On the Rainbow and Strong Rainbow Connection Numbers of the \textit{m}-Splitting of the Complete Graph $K_n$

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Abstract

An edge-coloring of a graph is called rainbow if any two vertices are connected by a path consisting of edges of different colors. The least number of colors in such a coloring is called the rainbow connection number of $G$, denoted by $rc(G)$. An edge-coloring of a graph is called strong rainbow if any two vertices are connected by a geodesic consisting of edges of different colors. The least number of colors in such a coloring is called the strong rainbow connection number of $G$, denoted by $src(G)$. In this paper we study the $rc$ and $src$ of the $m$-splitting of a graph. In particular we study $\text{Spl}_m(K_n)$. We present the exact values of its $rc$ and $src$ in several cases, and we prove several bounds in the other cases.

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Peer-review under responsibility of the Organizing Committee of ICGTIS 2015

Keywords: Edge coloring, rainbow connection, $m$-splitting, complete graph.

2010 MSC: 05C15, 05C40

1. Introduction

Connectedness is an important property in graph theory. The graphs we consider in this paper are finite, simple, and undirected. A graph $G$ is called connected if any two of its vertices are linked by a path in $G$. One way to modify this concept is by adding the notion of a graph coloring or labelling. A graph labelling is an assignment of integers to the vertices or edges of a graph, or to both, subject to certain conditions\cite{5}. A variant of edge-labelling that is related to connectedness was studied by Chartrand et. al.\cite{4}. We call any map from the edge-set $E(G)$ of a non-trivial connected graph $G$ into $\{1, \ldots, k\}$ as an edge-coloring with $k$ colors, or simply a $k$-coloring. Given such a $k$-coloring on $G$, a rainbow path is a path in $G$ whose edges all have different colors. The coloring is called a rainbow coloring if any two vertices in $G$ are linked by a rainbow path in $G$. There is an obvious way to rainbow-color any graph, by giving a different color to each edge. We call this the trivial coloring. There may be more efficient colorings. For example, the Petersen graph with 15 edges may be rainbow-colored using only three colors, but not less\cite{4}. The smallest $k$ for which we have a rainbow $k$-coloring on $G$ is called the rainbow connection number of $G$, and is denoted by $rc(G)$.

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The concept can be strengthened further by adding the notion of shortest paths, formally known as geodesics. A path in \( G \) with endpoints \( x, y \) and length \( l \) is called a geodesic in \( G \) if the distance of \( x \) and \( y \) in \( G \) is \( l \), or equivalently, if any path linking \( x \) and \( y \) in \( G \) has length at least \( l \). A rainbow coloring on \( G \) is called strong if any two vertices of \( G \) are linked by a rainbow geodesic in \( G \). For example, the trivial coloring is actually strong rainbow. There may be more efficient ways to strongly-rainbow-color a graph. For example, the Petersen graph has a strong rainbow coloring with four colors, but not less\[^4\]. The smallest \( k \) for which we have a strong rainbow \( k \)-coloring on \( G \) is called the strong rainbow connection number of \( G \), and is denoted by \( \text{src}(G) \).

Studying the \( rc \) and \( src \) of an arbitrary graph is apparently a difficult task. It has been proved\[^2\] that deciding \( rc(G) = 2 \) is NP-complete, and computing the exact values of rc is NP-hard. In fact, deciding \( rc(G) \leq k \) (or \( src(G) \leq k \)) for any \( k \geq 3 \) is already NP-hard\[^1\]. Deciding whether a given coloring is rainbow is NP-complete\[^2\].

The goal of this work is to study the \( rc \) and \( src \) of a particular class of graph arising from a construction known as \( m \)-splitting. Intuitively, this construction clones each vertex into \( m \) new vertices such that each clone of a vertex \( x \) is adjacent to all original neighbors of \( x \). The resulting new graph is denoted by \( \text{Spl}_m(H) \). The particular case \( m = 1 \) has been studied in many contexts of graph labelings\[^5\]. In this paper we study the \( rc \) and \( src \) of \( \text{Spl}_m(K_n) \). It may be worth noting that \( \text{Spl}_m(K_n) \) is a split graph, a graph whose vertex-set can be partitioned into an independent set and a clique. A nearly optimal algorithm has been announced\[^3\] that gives a rainbow coloring of any split graph \( G \) using at most \( rc(G) + 1 \) colors in linear time. However, it is unlikely that any polynomial time optimal algorithm can be found, because deciding \( rc(G) \leq 3 \) is already NP-hard even when \( G \) is restricted to be in the class of split graphs\[^3\].

In this work, we present the exact values of \( rc \) and \( src \) of \( \text{Spl}_m(K_n) \) for several families of \( m \) and \( n \), and we also prove several bounds. We will use the following description and notation for the \( m \)-splitting of a graph \( H \). The vertices of \( H \) are denoted by \( h_1, \ldots, h_n \) and they are called inner vertices. The \( m \) clones of \( h_i \) in \( \text{Spl}_m(H) \) are denoted by \( v_{i1}, v_{i2}, \ldots, v_{im} \) and they are called outer vertices. The lower indices are to be considered modulo \( n \). Thus, the edges of \( \text{Spl}_m(K_n) \) are \( h_i h_j \) and \( v_{ij} v_{lj} \) for all \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \) and \( 1 \leq k \leq m \).

The rest of the paper is organized as follows. Several preliminary facts and bounds are presented in Section 2. In Section 3, we state and prove our main results. Section 4 concludes this paper.

### 2. Preliminary results

In this section we collect several facts that will be used in the proofs of our main results.

#### 2.1. Basic bound

The following chain of inequalities\[^4\] gives a relationship between the diameter, \( rc \), \( src \) and size of a graph.

\[
\text{diam}(G) \leq \text{rc}(G) \leq \text{src}(G) \leq |E(G)|.
\] (1)

The left inequality is true because any rainbow path between two vertices of distance \( \text{diam}(G) \) must use at least \( \text{diam}(G) \) different colors. The middle inequality is true because any strong rainbow coloring is a rainbow coloring. The right inequality is true because the trivial coloring is strong rainbow.

#### 2.2. Complete graphs, trees, and complete multipartite graphs

We need the following results. The only graphs having \( rc \) or \( src \) equal to 1 are the complete graphs\[^4\]. The only graphs having \( rc \) or \( src \) equal to the number of edges are trees\[^4\]. Let \( K_{n_1, \ldots, n_k} \) be the complete \( k \)-multipartite graph, whose vertex-set is partitioned into \( k \geq 2 \) disjoint subsets of cardinalities \( n_1 \leq \ldots \leq n_k \) such that there is an edge between two vertices if and only if they belong to different subsets. Let \( s := n_1 + \ldots + n_{k-1} \). If \( s \leq n_k \), then\[^4\]

\[
\text{src}(K_{n_1, \ldots, n_k}) = \left\lceil \sqrt[n_k]{s} \right\rceil.
\] (2)
2.3. Equivalence of rc and src when either has value 2

We have the following equivalence \([4]\).

\[
\text{rc}(G) = 2 \iff \text{src}(G) = 2. \tag{3}
\]

This is true because any path of length 2 between non-adjacent vertices must be a geodesic.

2.4. General upper bound

Here we prove an upper bound for the rc of an \(m\)-splitting in terms of the rc of the original graph.

**Theorem 1.** For any connected graph \(H\) with minimum degree at least 2, and any \(m \in \mathbb{N}\), it holds that

\[
\text{rc}(\text{Spl}_m(H)) \leq \text{rc}(H) + 2. \tag{4}
\]

**Proof.** Let \(c : E(H) \rightarrow \{1, \ldots, k\}\) be a rainbow coloring, where \(k = \text{rc}(H)\). For each inner vertex \(h_i\), choose two of its neighbors \(h'_i, h''_i\). Define \(d : E(\text{Spl}_m(H)) \rightarrow \{1, \ldots, k + 2\}\) as follows.

- On \(E(H)\), \(d\) coincides with \(c\).
- For any outer vertex \(v_j\), we define \(d(v'_i h'_i) = k + 1\) and \(d(v'_i h''_i) = k + 2\).
- On the other edges, define \(d\) to have value 1 (or any other value, this part is not really relevant).

Now we prove that \(d\) is rainbow. Take any two non-adjacent vertices \(x\) and \(y\) in \(\text{Spl}_m(H)\). If they are both inner, then we already have a rainbow path from \(x\) to \(y\) in \(H\). If \(x\) is outer and \(y\) is inner, say \(x = v'_i\), then since \(x, y\) are not adjacent it must be the case that \(y \neq h'_i\). Find a rainbow path in \(H\) from \(h'_i\) to \(y\). Precomposing this path with the edge \(v'_i h'_i\) gives a rainbow path in \(\text{Spl}_m(H)\) from \(x\) to \(y\).

Next suppose that \(x, y\) are both outer, say \(x = v'_i\) and \(y = v''_a\). If \(h'_i = h''_a\), then the path \(v'_i - h'_i - v''_a\) has color sequence \(k + 1, k + 2\) and so it is a rainbow path. If \(h'_i \neq h''_a\), then precomposing a rainbow path in \(H\) from \(h'_i\) to \(h''_a\) with the edge \(v'_i h'_i\) and postcomposing it with the edge \(h''_a v''_a\) gives a rainbow path in \(\text{Spl}_m(H)\) from \(x\) to \(y\). The proof is complete. \(\square\)

The bound above is most efficient when \(H\) is complete. In fact, if \(n \geq 3\) then \(\delta(K_n) \geq 2\), so by the Theorem \(\text{rc}(\text{Spl}_m(K_n)) \leq 3\). Therefore, in view of (3), we have the following equivalences for \(n \geq 3\).

- \(\text{rc}(\text{Spl}_m(K_n)) = 2 \iff \text{src}(\text{Spl}_m(K_n)) \leq 2\).
- \(\text{rc}(\text{Spl}_m(K_n)) = 3 \iff \text{src}(\text{Spl}_m(K_n)) \geq 3\).

Note that \(\text{Spl}_m(K_1)\) consists of isolated vertices, and \(\text{Spl}_m(K_2)\) is a tree. So indeed the non-trivial cases are when \(n \geq 3\).

2.5. Specific upper bound

Here we prove an upper bound for the src of \(\text{Spl}_m(K_n)\) that depends only on \(m\).

**Theorem 2.** For any \(n \geq 4\) and any \(m\), it holds that

\[
\text{src}(\text{Spl}_m(K_n)) \leq \left\lceil \sqrt{m + \frac{1}{4} + \frac{1}{2}} \right\rceil. \tag{5}
\]

**Proof.** Let \(U\) denote the right hand side. It is not hard to see that \(m \leq U(U - 1)\). So we can choose \(m\) different ordered pairs \((a_1, b_1), \ldots, (a_m, b_m)\) such that \(a_j \neq b_j\) and \(a_j, b_j \in \{1, \ldots, U\}\) for each \(j \leq m\). Define a \(U\)-coloring \(c\) on \(\text{Spl}_m(K_n)\) as follows.

- On \(E(K_n)\) we define the value of \(c\) to be 1.
- For any \(i\), and for \(r = 1\) or for any even \(r \geq 2\), we define \(c(v'_i h_{i+r}) = a_j\).
For any $i$, and for any odd $r \geq 3$, we define $c(v^j_i h_{i+r}) = b_j$.

We show that $c$ is strong rainbow. Let $x, y \in V(Spl_m(K_n))$ be non-adjacent. If $x$ is outer and $y$ is inner, then $x = v^j_i$ and $y = h_i$ for some $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. The geodesic $v^j_i - h_{i+1} - h_i$ has color sequence $a_j, 1$, and the geodesic $v^j_i - h_{i+3} - h_i$ has color sequence $b_j, 1$. Since $a_j \neq b_j$, one of these geodesics is rainbow.

Now suppose that $x$ and $y$ are both outer, say $x = v^j_i$ and $y = v^j_i$ with $i \leq p$. We consider two cases.

**Case 1.** $0 \leq p - i \leq 2$.

Consider the paths $L_1 : v^j_i - h_{i+3} - v^j_i$ and $L_2 : v^j_i - h_w - v^j_i$ where $w$ is to be chosen.

If $p - i = 0$ then choose $w = i + 1$, so that $L_1$ has color sequence $b_j, b_q$ while $L_2$ has color sequence $a_j, a_q$. Because $(a_i, b_i) \neq (a_j, b_j)$, at least one of $L_1$ or $L_2$ is rainbow.

If $1 \leq p - i \leq 2$ then choose $i + 1 \leq w \leq i + 2$ with $w \neq p$, so that $L_1$ has color sequence $b_j, a_q$ while $L_2$ has color sequence $a_j, a_q$. Because $a_j \neq b_j$, at least one of $L_1$ or $L_2$ is rainbow. □

**2.6. Specific lower bound**

Here we prove a lower bound for the src of Spl$_m(K_n)$.

**Theorem 3.** For $m \geq n - 2$ it holds that
\[
\text{src}(\text{Spl}_m(K_n)) \geq \lceil \sqrt[3]{m} + 1 \rceil.
\] (6)

**Proof.** Consider the subgraph $A$ induced by $K_n \cup \{v^1_1, \ldots, v^m_1\}$. This is a complete multipartite graph $K_{1, \ldots, 1, m+1}$ with src equal to $\lceil \sqrt[3]{m} + 1 \rceil$, because $m + 1 \geq n - 1$. Observe that $A$ contains all geodesics in Spl$_m(K_n)$ between any pair of its vertices. Therefore, any strong rainbow coloring of Spl$_m(K_n)$ restricts to a strong rainbow coloring of $A$. This implies src(Spl$_m(K_n)) \geq$ src(Spl$_m(A))$. The proof is complete. □

3. Main results

**Theorem 4.** Let $G = \text{Spl}_m(K_n)$. The following are true.

1. For any $m$ and $n = 2$, it holds that $\text{rc}(G) = \text{src}(G) = 2m + 1$.
2. For any $m$ and $n = 3$, it holds that
\[
\text{rc}(G) = \begin{cases} 
2 & \text{if } m = 1 \\
3 & \text{if } m \geq 2 
\end{cases}
\]
and
\[
\text{src}(G) = \begin{cases} 
2 \left\lfloor \sqrt{m} \right\rfloor & \text{if } \left\lfloor \sqrt{m} \right\rfloor^2 = m \\
2 \left\lfloor \sqrt{m} \right\rfloor + 1 & \text{if } \left\lfloor \sqrt{m} \right\rfloor^2 < m \leq \left\lfloor \sqrt{m} \right\rfloor^2 + \left\lfloor \sqrt{m} \right\rfloor \\
2 \left\lfloor \sqrt{m} \right\rfloor + 2 & \text{if } \left\lfloor \sqrt{m} \right\rfloor^2 + \left\lfloor \sqrt{m} \right\rfloor < m
\end{cases}
\] (8)

3. For $m \in \{1, 2\}$ and $n \geq 4$, it holds that $\text{rc}(G) = \text{src}(G) = 2$.
4. For $m \geq 2^{n-1}$ and $n \geq 4$, it holds that $\text{rc}(G) = 3$.

**Proof.** The first point is true because Spl$_m(K_2)$ is a tree (in fact, it can be shown that Spl$_m(H)$ is a tree if and only if $H = K_2$). So its rc and src are both equal to the number of edges.
The third point is true because of Theorem 2 and the fact that \( U = 2 \) when \( m \in \{1, 2\} \).

To prove the fourth point, we note that the assumption \( m \geq 2^{n-1} \) implies both \( m \geq n - 2 \) and \( \lceil \sqrt[m]{m+1} \rceil \geq 3 \), and then we use Theorem 3 to get \( \text{src}(G) \geq 3 \).

The second point is proved in the next subsections. We first translate the problem into an algebra problem, and then we use number theory to find the solution. In the following, we fix \( G = \text{Spl}_m(K_3) \).

3.1. System of integer inequalities

We begin by proving the existence of a solution to a system of inequalities.

**Theorem 5.** For any natural number \( m \), there is a natural number \( S \) for which we can choose natural numbers \( s_{i,k} \leq S - 1 \), where \( i \in \{1, 2, 3\} \) and \( k \in \{1, 2\} \), that satisfy the following conditions for each \( i \in \{1, 2, 3\} \),

\[
\begin{align*}
s_{i,1} + s_{i-1,2} & \leq S, \quad (9) \\
s_{i,1}s_{i,2} & \geq m, \quad (10)
\end{align*}
\]

where the index \( i - 1 \) in (9) is viewed modulo 3.

**Proof.** Since this is a statement of existence, it does not matter which value of \( S \) we find, as long as we can choose the numbers \( s_{i,k} \in \{1, \ldots, S - 1\} \) satisfying (9) and (10). Choose \( S = m + 1 \). For each \( i \in \{1, 2, 3\} \), we choose \( s_{i,1} = 1 \) and \( s_{i,2} = m \). Then \( s_{i,1} + s_{i-1,2} = S \) and \( s_{i,1}s_{i,2} = m \). The proof is complete.

The proof above gives an obvious choice of \((S, s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}, s_{3,1}, s_{3,2})\). Other choices are possible. For example when \( m = 4 \), each of \((5,1,4,1,4,1,4),(5,3,2,3,2,3,2)\) and \((4,2,2,2,2,2,2)\) is a valid choice. We will be concerned with the smallest \( S \) satisfying Theorem 5, which we denote by \( S(m) \). We will prove that \( S(m) = \text{src}(G) \).

Note that the proof above remains valid even if we reverse the direction of (9) and (10). If we do so, then we simply get a different set of values of \( S \) and \( s_{i,k} \), which are relevant to our problem.

3.2. Necessity

The proof of the following theorem explains the motivation behind Theorem 5.

**Theorem 6.** \( \text{src}(G) \geq S(m) \).

**Proof.** Let us denote \( S(m) \) simply by \( S \). Suppose there is a strong rainbow \((S-1)\)-coloring \( c \) of \( G \). For each \( i \in \{1, 2, 3\} \) and \( k \in \{1, 2\} \), define the following set of colors

\[
S_{i,k} = \{ c(v^j_ih^k_i) \mid 1 \leq j \leq m \}.
\]

Between any pair \( v^j_i, v^k_i \) there is only one geodesic in \( G \), namely \( v^j_i - h^k_i - v^k_i \). This geodesic must be rainbow, so \( c(v^j_ih^k_i) \neq c(v^k_ih^k_i) \). Since this is true for any \( j \) and \( k \), we must have \( S_{1,1} \cap S_{3,2} = \emptyset \). Similarly,

\[
S_{2,1} \cap S_{1,2} = S_{3,1} \cap S_{2,2} = S_{1,1} \cap S_{3,2} = \emptyset.
\]

This implies that \( |S_{i,1}| + |S_{i-1,2}| \leq S - 1 \) for each \( i \in \{1, 2, 3\} \). Because each \( S_{i,k} \) is not empty, (12) also implies that each \( S_{i,k} \) is a proper subset of \( \{1, \ldots, S - 1\} \). So \( |S_{i,k}| \in \{1, \ldots, S - 2\} \).

Define the color code of an outer vertex \( v^j_i \) as an ordered pair \( \text{code}(v^j_i) = (c(v^j_ih^k_i), c(v^j_ih^l_i)) \). For any distinct \( x, y \in \{v^1_i, v^2_i, \ldots, v^m_i\} \), the only geodesics in \( G \) from \( x \) to \( y \) are \( x - h^k_i - y \) and \( x - h^l_i - y \). At least one of these must be rainbow, so \( \text{code}(x) \neq \text{code}(y) \). Therefore \( v^1_i, \ldots, v^m_i \) all have different codes, so \( |S_{i,1}| + |S_{i,2}| \geq m \).

Now if we let \( s_{i,k} = |S_{i,k}| \) then the numbers \( s_{i,k} \) satisfy Theorem 5 where \( S \) is replaced with \( S - 1 \). This contradicts the minimality of \( S = S(m) \). The proof is complete.
3.3. Sufficiency

Here we prove that $S(m)$ colors suffice to provide a strong rainbow coloring on $G$.

**Theorem 7.** $src(G) \leq S(m)$.

**Proof.** Let $s_{i,k}$ be the numbers in Theorem 5. Because of Inequality (9), we can choose proper subsets $S_{i,k}$ of $\{1, \ldots, S\}$ such that $|S_{i,k}| = s_{i,k}$ for each $i \in \{1, 2, 3\}$ and $k \in \{1, 2\}$, and such that $S_{i,1} \cap S_{i,2} = \emptyset$. For each $i \in \{1, 2, 3\}$ we choose $m$ different ordered pairs $(a^i_1, b^i_1), \ldots, (a^i_m, b^i_m)$ such that $a^i_1 \in S_{i,1}$ and $b^i_j \in S_{i,2}$ for each $j \in \{1, \ldots, m\}$. This is possible because of Inequality (10). We also choose some $d_i \in \{1, \ldots, S\}$ that is not a member of $S_{i,1}$. Now we define a map $c : E(G) \to \{1, \ldots, S\}$ as follows. For each $i \in \{1, 2, 3\}$ and $j \in \{1, \ldots, m\}$, define:

- $c(h_i, h_{i+1}) = d_i$
- $c(v^i_j, h_{i+1}) = a^i_j$
- $c(v^i_j, h_{i-1}) = b^i_j$

Now we show that $c$ is strong rainbow. Take any two non-adjacent vertices $x$ and $y$. If $x$ is outer and $y$ is inner, then (since they are are non-adjacent) we must have $x = v^i_j$ and $y = h_i$ for some $i \in \{1, 2, 3\}$ and $j \in \{1, \ldots, m\}$. The geodesic $v^i_j - h_{i+1} - h_i$ has color sequence $a^i_j, d_i$, so it is a rainbow geodesic.

Now suppose that $x$ and $y$ are both outer, say $x = v^i_j$ and $y = v^i_k$. By symmetry, we need only to consider two cases.

**Case 1.** $r = 1$.

The geodesics $v^i_j - h_{i+1} - v^i_k$ and $v^i_j - h_{i-1} - v^i_k$ have color sequences $a^i_j, a^i_k$ and $b^i_j, b^i_k$ respectively. Because $(a^i_j, b^i_j) \neq (a^i_k, b^i_k)$, one of these geodesics is rainbow.

**Case 2.** $r = i + 1$.

The geodesic $v^i_j - h_{i-1} - v^i_{i+1}$ has color sequence $a^i_j, a^i_{i+1}$. Because $b^i_j \in S_{i,2}$ and $a^i_{i+1} \in S_{i+1,1}$ but $S_{i+1,1} \cap S_{i,2} = \emptyset$, we must have $b^i_j \neq a^i_{i+1}$. So the geodesic is rainbow. The proof is complete.

3.4. Double bound

In Theorems 6 and 7 we have established that $src(G) = S(m)$. To determine the explicit form of $S(m)$, the key step is the following double bound which can in fact be used to determine $S(m)$ recursively.

**Theorem 8.** For any $m$ it holds that

$$\left\lceil \frac{S(m) - 1}{2} \right\rceil \leq m \leq \left\lceil \frac{S(m)}{2} \right\rceil.$$  \hspace{1cm} (13)

**Proof.** Consider the function $f(T) = \left\lceil \frac{T}{2}\right\rceil \left\lceil \frac{T}{2}\right\rceil$. This function is strictly increasing on $\mathbb{N}$, so there is a unique $T \in \mathbb{N}$ that satisfies $f(T-1) < m \leq f(T)$. We will show that $T = S(m)$.

If we choose $s_{i,1} = \left\lceil \frac{T}{2}\right\rceil$ and $s_{i,2} = \left\lceil \frac{T}{2}\right\rceil$ for each $i \in \{1, 2, 3\}$, then $s_{i,1} + s_{i-1,2} = T$ and $s_{i,1}s_{i,2} = f(T) \geq m$. This proves that $S(m) \leq T$.

Now suppose that $S(m) \leq T - 1$. Then there are $s_{i,k} \in \{1, \ldots, T - 2\}$ such that $s_{i,1} + s_{i-1,2} \leq T - 1$ and $s_{i,1}s_{i,2} \geq m$ for each $i \in \{1, 2, 3\}$. From the arithmetic-geometric mean inequality, we have $\sqrt{s_{i,1}s_{i,2}} \leq \frac{s_{i,1} + s_{i,2}}{2}$. Therefore

$$3\sqrt{m} \leq \sqrt{s_{1,1}s_{1,2}} + \sqrt{s_{2,1}s_{2,2}} + \sqrt{s_{3,1}s_{3,2}} \leq \frac{s_{1,1} + s_{3,3} + s_{2,1} + s_{1,2} + s_{3,1} + s_{2,1}}{2} \leq \frac{3(T - 1)}{2}. \hspace{1cm} (14)$$

So we have $T - 1 \geq 2\sqrt{m}$.

Because $T$ is an integer, $\left\lfloor \frac{T-1}{2} \right\rfloor = \frac{T-1}{2} - \alpha$ for some $\alpha \in [0, \frac{1}{2})$. This implies that $\left\lceil \frac{T-1}{2} \right\rceil = \frac{T-1}{2} + \alpha$. Therefore

$$f(T-1) = \left\lceil \frac{T-1}{2} \right\rceil \left\lceil \frac{T-1}{2} \right\rceil = \left(\frac{T-1}{2}\right)^2 - \alpha^2 \geq m - \alpha^2 > m - 1. \hspace{1cm} (15)$$

But $f(T-1) < m$ and $f(T-1)$ is an integer, so $f(T-1) \leq m - 1$. This contradicts (15).

So the assumption that $S(m) \leq T - 1$ is false, and this completes the proof of Theorem 8.
3.5. Explicit form

Let \( f \) be as in the proof of Theorem 8. Observe that \( f(2k - 1) = k^2 - k \), \( f(2k) = k^2 \), \( f(2k + 1) = k^2 + k \), and \( f(2k + 2) = (k + 1)^2 \) for any integer \( k \). Now let \( k = \lceil \sqrt{m} \rceil \). Then \( k^2 \leq m < (k + 1)^2 \).

- If \( k^2 = m \), then \( f(2k - 1) < m = f(2k) \) so \( S(m) = 2k \).
- If \( k^2 < m \leq k^2 + k \), then \( f(2k) < m \leq f(2k + 1) \) so \( S(m) = 2k + 1 \).
- If \( k^2 + k < m \), then \( f(2k + 1) < m < f(2k + 2) \) so \( S(m) = 2k + 2 \).

This gives the value of \( \text{src}(G) \) as stated in Theorem 4. This also gives the value of \( \text{rc}(G) \) as follows. When \( m = 1 \), \( \text{src}(G) = 2 \) so \( \text{rc}(G) = 2 \). When \( m \geq 2 \), \( \text{src}(G) \geq 3 \) so \( \text{rc}(G) = 3 \). The proof of Theorem 4 is complete.

4. Concluding remarks

In this paper we have discussed the rainbow connection number and strong rainbow connection number of the \( m \)-splitting of the complete graph \( K_n \) for several sets of values of \((m, n)\). Complete solutions were obtained for \( n = 2, n = 3, m = 1, \) and \( m = 2 \). Much work remains to be done to completely solve the remaining cases. However, we can make some general remarks about the \( \text{rc} \) and \( \text{src} \) of \( \text{Spl}_m(K_n) \) as functions of \( m \) and \( n \). For instance, Theorem 2 shows that if we fix \( m \) and let \( n \) vary, then the \( \text{rc} \) is bounded. On the contrary, if we fix \( n \) and let \( m \) grow, then Theorem 3 shows that the \( \text{src} \) is unbounded. In fact, the \( \text{src} \) is increasing on \( m \). For \( n = 3 \), this fact can be seen from the explicit formula for the \( \text{src} \). In general, the following relationship holds for any \( m \) and \( n \geq 2 \).

\[
\text{src}(\text{Spl}_m(K_n)) \leq \text{src}(\text{Spl}_{m+1}(K_n)).
\]

This is because \( \text{Spl}_m(K_n) \) embeds as an induced subgraph \( A \) of \( \text{Spl}_{m+1}(K_n) \) with the property that \( A \) contains all geodesics in \( \text{Spl}_{m+1}(K_n) \) between any two vertices of \( A \), analogous to the proof of Theorem 3. It may be interesting to investigate (16) when \( K_n \) is replaced with other graphs.

The minimum degree requirement in Theorem 1 cannot be relaxed, as demonstrated by \( H = K_2 \). The proof of Theorem 2 is in some sense optimal, because actually \((U - 1)(U - 2) < m \leq U(U - 1)\). Theorem 3 may be modified by finding other subgraphs that are "closed under taking geodesics".

There would be several difficulties if we attempt to generalize directly our solution for \( n = 3 \) to larger values of \( n \). We have to formulate and solve a system of inequalities analogous to Theorem 5, but more complicated and with more variables. And then, analogous to Theorem 7, we also have to build a strong rainbow coloring using only numerical data provided by the inequalities. Alternatively, because \( K_3 = C_3 \) and \( \text{Spl}_m(C_n) \) has fewer geodesics compared to \( \text{Spl}_m(K_n) \), perhaps the "correct" setting to generalize the proof is the \( m \)-splitting of \( C_n \).

Acknowledgement. This work was partially supported by the 2015 Cluster Research Grant of Universitas Indonesia.

References