Nearly equal distances and Szemerédi’s regularity lemma

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Received 22 November 2004; accepted 29 June 2005
Available online 5 October 2005
Communicated by J. Akiyama, M. Kano and X. Tan

Abstract

A point set is separated if the minimum distance between its elements is one. Two numbers are called nearly equal if they differ by at most one. If a fixed positive percentage of all pairs of points belonging to a separated set of size \( n \) in \( \mathbb{R}^3 \) determine nearly equal distances, then the diameter of the set is at least constant times \( n \). This proves a conjecture of Erdős.

Keywords: Repeated distances; Szemerédí’s regularity lemma; Separated sets

1. Introduction

In 1946, Erdős [6] raised the following problem on repeated distances determined by a point set: Given \( n \) points in the plane (or, more generally, in \( \mathbb{R}^d \)), at most how many of the \( \binom{n}{2} \) interpoint distances can coincide? It is conjectured that in the plane this maximum is \( n^{1+\frac{\log \log n}{\log n}} \), which is asymptotically sharp, for example for a \( \sqrt{n} \times \sqrt{n} \) piece of the integer lattice. The best known upper estimate is only \( O(n^{4/3}) \) [14,15]. In 3-space, the best known upper bound is \( n^{3/2} \beta(n) \), where \( \beta(n) \) is an extremely slowly increasing function related to the inverse Ackermann function [5]. However, the truth is probably closer to \( n^{4/3} \). In higher dimensions, Lenz construction gives the asymptotically tight answer, which is quadratic in \( n \) (e.g. see [4,11]). These questions are intimately related to various problems concerning incidences between points and curves, surfaces, etc. (See [2,13].)

Erdős observed that the answer to the above problem does not remain the same if one counts the number of distances that are nearly equal, where several distances are said to be nearly equal if they differ by at most 1, i.e., they all lie in an interval \([t, t+1]\) for some \( t > 0 \). Clearly, to exclude trivial examples, one needs to consider only separated point sets, i.e., point sets where the minimum distance between two points is at least 1. Erdős et al. [8] (see also [7,10]) proved that for any \( t > 0 \), \( d \geq 2 \), and for any separated set \( P \) of \( n \) points in \( \mathbb{R}^d \), where \( n \) is sufficiently

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1 Research supported by NSF Grant CCR-00-98246 and grants from NSA, PSC-CUNY, Hungarian Research Foundation, and BSF.
2 Research supported by NSF grant DMS-0503184.

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large, the number of point pairs in \( P \) whose distance lies in the interval \([t, t + 1]\) is at most \( T(d, n) = \frac{n^2}{2}(1 - \frac{1}{d} + o(1)) \). Here, \( T(d, n) \) denotes the number of edges of a balanced \( d \)-partite graph on \( n \) vertices [3], which is known to be the maximum number of edges in any graph of \( n \) vertices that does not contain a complete subgraph with \( d + 1 \) vertices. Moreover, this bound can be attained for every \( t \geq t(d, n) \), as shown by the following construction (described here only for \( d = 3 \)). Let \( t \) be a sufficiently large number, and let \( v_1, v_2, v_3 \) be the vertices of an equilateral triangle in the plane \( x_3 = 0 \), with edge length \( t \). At each \( v_i \) draw a line perpendicular to the plane \( x_3 = 0 \), and on each of these lines pick \( \lfloor n/3 \rfloor \) or \( \lceil n/3 \rceil \) distinct points whose \( x_3 \)-coordinates are integers between 0 and \( n/3 \), so that the total number of points is \( n \) (see Fig. 1). If \( t \) is sufficiently large depending on \( n \) (roughly \( \frac{1}{18} n^2 \)), the distance between any pair of points selected on different perpendicular lines belongs to the interval \([t, t + 1]\).

The question arises, what is the minimal diameter of a separated set of \( n \) points in \( \mathbb{R}^3 \) with \( \Omega(n^2) \) nearly equal distances? In the plane the answer is \( \Theta(n^2) \), by the Pythagorean theorem. The problem becomes more interesting in three dimensions. Notice that the diameter of the configuration depicted in Fig. 1 is \( \Omega(n^2) \). However, it is easy to find a set of \( n \) points in \( \mathbb{R}^3 \) with \( n^2/4 \) nearly equal distances, whose diameter is \( O(n) \): Take two \( \sqrt{n}/2 \times \sqrt{n}/2 \) integer grids in two parallel planes at distance \( n/2 \) from each other (see Fig. 2). Erdős conjectured that there is no such example with diameter \( o(n) \).

The aim of this note is to prove this conjecture.

**Theorem 1.1.** Let \( \varepsilon > 0 \) be fixed and let \( P \) be a separated set of \( n \) points in \( \mathbb{R}^3 \) containing at least \( \varepsilon n^2 \) pairs \((u, v)\), \( u, v \in P \), with \( \|u - v\| \in [t, t + 1] \) for some fixed real number \( t > 0 \). Then the diameter of \( P \) satisfies \( \text{diam}(P) = \Omega(n) \).

The proof is based on Szemerédi’s regularity lemma [9] and on a Ramsey-type result for dot products of vectors, derived in [1]. The geometric component of the proof (see Section 3) does not easily generalize to higher dimensions. We will return to the higher-dimensional analogue of Theorem 1.1 in a subsequent paper [12].
2. Using Szemerédi’s regularity lemma

In this section, we prove the following theorem.

**Theorem 2.1.** Let $\varepsilon > 0$ and let $P$ be a set of $n$ points in $\mathbb{R}^3$ containing at least $\varepsilon n^2$ pairs $(u, v)$, $u, v \in P$, with $\|u - v\| \in [t, t + 1]$ for some fixed real number $t > 0$. Then there exists a constant $c := c(\varepsilon) > 0$ and there are two subsets $Q, R \subset P$, such that $|Q| = |R| = cn$ and $\|u - v\| \in [t, t + 1]$ for all $u \in Q$, $v \in R$.

**Proof.** Let $G = (V(G), E(G))$ be the graph on the vertex set $V(G) := P$ in which two vertices $u, v \in V(G)$ are connected by an edge if and only if $\|u - v\| \in [t, t + 1]$. By the assumptions, we have $|E(G)| \geq \varepsilon n^2$.

Before we could state the “degree form” of Szemerédi’s regularity lemma (see e.g., [9]), we need a definition.

**Definition 2.2.** Let $\delta > 0$. Given a graph $G$ and two disjoint vertex sets $A \subset V, B \subset V$, we say that the pair $(A, B)$ is $\delta$-regular if for every $X \subset A$ and $Y \subset B$ satisfying $|X| \geq \delta |A|$ and $|Y| \geq \delta |B|$, we have $|d(X, Y) - d(A, B)| < \delta$.

Here, $d$ stands for the standard density function $d(X, Y) = \frac{|E(X, Y)|}{|X||Y|}$, where $E(X, Y)$ denotes the set of edges between $X$ and $Y$, two disjoint sets of vertices.

**Lemma 2.3** (Szemerédi’s regularity lemma). For every $\delta > 0$, there is an $M := M(\delta)$ such that if $G = (V, E)$ is any graph and $\rho \in [0, 1]$ is any real number, then there is a partition of the vertex set $V$ into $k + 1$ clusters $V_0, V_1, \ldots, V_k$, and there is a subgraph $G' \subset G$ with the following properties:

1. $k \leq M$,
2. $|V_0| \leq \delta |V|$,
3. all clusters $V_i, i \geq 1$, are of the same size $m \leq \delta |V|$, 
4. $\deg_G(v) \geq \deg_G(v) - (\rho + \delta)|V|$ for all $v \in V$,
5. $E(G(V_i)) = \emptyset$ for all $i \geq 1$,
6. all pairs $G'(V_i, V_j)$, $1 \leq i < j \leq k$, are $\delta$-regular, each with a density either 0 or greater than $\rho$.

Consider the graph $G$ and set parameters $\rho = 0, \delta = \min\{1/12, 2\varepsilon/5\}$. Using Lemma 2.3, we obtain a partition $V_0, V_1, \ldots, V_k$ meeting the requirements. Delete all elements in $V_0$ to obtain a “pure” graph $G''$ with $\geq (1 - \delta)|V|$ vertices, whose vertex set $V(G'') = V(G') \setminus V_0$ is partitioned into $k$ clusters $V_1, V_2, \ldots, V_k$, where $k \leq M(\delta)$, $|V_i| = m$ and $E(G''(V_i)) = \emptyset$ for all $i \geq 1$, and all pairs $G''(V_i, V_j)$, $1 \leq i < j \leq k$, are $\delta$-regular. The pure graph $G''$ still contains most of the original edges of $G$. Indeed, we have $\deg_{G''}(v) > \deg_G(v) - \delta n$ for every $v$, whence $|E(G'')| > |E(G)| - \delta n^2$. To obtain $G''$, we remove another set of at most $|V_0|n \leq \delta n^2$ edges, which implies

$$|E(G'')| > |E(G)| - 2\delta n^2 \geq (\varepsilon - 2\delta)n^2.$$ 

We claim that there is a pair $(V_i, V_j)$ of clusters in $G''$ with density $d(V_i, V_j) \geq \alpha := 2(\varepsilon - 2\delta)$. Indeed, otherwise, using the fact that $E(G''(V_i)) = \emptyset$, we conclude that $|E(G'')|$ is too small:

$$|E(G'')| \leq \left(\frac{k}{2}\right)\alpha m^2 \leq \frac{\alpha k^2 m^2}{2} \leq (\varepsilon - 2\delta)n^2.$$ 

Let $(V_i, V_j)$ be a $\delta$-regular pair with density $d(V_i, V_j) \geq 2(\varepsilon - 2\delta)$. Define maps $\omega_1, \omega_2 : V_i \cup V_j \mapsto \mathbb{R}^5$ as follows:

$$\omega_1(u) = (u_x, u_y, u_z, \|u\|^2 - t^2, 1),$$

$$\omega_2(u) = (u_x, u_y, u_z, \|u\|^2 - (t + 1)^2, 1),$$

$$\omega_1(v) = (-2v_x, -2v_y, -2v_z, 1, \|v\|^2),$$

$$\omega_2(v) = (2v_x, 2v_y, 2v_z, -1, -\|v\|^2),$$

for all $u = (u_x, u_y, u_z) \in V_i \subset \mathbb{R}^3, v = (v_x, v_y, v_z) \in V_j \subset \mathbb{R}^3$.

Then, for all $u \in V_i, v \in V_j$, the edge $(u, v)$ is in $E(G'')$, that is, $\|u - v\| \in [t, t + 1]$, if and only if $\omega_1(u) \cdot \omega_1(v) \geq 0$ and $\omega_2(u) \cdot \omega_2(v) \geq 0$. 

Recall the following lemma of Alon et al. [1], which can be proved using a Borsuk–Ulam type result in range searching, due to Yao and Yao [16].

**Lemma 2.4.** Let $U$ and $V$ be finite sets of vectors in $\mathbb{R}^d$. Then there are subsets $U' \subset U$ and $V' \subset V$ such that $|U'| \geq \frac{1}{2^d+1} |U|$, $|V'| \geq \frac{1}{2^d+1} |V|$ and either $u \cdot v \geq 0$ for all $u \in U'$, $v \in V'$, or $u \cdot v < 0$ for all $u \in U'$, $v \in V'$.

Applying Lemma 2.4 to the sets $U := \omega_1(V_i)$ and $V := \omega_1(V_j)$, we obtain two subsets $V'_i \subset V_i$ and $V'_j \subset V_j$ such that $|V'_i| \geq \frac{1}{2^d} |V_i|$, $|V'_j| \geq \frac{1}{2^d} |V_j|$, and either $\omega_1(u) \cdot \omega_1(v) \geq 0$ for all $u \in V'_i$, $v \in V'_j$, or $\omega_1(u) \cdot \omega_1(v) < 0$ for all $u \in V'_i$, $v \in V'_j$.

We claim that $\omega_1(u) \cdot \omega_1(v) \geq 0$ for all $u \in V'_i$, $v \in V'_j$. Indeed, otherwise $\|u - v\| < \varepsilon$ holds for all $u \in V'_i$, $v \in V'_j$, which implies that $d(V'_i, V'_j) = 0$. However, by the $\delta$-regularity of the pair $(V_i, V_j)$, we have $d(V'_i, V'_j) > d(V_i, V_j) - \delta \geq 2(\varepsilon - 2\delta) - \delta > 0$, since $\delta = \min\left\{\frac{1}{2^d}, \frac{2\varepsilon}{3}\right\}$ and $|V'_i| \geq \frac{1}{2^d} |V_i| > \delta |V_i|$, $|V'_j| \geq \frac{1}{2^d} |V_j| > \delta |V_j|$.

Therefore, we have $\omega_1(u) \cdot \omega_1(v) \geq 0$ for all $u \in V'_i$, $v \in V'_j$.

Next, we apply Lemma 2.4 to the sets $U := \omega_2(V_i)$ and $V := \omega_2(V_j)$, and we obtain two subsets $V''_i \subset V_i$ and $V''_j \subset V_j$ such that $|V''_i| \geq \frac{1}{2^d} |V_i|$, $|V''_j| \geq \frac{1}{2^d} |V_j|$, and either $\omega_2(u) \cdot \omega_2(v) \geq 0$ for all $u \in V''_i$, $v \in V''_j$, or $\omega_2(u) \cdot \omega_2(v) < 0$ for all $u \in V''_i$, $v \in V''_j$.

We now claim that $\omega_2(u) \cdot \omega_2(v) \geq 0$ for all $u \in V''_i$, $v \in V''_j$. Indeed, otherwise $\|u - v\| > \varepsilon$ holds for all $u \in V''_i$, $v \in V''_j$, and we have $d(V''_i, V''_j) = 0$. However, by the $\delta$-regularity of the pair $(V_i, V_j)$, we obtain $d(V''_i, V''_j) > d(V_i, V_j) - \delta \geq 2(\varepsilon - 2\delta) - \delta > 0$, since $\delta = \min\left\{\frac{1}{2^d}, \frac{2\varepsilon}{3}\right\}$ and $|V''_i| \geq \frac{1}{2^d} |V_i| > \delta |V_i|$, $|V''_j| \geq \frac{1}{2^d} |V_j| > \delta |V_j|$.

Thus, we conclude that $\omega_1(u) \cdot \omega_1(v) \geq 0$ and $\omega_2(u) \cdot \omega_2(v) \geq 0$ for all $u \in V''_i$, $v \in V''_j$. Furthermore, both $V''_i$ and $V''_j$ are of size at least $\delta |V_i| = \delta |V_j| \geq 0.125 \cdot \frac{\varepsilon}{M(\varepsilon)} n$, where $\delta = \min\left\{\frac{1}{2^d}, \frac{2\varepsilon}{3}\right\}$. Therefore, we have found two subsets $Q_i := V''_i \subset P$ and $Q_j := V''_j \subset P$ such that $|P| = |Q| = c(\varepsilon) n$ and $\|u - v\| \in [t, t + 1]$ for all $u \in P$, $v \in Q$. This completes the proof of Theorem 2.1. □

### 3. Proof of Theorem 1.1

In the previous section we have established that there are two sets of points $Q, R$ of size $\Omega(n)$ such that all pairwise distances between points in $Q$ and $R$ are in the interval $[t, t + 1]$. In the rest of the paper, we conclude that this is impossible unless $t = \Omega(n)$. The proof proceeds in two steps.

1. We prove that $R$ must contain a special configuration, either two points at large distance or three points forming a triangle of large area and small circumradius.
2. We show that this configuration forces $Q$ to be contained in a region of volume $O(t^2 / n)$. Since $Q$ requires volume $\Omega(n)$, this yields $t = \Omega(n)$.

**Lemma 3.1.** Let $t \leq n/64$ and let $R$ be a separated set of $n \geq 316$ points in $\mathbb{R}^3$ such that $t \leq \|x\| \leq t + 1$ for all $x \in R$. Then we have $t \geq 2$, and either there are two points $x_1, x_2 \in R$ such that $\|x_1 - x_2\| > 0.24t$, or there are three points $y_1, y_2, y_3 \in R$ such that the area of triangle $\{y_1, y_2, y_3\}$ is at least $n/48$ and the radius of its circumscribed circle is at most $t/2$.

**Proof.** Let $B$ denote the family of $n$ balls of radius $1/2$, each centered at a point of $R$. Note that the interiors of any two balls in $B$ are disjoint, since $R$ is separated. Let $A(t)$ denote the spherical annulus $A(t) = \{x: t - 1/2 \leq \|x\| \leq t + 3/2\}$. Clearly, $A(t)$ contains $B$ and its volume is

$$\text{Vol}(A(t)) = \frac{4\pi}{3} \left(\left(t + 3 \cdot \frac{3}{2}\right)^3 - \left(t - \frac{1}{2}\right)^3\right) = \pi \left(8t^2 + 8t + \frac{14}{3}\right).$$

Since the balls in $B$ are pairwise disjoint and each has volume $\pi/6$, the volume of $A(t)$ must be at least $n\pi/6 \geq 316\pi/6$, which implies $t \geq 2$.  


Fig. 3. Points of $\phi(R)$ in the plane $H$, divided into horizontal strips.

Let $x_1, x_2$ denote two points in $R$ whose distance $h := \|x_1 - x_2\|$ is maximal. If $h > 0.24t$, we are done. Assume that $h \leq 0.24t$. By a similar volume argument, we obtain that $h$ must be at least $\sqrt{n}/4$. Otherwise, all balls belonging to $B$ are contained in a sphere of radius $\sqrt{n}/4 + 1/2$, which intersects $A(t)$ in a region of volume less than $2(\sqrt{n}/4 + 1/2)^2 \pi < n\pi/6$, which is a contradiction. Therefore, we obtain

$$h = \|x_1 - x_2\| \geq \frac{1}{2} \sqrt{n} \geq 2\sqrt{t},$$

which yields

$$x_1 \cdot (x_1 - x_2) = \frac{1}{2} (\|x_1\|^2 - \|x_2\|^2 + \|x_1 - x_2\|^2) \geq \frac{1}{2} (t^2 - (t + 1)^2 + 4t) > 0.$$

Similarly, we have $x_2 \cdot (x_1 - x_2) < 0$. Hence, there is a point $w = \beta x_1 + (1 - \beta) x_2$, $\beta \in [0, 1]$ such that $w \cdot (x_1 - x_2) = 0$. Let $H$ be a plane through the origin, orthogonal to vector $w$. Project the points of $R$ onto $H$, and let $\phi(x) \in H$ denote the projection of $x \in R$. Let $p = \phi(x_1)$, $q = \phi(x_2)$, and note that $\|p - q\| = \|x_1 - x_2\| = h$. Without loss of generality, we can assume that $pq$ is a vertical line in $H$. Note that all points in $R$ are at most $0.24t$ away from $x_1$ and $x_2$, so they project to within $0.24t$ of the origin. Divide the plane between $p$ and $q$ into horizontal strips of height 1. Every point in $\phi(R)$ is contained in one of these strips (see Fig. 3). In the $i$th strip, let $l_i$ and $r_i$ be the horizontal coordinate of the leftmost and rightmost points belonging to $\phi(R)$.

The balls in $B$ project to disks of radius $1/2$ in the plane $H$. The area of the region $D$ covered by these disks

$$\text{Area}(D) \leq \sum_{i=1}^{\lceil h \rceil} 2(r_i - l_i + 1).$$

Consider a stabbing line through a point of $D$, perpendicular to the plane $H$. In view of our choice of $H$, the distance of the origin from the line is at most $0.24t + 1/2$. On each side of $H$, such a line intersects $A(t)$ in an interval of length at most

$$\sqrt{(t + \frac{3}{2})^2 - (0.24t + \frac{1}{2})^2} - \sqrt{(t - \frac{1}{2})^2 - (0.24t + \frac{1}{2})^2},$$
which is less than 2.5, provided that \( t \geq 2 \). Since the points in \( R \) have distance at most 0.24\( t \) from \( x_1 \), they lie in the same halfspace of \( H \) as \( x_1 \). The balls in \( B \) are contained in the region of \( A(t) \) which projects to \( D \), and whose volume is therefore at most 2\( \frac{5}{6} \)Area\( (D) \). On the other hand, each of these \( n \) balls has volume \( \pi/6 \). Therefore,

\[
\frac{\pi n}{6} \leq 2.5 \text{Area}(D) = 5 \sum_{i=1}^{[h]} (r_i - l_i + 1),
\]

and, hence, using \( h \geq \sqrt{n}/4 > 4.44 \) and \( [h] \leq 1.2h \), we deduce that there exists a \( j \) such that

\[
r_j - l_j + 1 \geq \frac{5n}{30[h]} \geq \frac{5n}{36h}.
\]

Since \( h \leq t/4 \leq n/256 \), we obtain \( \pi n/36h \geq 22.2 \) and

\[
r_j - l_j \geq \frac{n}{12h}.
\]

Let \( s \) and \( t \) denote the leftmost and rightmost points in the \( j \)th horizontal strip, whose coordinates are \( l_j \) and \( r_j \), respectively. Let \( \theta \) denote the angle between the line \( st \) and the \( x \)-axis in \( H \) (see Fig. 3). Then, \( \tan \theta \in [-\frac{1}{20}, \frac{1}{20}] \).

We have

\[
\text{Area}\left(\{p, s, t\}\right) + \text{Area}\left(\{q, s, t\}\right) = \frac{1}{2}h(r_j - l_j) \geq \frac{n}{24}.
\]

Without loss of generality, assume that \( \text{Area}(\{p, s, t\}) \geq n/48 \). Then the distance of \( p \) from line \( st \) is at least \( \frac{1}{7}h \cos \theta \geq \frac{1}{2}h \sqrt{400/401} > 0.49h \). Set \( y_1 := \phi^{-1}(p) \), \( y_2 := \phi^{-1}(s) \), and \( y_3 := \phi^{-1}(t) \). Since the area cannot increase by projection, we have \( \text{Area}(\{y_1, y_2, y_3\}) \geq n/48 \).

Finally, we show that the radius of the circumscribed circle of the triangle \( \{y_1, y_2, y_3\} \) is not too large. Let \( h' \) denote the distance of \( y_1 \) from the line \( y_2y_3 \) and let \( \gamma \) denote the inner angle at \( y_3 \). We have \( h' > 0.49h \) and \( \sin \gamma = h'/\|y_1 - y_3\| \). The radius of the circumscribed circle is

\[
r = \frac{\|y_1 - y_2\|}{\sin \gamma} = \frac{\|y_1 - y_2\| \cdot \|y_1 - y_3\|}{h'} \leq \frac{h^2}{0.49h} \leq \frac{t}{2}. \quad \square
\]

**Lemma 3.2.** If \( x_1, x_2 \in \mathbb{R}^3 \), \( \|x_1 - x_2\| = a \), then the ring-like region

\[
R(x_1, x_2) := \left\{ x : \forall i = 1, 2; t - \frac{1}{2} \leq \|x - x_i\| \leq t + \frac{3}{2} \right\}
\]

has volume at most \( 4\pi (2t + 1)^2/a \).
Proof. We claim that \( R(\mathbf{x}_1, \mathbf{x}_2) \) is contained in the parallel slab \( S_a \) (bounded by two parallel planes) of thickness \( 2(2t + 1)a \), orthogonal to line \( \mathbf{x}_1 \mathbf{x}_2 \) (see Fig. 4). Indeed, assume \( \mathbf{x}_1 = -\mathbf{x}_2, \|\mathbf{x}_1\| = \|\mathbf{x}_2\| = \frac{a}{2}, \text{ and } \|\mathbf{x} - \mathbf{x}_1\|^2, \|\mathbf{x} - \mathbf{x}_2\|^2 \in [(t - \frac{1}{2})^2, (t + \frac{3}{2})^2]. \) Then,
\[
\|\mathbf{x} - \mathbf{x}_1\|^2 - \|\mathbf{x} - \mathbf{x}_2\|^2 = 2\mathbf{x} \cdot (\mathbf{x}_2 - \mathbf{x}_1) \leq 4t + 2,
\]
which means that \( \mathbf{x} \) can deviate from the plane of symmetry between \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) by at most \( \frac{2t + 1}{a} \).

Consider the annulus \( A(t) \) between the spheres of radius \( t - \frac{1}{2} \) and \( t + \frac{3}{2} \), centered at \( \mathbf{x}_1 \). Take the intersection of this annulus with \( S_a \). The planes of \( S_a \) are at distances \( l = \frac{a}{2} - \frac{2t + 1}{a} \) and \( r = \frac{a}{2} + \frac{2t + 1}{a} \) from \( \mathbf{x}_1 \). The resulting volume is
\[
\text{Vol}(R(\mathbf{x}_1, \mathbf{x}_2)) \leq \text{Vol}(A(t) \cap S_a)
\leq \int_{l}^{r} \left( \pi \left( \left( t + \frac{3}{2} \right)^2 - x^2 \right) - \pi \left( \left( t - \frac{1}{2} \right)^2 - x^2 \right) \right) dx
= \int_{l}^{r} 2\pi (2t + 1) dx = \frac{4\pi (2t + 1)^2}{a}.
\]

Lemma 3.3. Let \( t \geq 2 \), and let \( \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \) be a triangle of area \( A \geq 2t + 1 \) whose sides are of length at most \( t/4 \) and whose circumscribing circle has radius at most \( t/2 \). Then the region
\[
R(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \left\{ \mathbf{x}: \forall i = 1, 2, 3; t - \frac{1}{2} \leq \|\mathbf{x} - \mathbf{x}_i\| \leq t + \frac{3}{2} \right\}
\]
has volume at most \( \frac{16(2t + 1)^2}{A} \).

Proof. Consider two points \( \mathbf{x}_1, \mathbf{x}_2 \) at distance \( a \) and the region
\[
R(\mathbf{x}_1, \mathbf{x}_2) = \left\{ \mathbf{x}: \forall i = 1, 2; t - \frac{1}{2} \leq \|\mathbf{x} - \mathbf{x}_i\| \leq t + \frac{3}{2} \right\}.
\]
As before, $R(x_1, x_2)$ is contained in a slab $S_y$ of thickness $2(2t + 1)/a$, orthogonal to line $x_1x_2$. The same holds for $R(x_1, x_3)$, that is, if $\|x_1 - x_3\| = b$, then $R(x_1, x_3)$ is contained in a slab $S_b$ of thickness $2(2t + 1)/b$. Let $\gamma$ denote the angle between $x_1x_2$ and $x_1x_3$. Then the angle between the normal vectors to $S_a$ and $S_b$ is also $\gamma$. The area $A$ of the triangle $\{x_1, x_2, x_3\}$ is $A = \frac{1}{2}ab\sin\gamma$.

The intersection $S_a \cap S_b$ is an infinite prism which intersects the plane of the triangle in a parallelogram $R'$ (see Fig. 5). The sides of $R'$ are of length $\frac{2(2t+1)}{a\sin\gamma}$ and $\frac{2(2t+1)}{b\sin\gamma}$, and the angle between them is $\gamma$, which implies

$$\text{Area}(R') = \frac{4(2t + 1)^2}{ab\sin\gamma} = \frac{2(2t + 1)^2}{A}.$$ 

The center of $R'$ is the center of the circle circumscribed around $\{x_1, x_2, x_3\}$. Since the radius of the circumscribing circle is at most $t/2$, the distance between any point in $R'$ and $x_1$ is at most

$$\frac{t}{2} + \frac{2t + 1}{a\sin\gamma} \leq \frac{3t}{4},$$

where the first inequality follows from the assumption that $A \geq 2t + 1$. This implies that the prism $S_a \cap S_b$ intersects the annulus $A(t)$ centered at $x_1$ within distance $3t/4$ from $x_1$. Any line within distance $3t/4$ from the center intersects $A(t)$ in two intervals of length at most

$$\sqrt{\left(\frac{t + \frac{3}{2}}{2}\right)^2 - \left(\frac{3t}{4}\right)^2} - \sqrt{\left(\frac{t - \frac{1}{2}}{2}\right)^2 - \left(\frac{3t}{4}\right)^2},$$

which is smaller than 4 for $t \geq 2$. Therefore, we have

$$\text{Vol}(R(x_1, x_2, x_3)) \leq \text{Vol}(A(t) \cap S_a \cap S_b) \leq 8\text{Area}(R') = \frac{16(2t + 1)^2}{A}. \quad \square$$

**Theorem 3.4.** Let $Q$ and $R$ be two separated sets of points in $\mathbb{R}^3$, each of size $n \geq 316$, such that $t \leq \|x - y\| \leq t + 1$ for all $x \in Q$, $y \in R$. Then, $t \geq n/800$.

**Proof.** Suppose that $n \geq 316$ and $t < n/800$. Assume that one of the points in $Q$ is the origin. Then Lemma 3.1 implies that $t \geq 2$, and either there exist two points $x_1, x_2 \in R$ at distance at least 0.24$t$, or there exist three points $y_1, y_2, y_3 \in R$ such that the triangle $\{y_1, y_2, y_3\}$ has area at least $n/48$, its edges are of length at most $t/4$, and its circumradius is at most $t/2$. In the first case, Lemma 3.2 implies that the volume of $R(x_1, x_2)$ is at most $60(2t + 1)^2/t$. In the second case, Lemma 3.3 implies that the volume of $R(y_1, y_2, y_3)$ is at most $768(2t + 1)^2/n \leq (2t + 1)^2/t$. In either case, the region must contain a ball of radius $1/2$ around each point of $Q$, and these balls are pairwise disjoint. Therefore, the volume of the region must be at least $n\pi/6$. For $2 \leq t < n/800$ this number is at most $60(2t + 1)^2/t \leq 375t < n\pi/6$, which is a contradiction. $\square$

Now we are in a position to prove Theorem 1.1. Let $P$ be a separated $n$-point set in $\mathbb{R}^3$ such that there exist $en^2$ pairs $(u, v)$, $u, v \in P$, $s > 0$, with $\|u - v\| \in [t, t + 1]$ for some fixed real number $t > 0$. Then, by Theorem 2.1, there exist a constant $c := c(e) > 0$ and two subsets $Q, R \subset P$ such that $|Q| = |R| = cn$ and $\|u - v\| \in [t, t + 1]$ for all $u \in Q$, $v \in R$. Then, by Theorem 3.4, we obtain $t \geq cn/800$. Hence, we have $\text{diam}(P) = \Omega(n)$, as required.

**References**


