Some Ergodic Theorems for Piecewise Monotonic Transformations

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Let $S: [0, 1] \to [0, 1]$ be a piecewise monotonic transformation satisfying some conditions. We show that time averages $(1/n) \sum_{j=0}^{n-1} f \circ S^j$ converge strongly in $L^p(0, 1)$ and pointwisely almost everywhere for any $f \in L^p(0, 1)$, where $1 \leq p < \infty$.

1. INTRODUCTION

Since the publication of von Neumann's Mean Ergodic Theorem and Birkhoff's Individual Ergodic Theorem for measure preserving transformations in 1931, there have appeared various generalizations of these classic results to Cesaro sum bounded linear operators in Banach spaces (see [1, 4] for more details). Let $X$ be a function space and $T: X \to X$ a bounded linear operator. Mean ergodic theorems or pointwise ergodic theorems concern the convergence of the sequence of averages

$$A_n(T)f = \frac{1}{n} \sum_{j=0}^{n-1} T^j f$$

in $X$ or in the pointwise sense, where $f \in X$. Such considerations have many applications in ergodic theory, statistical physics, among others (see, e.g., [5] for many uses of averages of linear operators).

Let $S: [0, 1] \to [0, 1]$ be a piecewise monotonic transformation that is either piecewisely stretching or convex with a strong repellor. We shall prove a mean ergodic theorem and a pointwise ergodic theorem for the
composition operator $U_S: f \to f \circ S$ corresponding to $S$: the time average sequence

$$A_n(U_S)f = \frac{1}{n} \sum_{j=0}^{n-1} U_j^i f = \frac{1}{n} \sum_{j=0}^{n-1} f \circ S^j$$

converges strongly in $L^p(0, 1)$ and pointwisely almost everywhere for any $f \in L^p(0, 1)$ with $1 \leq p < \infty$.

The proof of the mean ergodic theorem will be given in the next section. Section 3 will present the pointwise ergodic theorem and explore some structure of the limit functions, and we conclude in Section 4.

2. THE MEAN ERGODIC THEOREM

In 1960, S. Ulam proposed [9] the following conjecture: if a transformation $S: [0, 1] \to [0, 1]$ satisfies $\inf_{x \in [0, 1]} |S'(x)| > 1$, then the Frobenius-Perron operator $P_S: L^2(0, 1) \to L^2(0, 1)$ associated with $S$ defined by

$$\int_A P_Sf \, dm = \int_{S^{-1}(A)} f \, dm, \quad A \subset [0, 1] \text{ measurable}$$

has a fixed density, where $m$ is the usual Lebesgue measure. In 1973, Lasota and Yorke [6] proved this conjecture for a class of piecewise $C^2$ and stretching mappings as the following theorem states.

**Theorem 2.1 (Lasota-Yorke).** Let $S: [0, 1] \to [0, 1]$ satisfy that

1. there is a partition $0 = a_0 < a_1 < \cdots < a_r = 1$ of $[0, 1]$ such that $S|_{(a_{i-1}, a_i)}$ is of $C^2$ for each $i = 1, \ldots, r$,
2. there exists a constant $\lambda > 1$ such that

$$|S'(x)| \geq \lambda \quad \text{for } x \neq a_i \ (i = 0, 1, \ldots, r),$$
3. and there exists a constant $c$ such that

$$\frac{|S''(x)|}{[S'(x)]^2} \leq c \quad \text{for } x \neq a_i \ (i = 0, 1, \ldots, r).$$
Then, for all $f \in L^1(0, 1)$, the sequence

$$A_n(P_S)f = \frac{1}{n} \sum_{j=0}^{n-1} P_S^j f$$

converges strongly in $L^1(0, 1)$ as $n \to \infty$ to some $\tilde{f} \in L^1(0, 1)$ of bounded variation with $P_S\tilde{f} = \tilde{f}$.

The Lasota–Yorke theorem can be viewed as a mean ergodic theorem for the Frobenius–Perron operator. In this section we prove another mean ergodic theorem for the composition operator. Namely we show that, under the conditions of Theorem 2.1, for any $f \in L^p(0, 1)$ with $1 \leq p < \infty$,

$$\frac{1}{n} \sum_{j=0}^{n-1} f \circ S^j$$

converge in the usual norm of $L^p(0, 1)$ to some $f^* \in L^p(0, 1)$.

Let $M(0, 1)$ be the vector space of all Lebesgue measurable functions on $[0, 1]$. Functions in $M(0, 1)$ that are equal almost everywhere are considered the same. For a nonsingular transformation $S: [0, 1] \to [0, 1]$, that is, $m(A) = 0$ implies $m(S^{-1}(A)) = 0$, the operator $U_S: M(0, 1) \to M(0, 1)$ which is defined by

$$U_Sf(x) = f(S(x)) \quad \text{for all } f \in M(0, 1)$$

is called the composition operator associated with $S$. The restriction $U_S^*: L^\infty(0, 1) \to L^\infty(0, 1)$ of the composition operator to $L^\infty(0, 1)$ with the usual norm is sometimes called the Koopman operator which is the dual of the Frobenius–Perron operator $P_S$ [5]. It is well-known that $U_S$ is an isometry from $L^p(0, 1)$ into itself if the Lebesgue measure $m$ is invariant under $S$, that is, $m(S^{-1}(A)) = m(A)$ for all measurable subsets $A$ of $[0, 1]$. But in general $U_S$ may not map $L^p(0, 1)$ into itself. A necessary and sufficient condition [2, Lemma VIII.5.7] for $U_S: L^p(0, 1) \to L^p(0, 1)$ to be well defined and continuous for every $1 \leq p < \infty$ is that

$$\sup \left\{ \frac{m(S^{-1}(A))}{m(A)} \right\} = M < \infty. \quad (1)$$

Moreover, if (1) is satisfied, then $\|U_S\|_p = M^{1/p}$, where $\|U_S\|_p$ is the operator norm of $U_S: L^p(0, 1) \to L^p(0, 1)$ with respect to the $L^p$-norm.
\[ \|f\|_p = \left( \int_0^1 |f|^p \, dm \right)^{1/p} \] of \( L^p(0,1) \). Furthermore, if there is a constant \( M \) for which

\[ \frac{1}{n} \sum_{j=0}^{n-1} m(S^{-j}(A)) \leq Mm(A), \quad n = 1, 2, \ldots \quad (2) \]

for all \( m \)-measurable subsets \( A \) of \([0,1]\), then the averages \( A_n(U_S) = (1/n) \sum_{j=0}^{n-1} U_S^j \) are uniformly bounded and strongly convergent in \( L^p(0,1) \) for every \( p \) with \( 1 \leq p < \infty \) [2, Theorem VIII.5.9]. In the following, let \( \nu = m \circ S^{-1} \) be the measure defined by \( \nu(A) = m(S^{-1}(A)) \) and let

\[ A_n(\nu) = \frac{1}{n} \sum_{j=0}^{n-1} \nu^j = \frac{1}{n} \sum_{j=0}^{n-1} m \circ S^{-j}. \]

Note that the condition (2) amounts to saying that \( A_n(\nu)(A) \leq Mm(A) \) for all \( n = 1, 2, \ldots, \) which gives a kind of “uniform absolute continuity” of the sequence of measures \( \{ A_n(\nu) \} \) with respect to the Lebesgue measure \( m \).

**Theorem 2.2.** Suppose \( S : [0,1] \to [0,1] \) satisfies all the conditions of the Lasota–Yorke theorem. Let \( 1 \leq p < \infty \) and \( f \in L^p(0,1) \). Then

\[ \lim_{n \to \infty} A_n(U_S) f = f^* \quad (3) \]

with \( U_S f^* = f^* \). Moreover,

\[ \lim_{n \to \infty} \int_0^1 f dA_n(\nu) = \int_0^1 f^* \, dm. \quad (4) \]

If, in addition, \( S \) is ergodic with respect to \( m \), then

\[ f^*(x) = \lim_{n \to \infty} \int_0^1 f dA_n(\nu) \quad m\text{-a.e.} \quad (5) \]

**Proof.** From Theorem VIII.5.9 in [2], to prove (3), it is sufficient to show that (2) is satisfied. From the definition of Frobenius–Perron operators, we have for any \( m \)-measurable subset \( A \subset [0,1] \),

\[ m(S^{-i}(A)) = \int_{S^{-i}(A)} 1 \, dm = \int_A P^i \, dm. \]
Hence, the left hand side of (2) is
\[ \int_A \frac{1}{n} \sum_{j=0}^{n-1} P_j^1 \, dm = \int_A A_n(P_S)1 \, dm. \]

From the proof of the Lasota–Yorke theorem (see [6] or [5] for more details), there is a constant $K$ such that the total variation of $P_j^1$ satisfies
\[ \sqrt{\frac{1}{0} P_j^1} \leq K, \quad n = 1, 2, \ldots. \]

Therefore, for all $n = 1, 2, \ldots$,
\[ \sqrt{\frac{1}{0} A_n(P_S)1} \leq \frac{n}{n} \sum_{j=0}^{n-1} \sqrt{\frac{1}{0} P_j^1} \leq \frac{n}{n} \sum_{j=0}^{n-1} K = K. \]

Now for each $n$, since $\|A_n1\|_1 = 1$ and $A_n1 \geq 0$, there is $y \in [0,1]$ such that $A_n(P_S)1(y) \leq 1$. Let $x \in [0,1]$. Then
\begin{align*}
A_n(P_S)1(x) - A_n(P_S)1(y) & \leq |A_n(P_S)1(x) - A_n(P_S)1(y)| \\
& \leq \sqrt{\frac{1}{0} A_n(P_S)1} \leq K,
\end{align*}

which implies that
\[ A_n(P_S)1(x) \leq A_n(P_S)1(y) + K \leq 1 + K. \]

It follows that
\[ \int_A A_n(P_S)1 \, dm \leq (1 + K) m(A). \]

That is, (2) is true with $M = 1 + K$.

To prove (4), note that the strong convergence of $A_n(U_S)f$ to $f^*$ in $L^p(0,1)$ implies the weak convergence of $A_n(U_S)f$ to $f^*$. Thus,
\[ \lim_{n \to \infty} \int_0^1 A_n(U_S)f \, dm = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \int_0^1 U_j^1 f \, dm = \int_0^1 f^* \, dm. \]

Using the change of variable theorem [2], we have $\int_0^1 U_j^1 f \, dm = \int_0^1 f \, dm \ast S^{-j} = \int_0^1 f \, d\nu$, which implies
\[ \lim_{n \to \infty} \int_0^1 A_n(U_S)f \, dm = \lim_{n \to \infty} \int_0^1 f \, dA_n(\nu). \]
Now if $S$ is ergodic, then $U_sf^* = f^*$ implies that $f^*$ is constant almost everywhere by Theorem 4.2.1 of [5]. Thus (5) follows.

The next result is about the mean ergodic theorem for piecewise convex mappings with a strong repellor $x = 0$. In this case the iterates $\{P^n_s\}$ of the Frobenius–Perron operator $P_s$ are asymptotically stable [7]. That is, $P_s$ has a unique fixed density $\hat{f}$ such that $\lim_{n \to \infty} P^n_sf = \hat{f}$ for every density $f \in L^1(0,1)$.

**Theorem 2.3.** Let $S: [0,1] \to [0,1]$ satisfy that

1. There is a partition $0 = a_0 < a_1 < \cdots < a_r = 1$ of $[0,1]$ such that $S_{i+1} = S_{i}^{-1}a_{i+1}$ for each $i = 1, \ldots, r$.
2. $S'(0) > 0$ and $S''(x) \geq 0$ for all $x \in [0,1]$, where $S'(a_i)$ and $S''(a_i)$ are right derivatives,
3. $S(a_i) = 0$ for each integer $i = 1, \ldots, r$, and
4. $S'(0) > 1$.

Then the conclusion of Theorem 2.2 is still true.

**Proof.** It was shown in [5, Theorem 6.3.1] that for any nonincreasing function $f \in L^1(0,1)$,

$$P^n_s f(x) \leq f(0) + \frac{\lambda K}{\lambda - 1}, \quad n = 1, 2, \ldots,$$

where $\lambda = S'(0) > 1$ and $K = \sum_{i=2}^{r} 1/(a_i - a_{i-1})$. In particular, we have $P^n_s 1(x) \leq 1 + \lambda K/(\lambda - 1)$ for all $n = 1, 2, \ldots$. It follows that

$$\int_{A} A_n(P_s) 1 \, dx \leq \left(1 + \frac{\lambda K}{\lambda - 1}\right)m(A).$$

Thus, (2) is true with $M = 1 + \lambda K/(\lambda - 1)$.

**Remark 2.1.** From the above proofs and Theorem VIII.5.9 in [2], for any nonsingular transformation $S: X \to X$ with $(X, \Sigma, \mu)$ a finite measure space, a general sufficient condition for $A_n(U_s)f$ to converge in $L^p(X, \Sigma, \mu)$ with $p \in [1, \infty)$ is that there is a constant $M$ such that

$$\int_{A} A_n(P_s) 1 \, d\mu \leq M\mu(A), \quad \forall A \in \Sigma.$$  \hspace{1cm} (6)
This condition in turn implies that the sequence of nonnegative functions \( \{A_n(P_s)1\} \) is weakly compact, hence from the Kakutani–Yosida theorem,

\[
\lim_{n \to \infty} A_n(P_s)1 = \tilde{f}
\]

with \( P_s \tilde{f} = \tilde{f} \) and \( \|\tilde{f}\|_1 = \mu(X) \).

Remark 2.2. Using the Uniform Boundedness Principle and Lemma VIII.5.8 in [2], we see that, for a nonsingular transformation \( S \) on a finite measure space \((X, \Sigma, \mu)\), if \( U_s : L^2(X, \Sigma, \mu) \to L^2(X, \Sigma, \mu) \) is well defined and bounded, then the strong convergence of \( A_n(U_s) \) in \( L^2(X, \Sigma, \mu) \) implies that (6), or equivalently

\[
\frac{1}{n} \sum_{j=0}^{n-1} \mu(S^{-j}(A)) \leq M\mu(A), \quad \forall A \in \Sigma
\]  

(7)

is true. Thus, the strong convergence of \( A_n(U_s) \) in \( L^2(X, \Sigma, \mu) \) is equivalent to (7), which is a stronger condition than the strong convergence of \( A_n(P_s)1 \) in \( L^2(X, \Sigma, \mu) \) which implies the existence of an invariant measure absolutely continuous with respect to \( \mu \). Hence, if \( S \) does not preserve an invariant measure which is absolutely continuous with respect to \( \mu \), then not only \( A_n(P_s) \), but also \( A_n(U_s) \) do not converge strongly in \( L^2(X, \Sigma, \mu) \). One example is

\[
S(x) = \begin{cases} 
  x/(1-x) & \text{for } x \in [0, \frac{1}{2}] \\
  2x - 1 & \text{for } x \in (\frac{1}{2}, 1]
\end{cases}
\]

for which \( x = 0 \) is a weak repellor.

Let \( N(U_s - I) \) and \( R(U_s - I) \) be the null space and the range of \( U_s - I \), respectively, where \( I \) is the identity operator. Combining Theorem 2.2 or Theorem 2.3 with Corollary VIII.5.2 in [2], we immediately have

\[
\text{Corollary 2.1. Under the conditions of Theorem 2.2 or Theorem 2.3,}
L^p(0,1) = N(U_s - I) \oplus R(U_s - I),
\]

(8)

and \( A_n(U_s) = (1/n) \sum_{j=0}^{n-1} U_s^j \) strongly converge to \( E \), where \( E \) is the projection of \( L^2(0,1) \) upon \( N(U_s - I) \) along \( R(U_s - I) \).

Remark 2.3. Corollary 2.1 may be viewed as a “nonorthogonal” version of the basic idea in von Neumann’s proof of his mean ergodic theorem.

We present another mean ergodic theorem to end this section.
**Theorem 2.4.** Let $(X, \Sigma, \mu)$ be a finite measure space and $S: X \to X$ a nonsingular transformation. Suppose $f \in L^1(X, \Sigma, \mu)$ is a nonnegative function such that $S$ preserves the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(A) = \int_A f \, d\mu, \quad \forall A \in \Sigma.$$  

and there exist two positive constants $\alpha$ and $\beta$ for which

$$\alpha \mu(A) \leq \tilde{\mu}(A) \leq \beta \mu(A), \quad \forall A \in \Sigma.$$  

Then for any $1 \leq p < \infty$ and $f \in L^p(X, \Sigma, \mu)$,

$$\lim_{n \to \infty} A_n(U_S)f = Ef,$$

where $E$ is the projection of $L^p(X, \Sigma, \mu)$ upon $N(U_S - 1)$ along $R(U_S - I)$. Furthermore,

$$\lim_{n \to \infty} \int_X f \, dA_n(\nu) = \int_X Ef \, d\mu,$$

where $\nu = \mu \circ S^{-1}$. If, in addition, $S$ is ergodic with respect to $\mu$, then

$$Ef(x) = \lim_{n \to \infty} \int_X f \, dA_n(\nu) \quad \mu\text{-a.e.}$$

In particular, if

$$0 < \alpha \leq \tilde{f}(x) \leq \beta < \infty, \quad x \in X, \mu\text{-a.e.},$$

the conclusion of the theorem is true.

**Proof.** The condition implies that $\mu(A) \leq \tilde{\mu}(A)/\alpha$ for all $A \in \Sigma$. Hence,

$$\frac{1}{n} \sum_{j=0}^{n-1} \mu(S^{-j}(A)) \leq \frac{1}{n} \sum_{j=0}^{n-1} \frac{\tilde{\mu}(S^{-j}(A))}{\alpha} = \frac{1}{\alpha n} \sum_{j=0}^{n-1} \tilde{\mu}(A) \leq \frac{\beta}{\alpha} \mu(A).$$

Now Theorem VIII.5.9 in [2] gives the result. □

**Remark 2.4.** The condition of Theorem 2.4 implies that $L^3(X, \Sigma, \mu) = L^3(X, \Sigma, \tilde{\mu})$ and the two norms $\| \cdot \|_\mu$ and $\| \cdot \|_{\tilde{\mu}}$ are equivalent.
3. THE POINTWISE ERGODIC THEOREM

Suppose $S: [0, 1] \to [0, 1]$ is a mapping that satisfies the conditions of Theorem 2.2 or Theorem 2.3. In this section we prove that the averages $A_n(U_f) f(x) = (1/n) \sum_{j=0}^{n-1} U_j f(x)$ converge for almost all $x \in [0, 1]$ for $f \in L^p(0, 1)$. The proof is based on the following lemma due to C. Ryll-Nardzewski (for a proof, see Theorem VIII.6.13 in [2]).

**Lemma 3.1.** Let $(X, \Sigma, \mu)$ be a finite measure space and $S: X \to X$ a nonsingular transformation. Then for every $f \in L^1(X, \Sigma, \mu)$ there is a $g \in L^1(X, \Sigma, \mu)$ for which the limit

$$g(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} U_j f(x)$$

exists for almost all $x \in X$, if and only if for some constant $M$,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(S^{-j}(A)) \leq M \mu(A), \quad A \in \Sigma. \quad (12)$$

**Remark 3.1.** Comparing (12) with (7), we know that the strong convergence of $A_n(U_f)$ in $L^1(X, \Sigma, \mu)$ is stronger than the pointwise convergence of $A_n(U_f)f$ for all $f \in L^1(X, \Sigma, \mu)$. Thus the following result is obvious.

**Theorem 3.1.** If $S: [0, 1] \to [0, 1]$ satisfies the conditions of Theorem 2.2 or Theorem 2.3, then for all $f \in L^p(0, 1)$,

$$\lim_{n \to \infty} A_n(U_f) f(x) = Ef(x), \quad x \in [0, 1] \text{ m-a.e.,} \quad (13)$$

where $E$ is the projection of $L^1(0, 1)$ upon $N(U_1 - I)$ along $R(U_1 - I)$ as is given in Corollary 2.1.

**Proof.** First note that $f \in L^1(0, 1)$. Since

$$\frac{1}{n} \sum_{j=0}^{n-1} m(S^{-j}(A)) \leq M \mu(A), \quad A \subset [0, 1] \text{ measurable, } n = 1, 2, \ldots,$$

from the proof of Theorem 2.2 or Theorem 2.3,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} m(S^{-j}(A)) \leq M \mu(A), \quad n = 1, 2, \ldots,$$

for all $m$-measurable $A \subset [0, 1]$. The result follows from Lemma 3.1. ❑
Theorem 3.2. Under the condition of Theorem 2.4, if \( f \in L^p(X, \Sigma, \mu) \), then
\[
\lim_{n \to \infty} A_n(U_S) f(x) = Ef(x), \quad x \in X \text{ } \mu\text{-a.e.} \tag{14}
\]

Although the integral \( \int_0^1 Ef \, dm \) has the expression (4) and \( Ef \) is a constant given by (5) if \( S \) is ergodic, in general, the limit function \( Ef \) in Theorem 3.1 is not easy to determine. Here we further explore the structure of \( Ef \) when there is only one discontinuity of \( S \) in the Lasota–Yorke theorem, that is, \( r = 2 \) in Theorem 2.1. The following lemma is useful for obtaining the expression of \( Ef(x) \).

Lemma 3.2. For any positive integer \( k \),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(S^j(S^k(x))) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(S^j(x))
\]
whenever the right hand side limit exists.

Proof. The lemma follows from
\[
\frac{f(S^k(x)) + \cdots + f(S^{n-1+k}(x))}{n} = \frac{n+k f(x) + \cdots + f(S^{n+k-1}(x))}{n+k} - \frac{f(x) + \cdots + f(S^{k-1}(x))}{n}
\]
by taking limit \( n \to \infty \).

Theorem 3.3. If \( S : [0, 1] \to [0, 1] \) satisfies the conditions of the Lasota–Yorke theorem with \( r = 2 \), then \( P_S \) has a unique fixed density \( f \in L^2(0, 1) \) and for all \( f \in L^p(0, 1) \),
\[
\lim_{n \to \infty} A_n(U_S) f(x) = Ef(x), \quad x \in [0, 1] \text{ } m\text{-a.e.}, \tag{15}
\]
where
\[
Ef(x) = \int_0^1 \tilde{f} \, dm, \quad x \in \bigcup_{n=0}^{\infty} S^{-n}(\text{supp } \tilde{f}) \text{ } m\text{-a.e.} \tag{16}
\]
In particular, if \( \bigcup_{n=0}^{\infty} S^{-n}(\text{supp } \tilde{f}) = [0, 1] \), then
\[
\lim_{n \to \infty} A_n(U_S) f(x) = \int_0^1 \tilde{f} \, dm, \quad x \in [0, 1] \text{ } m\text{-a.e.} \tag{17}
\]
Proof. The existence of $\tilde{f}$ is guaranteed by Theorem 2.1 while its uniqueness and the ergodicity of $S$ with respect to the invariant measure $\tilde{\mu}(A) = \int_A \tilde{f} dm$ have been established by Li and Yorke in [8]. By the Birkhoff Individual Ergodic Theorem, there is $N \subset \text{supp} \tilde{f}$ with $\tilde{\mu}(N) = 0$ such that for $f \in L^1_{\tilde{\mu}}(0,1)$,

$$\lim_{n \to \infty} A_n(U_N) f(x) = \int_0^1 f d\tilde{\mu} = \int_0^1 \tilde{f} dm, \quad x \in [0,1] \tilde{\mu}\text{-a.e.} \quad (18)$$

From $0 = \tilde{\mu}(N) = \int_N \tilde{f} dm$ and $N \subset \text{supp} \tilde{f}$, we see that $m(N) = 0$.

Since $\tilde{f}$ is of bounded variation by Theorem 2.1, $\tilde{f}$ is bounded and (18) is still valid for $f \in L^1(0,1)$. Denote $B = \bigcup_{n=0}^{\infty} S^{-n}(\text{supp} \tilde{f})$ and $C = \bigcup_{n=0}^{\infty} S^{-n}(N)$. Then $m(C) = 0$ from the nonsingularity of $S$. Now let $D = B - C$, the set of all elements in $B$ but not in $C$. Then $m(D) = m(B)$, and for any $x \in D$,

$$S^n(x) \in \text{supp} \tilde{f} - N.$$ 

Thus Lemma 3.2 and (18) give (16).

Remark 3.2. Since $\text{supp} \tilde{f} \subset S([0,1])$, and $S^{-1}(\bigcup_{n=0}^{\infty} S^{-n}(\text{supp} \tilde{f})) = \bigcup_{n=0}^{\infty} S^{-n}(\text{supp} \tilde{f})$,

$$\bigcup_{n=0}^{\infty} S^{-n}(\text{supp} \tilde{f}) \subset S([0,1]).$$

It would be interesting to see when they are equal.

4. CONCLUSIONS

With the help of the classic definition of variation, we proved the mean ergodic theorem and the pointwise ergodic theorem for the class of piecewise monotonic stretching mappings and the class of piecewise convex mappings with a strong repellor on the unit interval. It seems a new approach to use the Frobenius–Perron operator to study the asymptotic property of the composition operator. Using the modern notion of variation for functions of multi-variables, the same result can be obtained for multi-dimensional piecewise expanding mappings the existence of whose invariant measures have been obtained in [3]. Moreover, most results of the paper are still true if the measure space $(X, \Sigma, \mu)$ is $\sigma$-finite. Finally, it will be interesting to study more general nonsingular transformations for which the corresponding Frobenius–Perron operators are constrictive.
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