JOURNAL OF COMPUTER AND SYSTEM SCIENCES 9, 69–87 (1974)

Complexity Classes of Partial Recursive Functions*

EDWARD L. ROBERTSON

Department of Computer Science, University of Waterloo, Waterloo, Ontario, Canada

Received July 2, 1971

This paper studies possible extensions of the concept of complexity class of recursive functions to partial recursive functions. Many of the well-known results for total complexity classes are shown to have corresponding, though not exactly identical, statements for partial classes. In particular, with two important exceptions, all results on the presentation and decision problems of membership for the two most reasonable definitions of partial classes are the same as for total classes. The exceptions concern presentations of the complements and maximum difficulty for decision problems of the more restricted form of partial classes.

The last section of this paper shows that it is not possible to have an "intersection theorem," corresponding to the union theorem of McCreight and Meyer, either for complexity classes or complexity index sets.

1. PRELIMINARY DEFINITIONS

The following definitions and notations, many of which are in common usage, are established for this paper.

N	the natural numbers $\{0, 1, 2, \ldots\}$;
$\psi(x)\downarrow$	the computation of the partial function ψ on input x halts or is defined, read "$\psi(x)$ converges",
$\psi(x)\uparrow$	the computation of $\psi(x)$ is not defined, "$\uparrow$" is read "diverges",
φ_i	the ith partial recursive function in a Gödel indexing $\{\varphi_i\}$, $\varphi_i\colon N \to N$,
W_i	the domain of $\varphi_i = \{x \mid \varphi_i(x)\downarrow\}$,
$\mathscr{P}$	the partial recursive (pt. r.) functions $= \{\varphi_i \mid i \in N\}$,
$\mathscr{R}$	the totoal recursive (rec) function $= \{\varphi_i \mid W_i = N\}$,
$\mathscr{P}^\infty$	partial recursive function with infinite domain,
A^C	the complement of A (with respect to N or $\mathscr{P}$ as appropriate),
$\Omega\mathscr{C}$	for $\mathscr{C} \subseteq \mathscr{P}$, $\Omega\mathscr{C} = \{i \mid \varphi_i \in \mathscr{C}\}$.

* A preliminary version of this paper was presented at the Third Annual ACM Symposium on Theory of Computing, Shaker Heights, Ohio (May 1971). The research in this paper was supported by the University of Wisconsin and the NSF.

The "quantifier" $\exists^\infty$ is an abbreviation such that

$$(\overset{\infty}{\exists}x)[P(x)] \equiv (\forall y)(\exists x)[x \geqslant y \mathbin{\&} P(x)],$$

where P is a predicate with one free variable. The usage of "$\exists^\infty$" is similar to that of "$\exists!$" which occurs commonly in mathematical writing. In writing, where the variable quantified over is unspecified or understood, we use "i.o." (infinitely often) instead of "$\exists^\infty$". That is, "P i.o." will be taken as synonymous with "$(\exists^\infty x)[P(x)]$". Similarly, "$\forall^\infty$" or "a.e." (almost everywhere) is an abbreviation such that

$$(\overset{\infty}{\forall}x)[P(x)] \equiv (\exists y)(\forall x)[x > y \Rightarrow P(x)].$$

If A is a predicate over function and $\mathscr{C}$ is a class of functions, we use "for sufficiently large $f \in \mathscr{C}$, $A(f)$" for "$(\exists g \in \mathscr{R})(\forall f \in \mathscr{C})[f \geqslant g$ a.e. $\Rightarrow A(f)]$". Similarly, "for arbitrarily large $f \in \mathscr{C}$, $A(f)$" means "$(\forall g \in \mathscr{R})(\exists f \in \mathscr{C})[f \geqslant g$ a.e. and $A(f)]$".

We will reserve the word "class" for subsets of $\mathscr{P}$, using "set" to refer to subsets of N, in an attempt to clarify whether functions or specific algorithms for functions are being considered.

We assume familiarity with the concepts of Turing reducibility and 1–1 reducibility [14], which will be denoted "$\leqslant_T$" and "$\leqslant_1$", respectively (e.g., $A \leqslant_T B$ is A is Turing reducible to B). Also Σ_n and Π_n denote the levels of the Kleene hierarchy [14]. Certain standard sets are used as reference points within the Kleene hierarchy. These sets, along with their known positions in the hierarchy, are:

$K = \{i \mid i \in W_i\}$	Σ_1-complete
$\mathit{Total} = \{i \mid W_i = N\}$	Π_2-complete
$\mathit{Finite} = \{i \mid W_i \text{ finite}\}$	Σ_2-complete
$\mathit{Equal} = \{\langle i, j\rangle \mid \varphi_i = \varphi_j\}$	Π_2-complete
$\mathit{Bounded} = \{i \mid W_i = N \text{ and } (\exists z)(\forall x)[\varphi_i(x) \leqslant z]\}$	$\Pi_3 \cap \Sigma_3$
$\mathit{Cofinite} = \{i \mid W_i^C \text{ is finite}\}$	Σ_3-complete

where B is Σ_n (or Π_n)-*complete* if $B \in \Sigma_n$ (respectively Π_n) and, for all $C \in \Sigma_n$ (respectively, Π_n), $C \leqslant_1 B$.

DEFINITION. $\langle \varphi, \Phi\rangle$ will denote an *abstract measure of computational complexity* [2], where $\varphi = \{\varphi_i\}$ is a Gödel enumeration of $\mathscr{P}$ and $\Phi = \{\Phi_i\}$ satisfies

$$\varphi_i(x)\downarrow \text{ iff } \Phi_i(x)\downarrow, \tag{1}$$

$$\text{the predicate "}\Phi_i(x) = y\text{" is recursive in } i, x, \text{ and } y. \tag{2}$$

Unless otherwise stated, we assume a fixed enumeration for φ and write Φ instead of $\langle \varphi, \Phi\rangle$.

DEFINITION. The recursive relation between measures Φ and Φ^* is the function

$$r(z) = \max_{i,x \leqslant z} \{\Phi_i^*(x) \mid \Phi_i(x) \leqslant z\} + \max_{i,x \leqslant z} \{\Phi_i(x) \mid \Phi_i^*(x) \leqslant z\}.$$

This function has the important properties [2] that

$$\Phi_i^*(x) \leqslant r(\max\{i, x, \Phi_i(x)\})$$

and (3)

$$\Phi_i(x) \leqslant r(\max\{i, x, \Phi^*(x)\})$$

for all i and x.

In order to simplify notation, we make the following assumption, which will hold for the rest of this paper. Any results in this paper will hold without this assumption making conceptually simple but notationally messy modifications.

INPUT REPRESENTATION ASSUMPTION. For any i, y there is an x_0 such that $x \geqslant x_0$ implies $\Phi_i(x) \geqslant y$. A slightly stronger condition, requiring the existence of a nondecreasing and unbounded recursive f such that $\Phi_i(x) \geqslant f(x)$ for all i and x, is the natural condition that some resource is required simply to represent or to read the input. If we are considering as a measure the amount of tape used by a Turing machine, and those machines represent their input as "tallies", then $f = \lambda x[x]$. If the representation is binary, $f = \lambda x[\log_2(x)]$. One immediate example of the simplification provided by this assumption involves the recursive relation r between Φ and Φ^*. The above result (3) may be simply stated, that for all i,

$$\Phi_i^* \leqslant r \circ \Phi_i \text{ a.e.} \qquad \text{and} \qquad \Phi_i \leqslant r \circ \Phi_i^* \text{ a.e.}$$

2. COMPLEXITY CLASSES OF TOTAL FUNCTIONS

Almost all of the investigation of abstract complexity measures to data has been concerned only with total functions, and even with certain subclasses of these functions. Important concepts in the development of these investigations have been the *Φ-complexity index set of t*

$$I_i^{\Phi} = \{i \mid \varphi_i \in \mathscr{R} \text{ and } \Phi_i \leqslant t \text{ a.e.}\}$$

and the *Φ-complexity class of t*,

$$R_t^{\Phi} = \{\varphi_i \mid i \in I_t^{\Phi}\},$$

defined for any measure Φ and total function t. In order to have a notation in the integers for a class of functions $\mathscr{C} \subseteq \mathscr{P}$, we say $B \subset N$ is a *presentation of $\mathscr{C}$* if

$$\mathscr{C} = \{\varphi_i \mid i \in B\}.$$

The value of presentations as notations for complexity classes is indicated by the following results.

THEOREM 2.1 [3]. *For any complexity measure Φ there exists $b^{\Phi} \in \mathscr{R}$ such that, if $t \in \mathscr{R}$ satisfies $t \geqslant b^{\Phi}$ a.e., then there is an r.e. set W_i such that W_i is a presentation of R_t^{Φ} (R_t^{Φ} is then said to be recursively presentable).*

THEOREM 2.2 [7]. *For any complexity measure Φ and any $t \in \mathscr{R}$, there exists $i \in N$ such that W_i^C is a presentation of R_t^{Φ}.*

THEOREM 2.3 [7]. *For any Φ and $t \in \mathscr{R}$, $\mathscr{P} - R_t^{\Phi}$ is recursively presentable.*

3. EXTENSIONS OF COMPLEXITY CLASSES TO PARTIAL FUNCTIONS

There has to date been very little study of classes of partial functions. The original motivation for the construction of various hierarchies of computable functions (the "subrecursive hierarchies") was a problem specifically oriented to the total functions. In Rice [10, 11] and Dekker and Myhill [6], the first thorough investigations of questions about algorithms and functions, classification was done for all functions, not only total ones. Thus there is precedent for considering complexity classes and sets of all partial recursive functions.

The first difficulty is, simply: what is a partial complexity class? There are many ways in which partial classes can reflect the properties of total classes, or the properties of partial functions. For this reason two alternative definitions of partial classes are introduced and considered.

DEFINITION. For any measure Φ and function τ, the set of Φ, *τ-computable algorithms* is

$$I_\tau^{\Phi} = \{i \mid \mathrm{Dom}(\tau) \subseteq W_i \text{ and } \Phi_i \leqslant \tau \text{ a.e.}\}.$$

This is the obvious analog for some partial function τ of the set of Φ, t-computable algorithms for a total t. Observe that the notation is consistent, as I_t^{Φ}, t total, is the same class according to either definition. The predicate,

$$(\exists u)(\forall x)[\tau(x)\downarrow \Rightarrow \Phi_i(x) \leqslant \max(u, \tau(x))]$$

expresses "$i \in I_\tau^{\Phi}$."

Recall that the input representation assumption is in effect, allowing the simple predicate above.

DEFINITION. For any measure Φ and any function τ, the *partial Φ-complexity class of τ* is

$$P_\tau^\Phi = \{\varphi_i \mid i \in I_\tau^\Phi\}.$$

An alternative definition which will be considered is

$$\hat{P}_\tau^\Phi = \{\psi \mid \psi \in \mathscr{P} \;\&\; (\exists i)[i \in I_\tau^\Phi \;\&\; (\forall x)[\tau(x)\downarrow \Rightarrow \psi(x) = \Phi_i(x)]]\}.$$

Once again, for total t, it is true that $P_t^\Phi = R_t^\Phi$ and even $\hat{P}_t^\Phi = R_t^\Phi$. P_t^Φ was defined as a straight translation of "$R_t^\Phi = \{\Phi_i \mid i \in I_t^\Phi\}$," by far the most natural way to correspond classes of functions to sets of algorithms.

The definition of $\hat{P}_\tau^\Phi$ is motivated by considering τ to specify types of problems, and conditions on the solution of problems. The domain of interest in the solution of these problems is just the domain of τ; and all those values for which τ diverges are "don't care" conditions. A "partial algorithm," as suggested by Ullian [16], is an algorithm where we are interested in the effect on only a restricted set, and certainly we would only worry about the efficiency of the algorithm on this set. Many other examples occur, as in algebra where there exists an algorithm which decides if a set of universal equations implies a given equation if indeed this same set implies the theory of Abelian groups [15], but the question of whether the set implies the theory of Abelian groups is not itself decidable [10].

It is very easy to see that there are measures with the anomalous conditions that $\Phi_\tau^\Phi = I_\rho^\Phi$, for some τ and ρ, but $\hat{P}_\tau^\Phi \neq \hat{P}_\rho^\Phi$. Say, for example, that no algorithm except k has Φ-complexity equal to zero at any point, but $\Phi_k = \lambda x[0]$. Let $\tau \equiv 0$ and $\rho(2 \cdot x) = 0$, $\rho(2x + 1)\uparrow$. Then $I_\tau^\Phi = I_\rho^\Phi = \{k\}$, but obviously $\hat{P}_\rho^\Phi$ contains infinitely more functions than $\hat{P}_\tau^\Phi$.

One further complication which arises with partial functions is the cardinality of the domain. In particular if $\mathrm{Dom}(\tau)$ is finite, then

$$I_\tau = \Omega P_\tau = \Omega \hat{P}_\tau = \{i \mid \mathrm{Dom}(\tau) \subseteq W_i\}.$$

In this case the decision problem is known to be equivalent to K. Thus all further consideration will be only of $\tau \in \mathscr{P}^\infty$, partial functions with infinite domain.

Before continuing, we mention several obvious containment results which will be helpful. For any measures Φ, Φ^*; any ψ, $\xi \in \mathscr{P}$:

(*1) $\hat{P}_\psi^\Phi \supseteq P_\psi^\Phi$,

(*2) $\psi \geqslant \xi$ a.e. $\Rightarrow P_\psi^\Phi \supseteq P_\xi^\Phi$ and $\hat{P}_\psi^\Phi \supseteq \hat{P}_\xi^\Phi$.

If r is the recursive relation between Φ and Φ^*, recall the input representation assumption has the consequence that

$$\Phi_i \leqslant r \circ \Phi_i^* \text{ a.e.} \qquad \text{and} \qquad \Phi_i^* \leqslant r \circ \Phi_i \text{ a.e.}$$

Thus, if any $\Phi_i \leqslant \psi$ a.e., then $\Phi_i^* \leqslant r \circ \Phi_i \leqslant r \circ \psi$ a.e. Hence

(*3) $P_\psi{}^\Phi \subseteq P^{\Phi^*}_{r\circ\psi}$ and $\hat{P}_\psi{}^\Phi \subseteq \hat{P}^{\Phi^*}_{r\circ\psi}$.

Let L ($=\langle\varphi, L\rangle$) be the "standard"[1] tape measure, which does satisfy the input representation assumption.

The following results show that these two alternative versions of partial complexity classes agree for certain natural bounding functions.

PROPOSITION 3.1. *For any function Ψ which can be computed using Ψ tape,*

$$\hat{P}_\Psi{}^L \subseteq P_\Psi{}^L.$$

Proof. The proof depends heavily on the properties of the tape measure, particularly that many computations can be performed "in parallel" without using any extra tape.

Assume ξ such that $\xi \in \hat{P}_\Psi{}^L$, that is there exists $j \in I_\Psi{}^L$ such that

$$(\forall x)[\Psi(x)\downarrow \Rightarrow \xi(x) = \varphi_j(x)].$$

The following sketches the computation of φ_k, which can be seen to satisfy $k \in I_\Psi{}^L$ and $\varphi_k = \xi$. These conditions imply $\xi \in P_\Psi{}^L$, as required.

Compute $\varphi_k(x)$ as follows. Compute in parallel $\Psi(x)$, $\varphi_j(x)$, and $\xi(x)$, choosing appropriate algorithms and keeping track of the amount of tape by each. In particular, pick a computation of Ψ using exactly Ψ tape.

(1) If $\xi(x)$ converges using the least amount of tape, output $\xi(x)$.

(2) If $\varphi_j(x)$ converges using the least tape, continue computing until either (a) $\Psi(x)\downarrow$ or (b) $\xi(x)\downarrow$. If (a) occurs first, output $\varphi_j(x)$; otherwise output $\xi(x)$.

(3) The case that $\Psi(x)$ converges in (strictly) least amount of tape can only happen finitely often. In this case compute and output $\xi(x)$.

The reader may easily implement on a Turing machine the computation described above in such a way that the desired properties are apparent. ∎

COROLLARY 3.2. *For any i,*

$$\hat{P}^L_{L_i} = P^L_{L_i}.$$

Proof. Obvious, since L_i may easily be computed using L_i tape, also (*1). ∎

[1] Specifically, the number of squares read or written on by a Turing machine (Davis' model [5]) in the course of its compution.

THEOREM 3.3. *For any measure Φ there is an $s \in \mathscr{R}$ satisfying, for any i,*

$$\hat{P}^{\Phi}_{\varphi_i} \subseteq P^{\Phi}_{s\circ\Phi_i} \quad \textit{and} \quad \hat{P}^{\Phi}_{\Phi_i} \subseteq P^{\Phi}_{s\circ\Phi_i}.$$

Proof, Let r be the recursive relation between Φ and L, and let R be the tape-complexity of some algorithm for r. The property of Davis' model [5] that $L_i(x) \geqslant \varphi_i(x)$ for all i and x is used to simplify the following argument.

Then the following containments hold for any i,

$$\hat{P}^{\Phi}_{\varphi_i} \subseteq \hat{P}^{L}_{r\circ\varphi_i} \subseteq \hat{P}^{L}_{R\circ L_i} \subseteq P^{L}_{R\circ L_i} \subseteq P^{L}_{R\circ r\circ\Phi_i} \subseteq P^{\Phi}_{r\circ R\circ r\circ\Phi_i}.$$

The first and last of these containments hold using $(*3)$ above, the second using $(*2)$ and the properties of the model mentioned in Corollary 3.2, the fourth using $(*2)$ alone, and the third follows from 3.1, since $R \circ L_i$ may be computed using only that much tape if R is choosen to be increasing. Similarly,

$$\hat{P}^{\Phi}_{\Phi_i} \subseteq P^{\Phi}_{r\circ R\circ r\circ\Phi_i}.$$

Hence, the required function is

$$s = r \circ R \circ r. \quad \blacksquare$$

In light of the previous results, one might hope that a measure could be constructed with sufficiently strong properties so that the two definitions of partial class coincide. The following results show that this is not possible, indicating limits to which conditions can be imposed on measures.

DEFINITION. For any function ψ, let

$$\alpha_\psi(x) = \begin{cases} x & \text{if } \psi(x)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

THEOREM 3.4. *There exist arbitrarily large functions ψ satisfying*

$$\hat{P}^{L}_{\alpha_\psi} \not\subseteq P_\psi{}^{L}.$$

Proof. Let f be any arbitrarily large function, $f = L_j$ for some j. By the usual diagonalization method [2, Theorem 1] construct a 0–1-valued function $g \in \mathscr{R}$ such that

$$g \notin R_f{}^{L}.$$

Define

$$\xi(x) = \begin{cases} 1 & \text{if } g(x) = 1, \\ \uparrow & \text{if } g(x) = 0, \end{cases}$$

and

$$\psi(x) = f(x) \cdot \xi(x).$$

Claim ψ is as required, since $\xi \in \hat{P}^L_{\alpha_\psi}$ but $\xi \notin P_\psi{}^L$. Since $\psi(x) \geqslant x$, $\lambda x[1] \in P_\psi{}^L$; and $\xi \equiv 1$ on Dom(ψ). Hence $\xi \in \hat{P}^L_{\alpha_\psi}$.

Assume $\xi \in P_\psi{}^L$, then there is an $i \in I_\psi{}^L$, $\varphi_i = \xi$. Define (recalling $f = L_j$)

$$\varphi_k(x) = \begin{cases} 0 & \text{if } L_j(x) < L_i(x), \\ 1 & \text{if } L_i(x) \leqslant L_j(x). \end{cases}$$

Then it is easy to see that $\varphi_k = g$,[2] but, using the "parallel computability" property of L[3, 7], k may be taken so that $L_k \leqslant f$ a.e. Hence

$$g = \varphi_k \in R_f{}^L,$$

a contradiction. ▮

Corollary 3.5. *For any Φ, any recursive g, there exists arbitrarily large partial recursive ζ such that*

$$\hat{P}_\zeta{}^\Phi \not\subseteq P^\Phi_{g \circ \zeta}.$$

Proof. Let r be the recursive relation between Φ and L. Pick any recursive f, $f(x) \geqslant x$, to construct $\zeta \geqslant f$ a.e.

By the previous theorem, construct $\psi \geqslant r \circ g \circ r \circ f$ a.e. such that

$$\hat{P}^L_{\alpha_\psi} \not\subseteq P_\psi{}^L.$$

and let $\zeta = r \circ f \circ \alpha_\psi$. Now assume the desired property does not hold, that is

$$\hat{P}_\zeta{}^\Phi \subseteq P^\Phi_{g \circ \zeta}.$$

Then, using (*3), it follows that

$$\hat{P}^L_{\alpha_\psi} \subseteq \hat{P}^L_{f \circ \alpha_\psi} \subseteq \hat{P}_\zeta{}^\Phi \subseteq P^\Phi_{g \circ \zeta} \subseteq P^L_{r \circ g \circ \zeta} \subseteq P_\psi{}^L,$$

contradicting Theorem 3.4. ▮

The use of presentations as notations is as applicable to either definition of partial class as it is to total classes. With one important exception, the results for presentations carry over exactly from total classes to either definition of partial class. The results for classes $P_\tau{}^\Phi$ will be presented next.

[2] Using the above definition, this holds only a.e., but a finite control can be attached to φ_k to make the equality hold without affecting the amount of tape used.

THEOREM 3.6 [7]. *For any measure Φ and all sufficiently large φ_j, $P^{\Phi}_{\varphi_j}$ is recursively presentable.*

That there exist exceptional cases of Φ and t such that R_t^{Φ} is not recursively presentable [7, 8] obviously carries over to P_t^{Φ} (and $\hat{P}_t^{\Phi}$).

THEOREM 3.7 [7]. *For any measure Φ and any φ_k there exists a presentation V of $P^{\Phi}_{\varphi_k}$ such that V^C is recursively enumerable.*

The corresponding results hold for classes $\hat{P}_\tau^{\Phi}$, with proofs that are in many ways similar. In particular, the proof of Theorem 3.9 differs from that of 3.7 essentially by one line.

THEOREM 3.8. *For any measure Φ and sufficiently large φ_k, $\hat{P}^{\Phi}_{\varphi_k}$ is recursively presentable.*

As with Theorem 3.6, this result holds for all classes large enough to contain all a.e. zero functions. Invoking Theorem 3.3, we modify an enumeration of $P^{\Phi}_{s\cdot\Phi_k}$ so that an enumerated function is unchanged by the modification if it meets certain criteria for membership in $\hat{P}^{\Phi}_{\varphi_k}$, but becomes zero a.e. if it fails to meet these criteria. This modified enumeration is, of course, a presentation of $\hat{P}^{\Phi}_{\varphi_k}$.

THEOREM 3.9. *For any measure Φ and φ_k, there is a presentation Y of $\hat{P}^{\Phi}_{\varphi_k}$ such that Y^C is recursively enumerable.*

The following result corresponds to 2.3 for classes $\hat{P}_\tau^{\Phi}$. It is significant that it does not carry over to classes P_τ^{Φ}, as is shown in 4.4 below.

THEOREM 3.10. *For any measure Φ and any $\tau \in \mathscr{P}$; $\mathscr{P} - \hat{P}_\tau^{\Phi}$ is recursively presentable.*

Proof. The following is a sketch of a stage in the operation of a device which enumerates a presentation of $\mathscr{P} - \hat{P}_\tau^{\Phi}$. Say $\tau = \varphi_j$. Assume again $\tau \in \mathscr{P}^\infty$.

Stage n

(1) If $(\forall x \leqslant n)[\Phi_j(x) > n]$, go to stage $n + 1$.

(2) Enumerate functions diverging at some value where φ_j converges. This requires listing the domain of φ_j, which is done in stages corresponding to the stages of the larger device.

(3) Enumerate e_n, the index of an algorithm which is equal to φ_n if indeed $\varphi_n \in \mathscr{P} - \hat{P}^{\Phi}_{\varphi_j}$, and which is almost everywhere undefined otherwise. In particular, $\varphi_{e_n}(x)$ is computed as follows.

(a) Compute $\varphi_n(x)$ (hence $\varphi_{e_n}(x)$ is undefined if $\varphi_n(x)$ is).

(b) For each k, $0 \leqslant k \leqslant x$, check whether the following condition holds.

$$(\exists y \in W_j \cap W_n)$$
$$[\Phi_k(y) > \max(x, \varphi_j(y)) \vee (\Phi_k(y) \leqslant \max(x, \varphi_j(y)) \;\&\; \varphi_k(y) = \varphi_n(y))].$$

Naturally this condition is checked, for each k, by enumerating in some sequence $y \in W_j \cap W_n$ and checking the predicate comprising the rest of the condition. For a fixed $y \in W_j \cap W_n$, this predicate is clearly decidable. If no such y exists for a given k, however, then the checking does not terminate and $\varphi_{e_n}(x)$ is undefined.

(c) If the condition of (b) is successfully checked for each k, output $\varphi_n(x)$.

The reader may show that, except for possibly a finite initial set on which φ_n and φ_{e_n} agree,

$$\varphi_{e_n}(x) = \begin{cases} \varphi_n(x) & \text{if there is no } k \in I^{\Phi}_{\varphi_j} \text{ such that } \varphi_k = \varphi_n \text{ on } W_j, \\ \uparrow & \text{otherwise.} \end{cases}$$ ▮

4. Decision Problems for Partial Classes

An interesting approach to the study of total complexity classes was that taken by F. Lewis [8], who investigated the structure of an individual complexity class via a classical tool for studying complexity of another sort, the Kleene hierarchy. The following results show how the results for total classes carry over, or fail to carry over, to partial classes.

Theorem 4.1. *For any measure Φ and φ_k*

$$\Omega \hat{P}^{\Phi}_{\varphi_k} \textit{ is a } \Pi_3 \cap \Sigma_3 \textit{ set.}$$

Proof. The predicate

$$(\forall x)[\varphi_k(x)\downarrow \Rightarrow \varphi_i(x)\downarrow] \quad \& \quad (\exists j)[j \in I^{\Phi}_{\varphi_k} \;\&\; (\forall x)[\varphi_k(x)\downarrow \Rightarrow \varphi_i(x) = \varphi_j(x)]]$$

clearly expresses "$\varphi_i \in \hat{P}^{\Phi}_{\varphi_k}$." Noting that "$\varphi_i(x)\downarrow$" is Σ_1 and "$\varphi_i(x) = \varphi_j(x)$" may be rewritten as "$\varphi_i(x)\downarrow$ and $\varphi_j(x)\downarrow \Rightarrow \varphi_i(x) = \varphi_j(x)$",[3] the predicate is clearly a conjunction of Π_2 and Σ_2, and hence $\Sigma_3 \cap \Pi_3$. ▮

Under certain conditions a similar extension may be made for classes P_τ^{Φ}.

[3] This is sufficient, since if i and j do indeed satisfy the rest of the expression, then $\varphi_i(x)\downarrow$ and $\varphi_j(x)\downarrow$.

THEOREM 4.2. *For any measure Φ and $\tau \in \mathscr{P}^\infty$. If* (1) $P_\tau^\Phi \neq \varnothing$ *and* (2)$\mathscr{P} - P_\tau^\Phi$ *has a $\Pi_3 \cap \Sigma_3$ presentation, then $\Omega P_\tau^\Phi \equiv_T$ Equal.*

Proof. Let $\tau \in \mathscr{P}^\infty$ satisfying conditions 1 and 2. First show $\Omega P_\tau^\Phi \leqslant_T$ *Equal.*

We describe a machine with an "oracle" for *Equal* which performs two processes in parallel, the first of which will halt if $\varphi_i \notin P_\tau^\Phi$ (but which by itself deos not converge if $\varphi_i \notin P_\tau^\Phi$), the second determines $\varphi_i \in P_\tau^\Phi$.

To determine if $\varphi_i \notin P_\tau^\Phi$, enumerate e_0 , e_1 ,..., a presentation of $\mathscr{P} - P_\tau^\Phi$ and ask the oracle if $\langle i, e_0\rangle \in$ *Equal*, $\langle i, e_1\rangle \in$ *Equal*,... . By assumption such an enumeration is possible, since any $\Pi_3 \cap \Sigma_3$ set can be numerated with *Equal* as an oracle.

$\varphi_i \in P_\tau^\Phi$ is determined in a similar manner, with the enumeration always possible since P_τ^Φ is Π_1-presentable (3.7).

Now to show *Equal* $\leqslant_T \Omega P_\tau^\Phi$. It was assumed that ΩP_τ^Φ is nonempty, so say $\rho \in P_\tau^\Phi$. Define f so that

$$\varphi_{f(i,j)}(x) = \begin{cases} \rho(x) & \text{if } (\forall y \leqslant x)[(\Phi_i(y) \leqslant x \vee \Phi_j(y) \leqslant x) \\ & \quad \Rightarrow (\varphi_i(y)\downarrow \,\&\, \varphi_j(y)\downarrow \,\&\, \varphi_i(y) = \varphi_j(y))], \\ \uparrow & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\langle i, j\rangle \in \textit{Equal} \Leftrightarrow \varphi_{f(i,j)} = \rho \Leftrightarrow f(i,j) \in \Omega P_\tau^\Phi.$$

Thus, if f is made $1 - 1$, *Equal* $\leqslant_1 \Omega P_\tau^\Phi$. ▌

THEOREM 4.3. *There exists a measure Φ^* such that, for arbitrarily large $\tau \in \mathscr{P}^\infty$,*

$$\textit{Cofinite} \leqslant_1 \Omega P_\tau^{\Phi^*}.$$

Proof. Let Φ be any measure, with γ an effective procedure as described in the "honesty theorem" [9] such that, for all i, j, x, y

(1) "$\Phi_{\gamma(i)}(x) \leqslant y$" is decidable,

(2) $\Phi_j \leqslant \varphi_i$ a.e. iff $\Phi_j \leqslant \varphi_{\gamma(i)}$ a.e.

The following construction has the desired effect only for those $\varphi_{\gamma(i)}$ which converge on an infinite set of even x. A complete construction would require to define a set of indices $\{o_k^i\}$ analogous to the set $\{e_k^i\}$ define below, and modifying the defined measure on these indices as well.

In the following division is naturally integer division, where any remainder is truncated.

For each i, let e^i be the index of a function computed, for input x, in the following manner (after [2, Theorem 7]).

If x is odd, output 0.

If x is even, compute $\varphi_{\gamma(i)}(x)$ until (and if) it halts. Let k be the least integer $k \leqslant x$ such that $\Phi_k(x) \leqslant \varphi_{\gamma(i)}(x)$, but for no even $y < x$ is it the case that

$$\Phi_{\gamma(i)}(y) \leqslant x \quad \text{and} \quad \Phi_k(y) \leqslant \varphi_{\gamma(i)}(x) \quad \text{and} \quad k \text{ is checked off at } y \text{ by stage } x.$$

This k is now said to be "checked off at x." If $\varphi_k(x) = 0$, output 1. If $\varphi_k(x) \neq 0$ or no such k exists, output 0.

For each $\varphi_{\gamma(i)}$, φ_{e^i} is a 0–1-valued function with a domain of $\mathrm{Dom}(\varphi_{\gamma(i)}) \cup \{\text{odds}\}$ satisfying

$$(\forall j)[(\forall x)[\varphi_j(2x) = \varphi_{e^i}(2x)] \Rightarrow (\overset{\infty}{\forall} x)[\Phi_j(2x) > \varphi_{\gamma(i)}(2x) \vee \varphi_{\gamma(i)}(2x)\uparrow]].$$

Now define e_k^i such that

$$\varphi_{e_k^i}(x) = \begin{cases} \varphi_{e^i}(x) & \text{if } x \text{ even,} \\ 2^{i+1} & \text{if } x \text{ odd and } x/2 \notin D_k\,, \\ \uparrow & \text{if } x \text{ odd and } x/2 \in D_k\,, \end{cases}$$

where D_k is the kth set in an effective listing of all finite sets. The use of 2^{i+1} assures that modifications associated with one i do not interfere with other modifications. Finally, $\{e_k^i \mid i, k \in N\}$ should be recursive.

Define a new measure by

$$\Phi_n^*(x) = \begin{cases} \Phi_n(x) & \text{if } n \notin \{e_k^i\}, \\ \varphi_{\gamma(i)}(x) & \text{if } n = e_k^i \text{ and } x \text{ even,} \\ 0 & \text{if } x/2 \notin D_k \text{ and } n = e_k^i \text{ and } x \text{ odd,} \\ \uparrow & \text{if } x/2 \in D_k \text{ and } n = e_k^i \text{ and } x \text{ odd.} \end{cases}$$

Φ^* is a measure since membership in D_k and in $\{e_k^i\}$ is decidable, and since $\varphi_{\gamma(i)}$ is honest.

Now pick some arbitrarily large $\varphi_{\gamma(i)} \in \mathscr{P}^\infty$. Assume $\varphi_{\gamma(i)}$ converges i.o. on even input and define

$$\tau(x) = \begin{cases} \varphi_{\gamma(i)}(x) & \text{if } x \text{ even,} \\ \uparrow & \text{if } x \text{ odd.} \end{cases}$$

Then define h so that

$$\mathit{Cofinite} \leqslant_1 \Omega P_\tau^{\Phi^*}$$

via h. This is accomplished by h (padded to be 1–1) satisfying

$$\varphi_{h(j)}(x) = \begin{cases} \varphi_{e^i} & \text{if } x \text{ even,} \\ 2^{i+1} & \text{if } x \text{ odd and } \varphi_j([x/2])\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

Observe that, if φ_n agrees with φ_{e^i} on all even inputs then, by construction, $\varphi_n \in P_\tau^{\Phi*}$ iff $\varphi_n = \varphi_{e_k^i}$ for some k. But $\varphi_{h(j)} = \varphi_{e_k^i}$ iff $W_j = N - D_k$. Hence

$$h(j) \in \Omega P_\tau^{\Phi*} \Leftrightarrow j \in \textit{Cofinite}. \quad \blacksquare$$

COROLLARY 4.4. *For Φ and τ as in Theorem 4.3, $\mathscr{P} - P_\tau{}^\Phi$ is not recursively presentable.*

Proof. By 4.2, if $\mathscr{P} - P_\tau{}^\Phi$ were recursively presentable, then, using 4.3 also

$$\textit{Cofinite} \leqslant_1 \Omega P_\tau{}^\Phi \leqslant_T \textit{Equal},$$

which is a contradiction to known fact. $\blacksquare$

The above Theorem 4.3 states that $\Omega P_\tau{}^\Phi$ may be Σ-$_3$complete for some measures and functions. It is an open question whether such functions exist for all measures. It is easy to see, however, that this is worst case.

PROPOSITION 4.5. *For any measure Φ and $\varphi_k \in \mathscr{P}$,*

$$\Omega P_{\varphi_k}^\Phi \textit{ is } \Sigma_3 .$$

Proof. The reader may easily see that the following predicate is Σ_3 and expresses "$\varphi_i \in P_{\varphi_k}^\Phi$":

$$(\exists j)[j \in I_{\varphi_k}^\Phi \text{ and } \varphi_i = \varphi_j]. \quad \blacksquare$$

A stronger lower bound on the hierarchy classification of partial class is the following suggested by and including Lewis' [8] result for total classes.

THEOREM 4.6. *If $\tau \in \mathscr{P}^\infty$ and $P_\tau{}^\Phi$ is finitely invariant,*[4] *then Bound $\leqslant_1 \Omega P_\tau{}^\Phi$.*

Proof. Let W_j be an infinite recursive subset of Dom(τ), which may be found from τ by a standard construction. By Theorem 3.7 there exists $f \in \mathscr{R}$ which presents $P_\tau{}^\Phi$. Now define g such that

$$\varphi_{g(i)}(x) = \begin{cases} \varphi_{f(k)}(x) & \text{if } x \notin W_j \text{ or } \varphi_i(y) \in \{\varphi_i(0),..., \varphi_i(y-1)\}, \\ \varphi_{f(k)}(x) + 1 & \text{otherwise,} \end{cases}$$

where $y = | W_j \cap \{0,..., x-1\} |$ and $k = | \varphi_i(0),..., \varphi_i(y-1)\}|$, which is obviously undefined if φ_i diverges at some input less than y, causing $\varphi_{g(i)}(x)$ to diverge.

[4] A class of functions $\mathscr{C} \subseteq \mathscr{P}$ is finitely invariant if $\psi \in \mathscr{C}$ and $\xi = \psi$ a.e. and Dom(ξ) = Dom(ψ) together imply $\xi \in \mathscr{C}$.

Now claim: *Bound* $\leqslant_1 \Omega P_\tau^\Phi$ via g, which follows from

$$\varphi_i \in \textit{Bound} \Leftrightarrow \varphi_i \text{ total and } (\exists z)(\forall y)[y > z \Rightarrow \varphi_i(y) \in \{\varphi_i(0),..., \varphi_i(z)\}]$$
$$\Leftrightarrow (\exists k)[\varphi_{g(i)} = \varphi_{f(k)} \text{ a.e.}] \Leftrightarrow \varphi_{g(i)} \in P_\tau^\Phi. \quad \blacksquare$$

Finally, we show a result concerning decision problems not for functions but for algorithms.

THEOREM 4.7. *For any measure Φ there is a $g \in \mathscr{R}$ such that, for all $\tau \in \mathscr{P}^\infty$,*

$$\tau \geqslant g \text{ a.e.} \Rightarrow \textit{Finite} \leqslant_1 \Omega I_\tau^\Phi.$$

Proof. Unlike b^Φ (of Theorem 2.1), it is not possible to specify g by some simple criteria. Instead, g must majorize (in certain cases where convergence is guaranteed) functions which reduce *Finite* to I_τ^Φ for all possible τ.

First, introduce the notation

$$Q(i, k, x) \equiv (\exists y, w \leqslant x)[\Phi_i(y) = w \text{ and } (\forall z)[\max(w, y) < z < x \Rightarrow \Phi_k(z) > x]].$$

This predicate says "in a limited search, φ_i has been found to converge at some point after φ_k."

Assume $\varphi_k \in \mathscr{P}^\infty$ and examine the implications of this for Q. In particular, say $\Phi_i(y_0) = w_0$. By assumption φ_k converges above $\max(y_0, w_0)$, say z_0 is the least such value for which this occurs. Then $Q(i, k, z_0)$ holds. On the other hand, for $x > \max(z_0, \Phi_k(z_0))$, $Q(i, k, x)$ can hold only if $\varphi_i(y)\downarrow$ for some $y > y_0$ and not because $\varphi_i(y_0)\downarrow$. Hence,

(i) $\varphi_k \in \mathscr{P}^\infty$ and $\varphi_i \in \mathscr{P}^\infty \Rightarrow (\exists^\infty x)[\varphi_k(x)\downarrow \,\&\, Q(i, k, x)]$,

(ii) $\varphi_k \in \mathscr{P}^\infty$ and $\varphi_i \notin \mathscr{P}^\infty \Rightarrow (\exists x_0)(\forall x > x_0)[\neg Q(i, k, x)]$.

Now define $f \in \mathscr{R}$ so that $\varphi_{f(e)}$ enumerates (increasing) indices of functions such that

$$\varphi_{\varphi_{f(e)}(i,k)}(x) = \begin{cases} \uparrow & \text{if } Q(i, k, x) \text{ and } \varphi_{\varphi_e(i,k)}(x) \leqslant \max(\varphi_k(x), \Phi_k(x)), \\ 0 & \text{otherwise.} \end{cases}$$

Pick e_0 a fixed-point of f. Since φ_{e_0} enumerates indices, it is evidently total and, letting $h = \varphi_{e_0}$, the above may be rewritten as

$$\varphi_{h(i,k)}(x) = \begin{cases} \uparrow & \text{if } Q(i, k, x) \text{ and } \Phi_{h(i,k)}(x) \leqslant \max(\Phi_k(x), \varphi_k(x)), \\ 0 & \text{otherwise.} \end{cases}$$

The function $\lambda i[h(i, k)]$ will ultimately be shown to provide the reduction for appropriate $\varphi_k = \tau$.

Observe that

(iii) $\neg Q(i, k, x) \Rightarrow \varphi_{h(i,k)}(x)\downarrow$,

(iv) $Q(i, k, x)$ & $\varphi_k(x)\downarrow \Rightarrow \varphi_{h(i,k)}(x)\downarrow$ & $\Phi_{h(i,k)}(x) > \varphi_k(x)$.

Condition (iii) is immediate and (iv) must hold to avoid the contradiction that $\Phi_{h(i,k)}(x)$ is both bounded and undefined. Together (i) and (iii) imply

(v) $\varphi_k \in \mathscr{P}^\infty$ & $\varphi_i \in \mathscr{P}^\infty \Rightarrow (\exists^\infty x)[\varphi_k(x)\downarrow$ & $\Phi_{h(i,k)}(x) > \varphi_k(x)]$,

(vi) $\neg Q(i, k, x) \vee \varphi_k(x)\downarrow \Rightarrow \varphi_{h(i,k)}(x)\downarrow$.

From the latter if follows that

(vii) $\varphi_k \in \mathscr{P}^\infty \Rightarrow (\forall i)[W_{h(i,k)} \supset W_k]$.

Finally, define

$$g(x) = \max(0, \{\Phi_{h(i,k)}(x) \mid i, k \leqslant x \ \&\ \neg Q(i, k, x)\}).$$

By (iii), g is total. From (ii) and the definition of g,

(viii) $\varphi_k \in \mathscr{P}^\infty$ & $\varphi_i \notin \mathscr{P}^\infty \Rightarrow \Phi_{h(i,k)} \leqslant g$ a.e.

If $\tau = \varphi_k \in \mathscr{P}^\infty$ and $\tau \geqslant g$ a.e., (v) and (viii) imply

$$\varphi_{h(i,k)} \in I_\tau^\Phi \Leftrightarrow W_i \text{ finite},$$

which is exactly what is required for $\lambda i[h(i, k)]$ to reduce

$$\mathit{Finite} \leqslant_1 \Omega I_\tau^\Phi. \quad \blacksquare$$

As was noted above, I_τ^Φ is Σ_2 for any $\tau \in \mathscr{P}^\infty$, and thus under the above conditions it actually holds that

$$\mathit{Finite} \equiv_1 \Omega I_\tau^\Phi,$$

that is that I_τ^Φ is Σ_2-complete.

Although an original intent of this investigation was to suggest *a* definition for partial complexity class, there seems to be no absolute justification for choosing one of P_τ^Φ or $\hat{P}_\tau^\Phi$ over the other. Further study in this area is obviously desirable. It would be especially interesting to discover whether or not Theorem 4.3 can be generalized to all measures.

5. Infinite Intersections of Total Complexity Classes

In this section we return to complexity classes bounded by total functions to answer negatively two important question. McCreight and Meyer [9] showed that the family of complexity classes defined by total recursive functions was closed under the infinite union of "upward nested" sequences of complexity classes. It was then natural to ask whether the same result held for "downward nested" sequences under infinite intersections. This question was originally suggested to the author by L. Bass (personal communication).

DEFINITION. A sequence of functions $\{f_i\}$ is an *r.e. sequence of total functions* if $\lambda i, x[f_i(x)]$ is recursive. Such a sequence is said to be *increasing* (*decreasing*) if $f_i(x) \leqslant f_{i+1}(x)$ ($f_i(x) \geqslant f_{i+1}(x)$) for all $i, x \in N$.

THEOREM 5.1. (Union Theorem) [9]. *For any measure Φ and any r.e. increasing sequence of total functions $\{f_i\}$, there is a $g \in \mathscr{R}$ such that*

$$\bigcup_{i \in N} I_{f_i}^{\Phi} = I_g^{\Phi}.$$

This result extends immediately to classes R_f^{Φ} and to weaker conditions on the sequences $\{f_i\}$. On the other hand,

THEOREM 5.2.[5] *For $L =$ the tape measure, there is an r.e. decreasing sequence of functions $\{g_i\}$ such that, for no $h \in \mathscr{R}$ is it true that*

$$\bigcap_{i \in N} I_{g_i}^{L} = I_h^{L}.$$

Proof. Let g be a function such that $g(x) > x$ and which is computable in $g(x)$ squares. Define a recursive set of indices $\{e_j\}$ such that

$$\varphi_{e_j} \equiv 0$$

and the computation of $\varphi_{e_j}(x)$ operates as follows.

(1) Simulate the computation of $\varphi_j(0), \varphi_j(1), \ldots,$ on x squares to find the least z such that

$$\sum_{y=0}^{z} (L_j(y) + 1) > x.$$

(2) Calculate $g(x)$ (by the method using $g(x)$ squares of course), and move so that exactly $g(x) + (x \dot{-} z)$ squares are used by the entire computation, then halt with output 0.

Obviously φ_{e_j} is total and is the identically zero function. Now consider the relationship between the computations of φ_{e_j} and φ_j. In particular, observe that, if $\varphi_j(z)\uparrow$

[5] A similar result has been independently discovered by M. S. Paterson.

and this is the least z for which φ_j diverges, then for all x, $\sum_{y=0}^{z}(L_j(y)+1) > x$, and z is the least value for which this is true for arbitrarily large x. Thus $L_{e_j}(x) = g(x) + x \dot{-} z$ for almost all x. In general,

$$\varphi_j(z)\uparrow \Rightarrow L_{e_j}(x) \geqslant g(x) + x \dot{-} z \text{ a.e.}$$

On the other hand, if φ_j is total, then for every z there is an x_0 such that

$$x_0 \geqslant \sum_{y=0}^{z} (L_j(y) + 1).$$

Thus

$$x \geqslant x_0 \Rightarrow L_{e_j}(x) \leqslant g(x) + x \dot{-} z.$$

Now define $g_i(x) = g(x) + x \dot{-} i$ and it follows immediately from the above that

$$\varphi_j \text{ total} \Leftrightarrow (\forall i)[L_{e_j} \leqslant g_i \text{ a.e.}] \Leftrightarrow \varphi_{e_j} \in \bigcap_{i \in N} I_{g_i}^L .$$

Now assume the existance of $h \in \mathscr{R}$ such that

$$I_h{}^L = \bigcap_{i \in N} I_{g_i}^L$$

Then $j \in \mathit{Total} \Leftrightarrow e_j \in I_h{}^L$. Since we may easily make $\lambda j[e_j]$ a $1-1$ function, this implies

$$\mathit{Total} \leqslant_1 I_h{}^L.$$

But this is a contradiction, since Total and $I_h{}^L$ are, respectively, Π_2- and Σ_2-complete [7, 8]. ∎

This extends either by a direct proof or from Theorem 5.2 using the recursive relation between measures,[6] to

THEOREM 5.3. *For any measure Φ, and any $t \in \mathscr{R}$, there is an r.e. decreasing sequence functions $\{g_i\}$ such that, for all i,*

$$g_i \geqslant t \quad \text{a.e.,}$$

and such that there is no $h \in \mathscr{R}$ satisfying

$$I_h{}^\Phi = \bigcap_{i \in N} I_{g_i}^\Phi .$$

[6] The author is indebted to A. Borodin for observing this method of proof was applicable.

Proof. Let $\varphi_k = t$ and let φ_j be the recursive relation between L and Φ, define

$$T(x) = \max(\varphi_k(x), \Phi_k(x), x)$$

and

$$R(x) = \max(\varphi_j(x), \Phi_j(x)).$$

Using R^n to denote the n-fold composition of R, define

$$g_i(x) = R^{(x \dot{-} i)} \circ T(x).$$

These functions g_i may be seen to be as required, arguing largely as before in the measure L, but shifting to Φ for the final steps.

First observe that $\lambda x, z[g_z(x)]$ is computable by a Turing machine using exactly that amount of tape. Hence we may redefine (2) in the computation of φ_{e_i}:

(2′) use exactly $g_z(x)$ squares and output zero.

Then it follows as before that, if z is the least number such that $\varphi_j(z)\uparrow$,

$$L_{e_j} = \lambda x[g_z(x)] \text{ a.e.}$$

$$\varphi_j \text{ total} \Rightarrow (\forall i)[L_{e_j} \leqslant g_i \text{ a.e.}].$$

Using notation freely, we may carry the above facts over to the measure Φ obtaining: if z is the least number such that $\varphi_j(z)\uparrow$ (using $R^{-1}g_z$ for $\lambda x[R^{x \dot{-} z \dot{-} 1} \circ T(x)]$),

$$e_j \in I^{\Phi}_{R \circ g_z} - I^{\Phi}_{R^{-1} \circ g_z}$$

and

$$\varphi_j \text{ total} \Leftrightarrow (\forall i)[\Phi_{e_j} \leqslant g_i \text{ a.e.}].$$

The existence of an h yields the same contradiction as before. ▮

Unlike the case with the "union theorem," there can be no general implications concerning infinite intersections from $I_t{}^{\Phi}$ to $R_t{}^{\Phi}$ or vice versa. Hence the following theorem must be proved independently of the previous theorem. This important result is due to L. Bass [1].

THEOREM 5.4. *For any measure Φ there is an r.e. decreasing sequence of total functions $\{q_i\}$ such that, for no $t \in \mathscr{R}$,*

$$\bigcap_{i \in N} R^{\Phi}_{q_i} = R_t{}^{\Phi}.$$

Although Theorems 5.2, 5.3, and 5.4 prohibit the existence of recursive functions with certain properties, functions higher in the Kleene heirarchy always exist with these properties [13].

References

1. Leonard J. Bass, A note on the intersection of complexity classes of functions, *SIAM J. Comput.* 1 (1972), 288–289.
2. Manuel Blum, Machine independent theory of the complexity of recursive functions, *J. Assoc. Comput. Mach.* **14** (1967), 322–336.
3. Allan B. Borodin, Computational complexity and the existance of complexity gaps, *in* "Proc. ACM Symp. on Theory of Computing," (May 1969), pp. 67–78.
4. Robert L. Constable, Extending and refining heirarchies of computable functions, Computer Sciences Tech. Rept. #25, Univ. of Wisconsin, Madison, WI, 1968.
5. Martin Davis, "Computability and Unsolvability," McGraw-Hill, New York, 1958.
6. J. C. E. Dekker and J. Myhill, Some theorems on classes of recursively enumerable sets, *Trans. Amer. Math. Soc.* **89** (1958), 25–59.
7. L. H. Landweber and E. L. Robertson, Recursive properties of abstract complexity classes, *J. Assoc. Comput. Mach.* **19** (1972), 296–308.
8. Forbes D. Lewis, Unsolvability considerations in computational complexity, *in* "Proc. 2nd ACM Symp. on Theory of Computing," May 1970, pp. 22–30.
9. E. M. McCreight and A. R. Meyer, Classes of computable functions defined by bounds on computation, *in* "Proc. ACM Symp. on Theory of Computing," May 1969, pp. 79–88.
10. V. L. Murskij, Nondiscernible properties of finite systems of identity relations, *Soviet Math. Dokl.* **12** (1971), 183–186.
11. H. G. Rice, Classes of recursively enumerable sets and their decision problems, *Trans. Amer. Math. Soc.* **74** (1953), 358–366.
12. H. G. Rice, On completely recursively enumerable classes and their key arrays, *J. Symbolic Logic* **21** (1956), 304–308.
13. Edward Robertson, Properties of Complexity Classes and Sets in Abstract Computational Complexity, Thesis, Aug. 1970, Univ. of Wisconsin, Madison, WI.
14. Hartley Rogers, "Theory of Recursive Functions and Effective Computability," McGraw-Hill, New York, 1970.
15. W. Szmielew, Elementary properties of Abelian groups, *Fund. Math.* **41** (1955), 203–271.
16. J. S. Ullian, Partial algorithm problems for context free languages, *Information and Control* **11** (1967), 80–101.