If $n<p$, then $\widehat{t}^{\,l}_{P_n}(n,p)=0$, and moreover $\widehat{t}^{\,l}_{P_n}(n,1)=1$ if $n\leqslant l+1$ and $\widehat{t}^{\,l}_{P_n}(n,1)=0$ if $n\geqslant l+2$.

Let $n\geqslant p\geqslant 2$. Then every subset $S=S'\cup\{t_n\}$ is a p-element l-dominating set of P_n if S' is an arbitrary $(p-1)$-element l-dominating set of P_{n-1} or S' is not a $(p-1)$-element l-dominating set of P_{n-1} but it is a $(p-1)$-element l-dominating set of the graph P_{n-k}, $k=l+2,\ldots,2l+1$, containing the vertex t_{n-k}. Hence by previous considerations we obtain that $\widehat{t}^{\,l}_{P_n}(n,p)=t^l_{P_{n-1}}(n-1,p-1)+\sum_{k=l+2}^{2l+1}\widehat{t}^{\,l}_{P_{n-k}}(n-k,p-1)$ that completes the proof. $\square$

Corollary 6. *Let $l\geqslant 1$, $n\geqslant 1$ be integers. Then the number $T^l(P_n)$ satisfies the following recurrence relations*:

$$T^l(P_n)=\sum_{i=0}^{l}\widehat{T}^{\,l}(P_{n-i})\quad \textit{for } n\geqslant l+1,$$

$$\widehat{T}^{\,l}(P_n)=T^l(P_{n-1})+\sum_{k=l+2}^{2l+1}\widehat{T}^{\,l}(P_{n-k}),\quad n\geqslant l+2$$

with the initial conditions

$$T^l(P_n)=2^n-1\quad \textit{if } n=1,\ldots,l$$

and

$$\widehat{T}^{\,l}(P_n)=T^l(P_{n-1})+1\quad \textit{if } n=2,\ldots,l+1$$

and

$$\widehat{T}^{\,l}(P_1)=1.$$

Proof. If $n=1,\ldots,l$, then

$$T^l(P_n)=\sum_{p\geqslant 1}^{n}t^l_{P_n}(n,p)=\sum_{p\geqslant 1}^{n}\binom{n}{p}=2^n-1.$$

For $n\geqslant l+1$ we obtain that

$$T^l(P_n)=\sum_{p\geqslant 1}t^l_{P_n}(n,p)=\sum_{p\geqslant 1}\left(\sum_{i=1}^{l}\widehat{t}^{\,l}_{P_{n-i}}(n-i,p)\right)=\sum_{i=1}^{l}\left(\sum_{p\geqslant 1}\widehat{t}^{\,l}_{P_{n-i}}(n-i,p)\right)=\sum_{i=1}^{l}\widehat{T}^{\,l}(P_{n-i}).$$

Now, we calculate the number $\widehat{T}^{\,l}(P_n)$, $n\geqslant 1$. If $n=1$, then evidently $\widehat{T}^{\,l}(P_1)=1$.

If $2\leqslant n\leqslant l+1$, then

$$\begin{aligned}\widehat{T}^{\,l}(P_n)&=\sum_{p\geqslant 1}\widehat{t}^{\,l}_{P_n}(n,p)=\widehat{t}^{\,l}_{P_n}(n,1)+\sum_{p\geqslant 2}\widehat{t}^{\,l}_{P_n}(n,p)=1+\sum_{p\geqslant 2}\widehat{t}^{\,l}_{P_n}(n,p)\\&=1+\sum_{p\geqslant 2}t^l_{P_{n-1}}(n-1,p-1)=1+\sum_{r=p-1\geqslant 1}t^l_{P_{n-1}}(n-1,r)=1+T^l(P_{n-1}).\end{aligned}$$

For $n\geqslant l+2$ we have

$$\begin{aligned}\widehat{T}^{\,l}(P_n)&=\sum_{p\geqslant 1}\widehat{t}^{\,l}_{P_n}(n,p)=\sum_{p\geqslant 1}\left[t^l_{P_{n-1}}(n-1,p-1)+\sum_{k=l+2}^{2l+1}\widehat{t}^{\,l}_{P_{n-k}}(n-k,p-1)\right]\\&=\sum_{r=p-1\geqslant 0}\left[t^l_{P_{n-1}}(n-1,r)+\sum_{k=l+2}^{2l+1}\widehat{t}^{\,l}_{P_{n-k}}(n-k,r)\right].\end{aligned}$$

Because every l-dominating set has at least one vertex, so we can put that

$$\widehat{T}^l(P_n) = \sum_{r\geqslant 1}\left[t^l_{P_{n-1}}(n-1,r) + \sum_{k=l+2}^{2l+1}\widehat{t}^l_{P_{n-k}}(n-k,r)\right] = T^l(P_{n-1}) + \sum_{k=l+2}^{2l+1}\widehat{T}^l(P_{n-k}),$$

which ends the proof. □

If $l = 1$, by simple calculations we obtain that the total number of dominating sets of P_n can be calculated using the third-order linear recurrence relations $T(P_n) = T(P_{n-1}) + T(P_{n-2}) + T(P_{n-3})$, $n\geqslant 4$ with the initial conditions $T(P_1) = 1$, $T(P_2) = 3$, $T(P_3) = 5$.

From the Theorems 19, 21 and Corollary 5 immediately follows:

Corollary 7. *Let $l > 1$, $n\geqslant 2$, $x\geqslant 1$, $p\geqslant 1$ be integers. Then for an arbitrary sequence $h_n = (H_i)_{i\in\{1,\ldots,n\}}$ of vertex disjoint graphs on x holds $T^l(P_n[h_n]) = \sum_{p\geqslant 1} t^l_{P_n}(n, p)(2^x - 1)^p$.*

Corollary 8. *Let $n\geqslant 2$, $x\geqslant 1$, $p\geqslant 1$ be integers. Then $T(P_n[K_x]) = \sum_{p\geqslant 1} t_{P_n}(n, p)(2^x - 1)^p$.*

6. Concluding remarks

Note that while every maximal k-independent set of a graph G is a (k, l)-kernel of G, for $l\geqslant k-1$ there are some difficulties in finding a characterization of graphs having a (k, l)-kernel for $l < k-1$ and we do not know a complete characterization of them. So far only for specific graphs the problem of the existence of a (k, l)-kernel is solved. There are a number of interesting open problem related to this area. Among all (k, l)-kernels with $l < k-1$ the most interesting are $(2s+1, s)$-kernels, $s\geqslant 1$, which generalize efficient dominating sets. It is natural to ask about the characterization of graphs having a $(2s+1, s)$-kernel (in particular for fixed s).

References

[1] D.W. Bange, A.E. Barkauskas, P.J. Slater, Efficient Dominating Sets in Graphs, Application of Discrete Mathematics, SIAM, Philadelphia, PA, 1988 pp. 189–199.

[2] C. Berge, Principles of Combinatorics, Academic Press, New York, London, 1971.

[3] G.H. Fricke, S.T. Hedetniemi, M.A. Henning, Distance independent domination of graphs, Ars Combinatoria 41 (1995) 34–44.

[4] G. Hopkins, W. Staton, Some identities arising from the Fibonacci numbers of certain graphs, Fibonacci Quart. 22 (1984) 225–228.

[5] M. Kwaśnik, On (k, l)-kernels in graphs and their products, Doctoral dissertation, Technical University of Wrocław, Wrocław, 1980.

[6] M. Kwaśnik, I. Włoch, The total number of generalized stable sets and kernels of graphs, Ars Combinatoria 55 (2000) 139–146.

[7] A.S. Pedersen, P.D. Vestergaard, Bounds of the number of vertex independent sets in a graph, Taiwanese J. Math. 10 (6) (2006) 1575–1587.

[8] H. Prodinger, R.F. Tichy, Fibonacci numbers of graphs, Fibonacci Quart. 20 (1982) 16–21.

[9] B.E. Sagan, A note on independent sets in trees, SIAM J. Algebraic Discrete Math. 1 (1) (1988) 105–108.

[10] B. West, Introduction to Graph Theory, Prentice-Hall, Upper Saddle River, NJ, 1996.

[11] I. Włoch, Generalized Fibonacci polynomial of graphs, Ars Combinatoria 68 (2003) 49–55.

[12] A. Włoch, I. Włoch, On (k, l)-kernels in generalized products of graphs, Discrete Mathematics 164 (1996) 295–301.

[13] A. Włoch, I. Włoch, The total number of maximal k-independent sets in the generalized lexicographic product of graphs, Ars Combinatoria 75 (2005) 163–170.