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Necessary and Sufficient Conditions for Permanence and Global Stability of a Lotka–Volterra System with Two Delays

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In this paper we seek necessary and sufficient conditions for the permanence and the global asymptotic stability of a positive equilibrium for a Lotka-Volterra system with two delays.

1. INTRODUCTION

We consider the following symmetrical Lotka–Volterra-type predator–prey system with two delays τ_1 and τ_2:

$$\begin{aligned} x'(t) &= x(t)\left[r_1 + ax(t) + \alpha x(t-\tau_1) - \beta y(t-\tau_2)\right] \\ y'(t) &= y(t)\left[r_2 + ay(t) + \beta x(t-\tau_1) + \alpha y(t-\tau_2)\right]. \end{aligned} \tag{1.1}$$

The initial condition of (1.1) is given as

$$\begin{aligned} x(s) &= \phi(s) \geq 0, \qquad -\tau_1 \leq s \leq 0; \quad \phi(0) > 0 \\ y(s) &= \psi(s) \geq 0, \qquad -\tau_2 \leq s \leq 0; \quad \psi(0) > 0. \end{aligned} \tag{1.2}$$

Here a, α, β, r_1, r_2, τ_1, and τ_2 are constants with $a < 0$, $\tau_1 \geq 0$, and $\tau_2 \geq 0$, and ϕ, ψ are continuous functions. Obviously, we can take $\beta \geq 0$ without loss of generality. We assume that (1.1) has a positive equilibrium

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(x^*, y^*), that is,

$$x^* = \frac{-(a+\alpha)r_1 - \beta r_2}{(a+\alpha)^2 + \beta^2} > 0,$$

$$y^* = \frac{\beta r_1 - (a+\alpha)r_2}{(a+\alpha)^2 + \beta^2} > 0.$$

We say that the system (1.1) is permanent if there exists some compact set D in the interior of R_+^2 such that any solution of (1.1) with (1.2) will ultimately stay in D. The positive equilibrium (x^*, y^*) is said to be globally asymptotically stable if (x^*, y^*) is stable and attracts any solution of (1.1) with (1.2). Our purpose is to seek sharp conditions for the permanence of (1.1) and the global asymptotic stability of (x^*, y^*) for all τ_1 and τ_2, making the best use of the symmetry of (1.1). In this paper we first give the following necessary and sufficient condition for the permanence of (1.1) for all delays $\tau_1 \geq 0$ and $\tau_2 \geq 0$:

THEOREM 1.1. *The system* (1.1) *is permanent for all* $\tau_1 \geq 0$ *and* $\tau_2 \geq 0$ *if and only if*

$$a + \alpha < 0$$

holds.

Then we also establish the following necessary and sufficient condition for the global asymptotic stability of (x^*, y^*) for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$:

THEOREM 1.2. *The positive equilibrium* (x^*, y^*) *of* (1.1) *is globally asymptotically stable for all* $\tau_1 \geq 0$ *and* $\tau_2 \geq 0$ *if and only if*

$$\sqrt{\alpha^2 + \beta^2} \leq -a$$

holds.

When the system (1.1) has no delay, that is, $\tau_1 = \tau_2 = 0$, it is easy to see that (x^*, y^*) is globally asymptotically stable if and only if $a + \alpha < 0$ (cf. Appendix). So we can see that the condition $\sqrt{\alpha^2 + \beta^2} \leq -a$ in Theorem 1.2 reflects the delay effects.

The permanence of (1.1) with $\alpha \leq 0$ has been well studied (see, for example, [5]). Wang and Ma [10] showed that (1.1) is permanent for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$ under conditions $a < 0$ and $\alpha \leq 0$. Thus, Theorem 1.1 generalizes their result for (1.1).

In the case $\alpha > 0$, we notice that the positive delayed feedback terms $\alpha x(t - \tau_1)$ and $\alpha y(t - \tau_2)$ on the right-hand side of (1.1) play a role of *destabilizer* of the system. Biologically, $\alpha x(t - \tau_1)$ and $\alpha y(t - \tau_2)$ with $\alpha > 0$ may be viewed as the *recycling* of population.

Gopalsamy [2] showed that if $|\alpha| + |\beta| < -a$ holds, then the positive equilibrium (x^*, y^*) is globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$. It is clear that Theorem 1.2 improves the Gopalsamy condition for (1.1). Recently, Lu and Wang [8] also considered the global asymptotic stability of (x^*, y^*) for (1.1) with $\alpha = 0$.

The proofs of the global existence of the solutions of (1.1) and Theorems 1.1 and 1.2 are given in Sections 2 and 3, based on the well-known comparison principle and the methods developed in [5, 6, 8, and 10]. To prove Theorem 1.1, a similar method in [10] is used. However, we see that our proof is simpler than that in [10]. In the proof of the sufficiency of Theorem 1.2, we use an extended LaSalle's invariance principle (also see [9] and [11] for ODE), by which our proof is more complete than that in [8].

2. PERMANENCE

In this section, we first prepare the following elementary result.

LEMMA 2.1. *If* $a + \alpha < 0$, *then every solution* $(x(t), y(t))$ *of* (1.1) *with the initial condition* (1.2) *exists on* $[0, \infty)$ *and is positive*.

Proof. We first show that $x(t) > 0$ as long as it is defined. In fact, if not, there exists some $\bar{t} > 0$ such that

$$\bar{t} = \inf\{t | x(t) = 0,\, t > 0\}.$$

Clearly, $x(\bar{t}) = 0$. Thus, we have

$$x(t) = x(0)\exp\left\{\int_0^t [r_1 + ax(s) + \alpha x(s - \tau_1) - \beta y(s - \tau_2)]\,ds\right\}$$

on $[0, \bar{t})$. By the continuity of $x(t)$, we have

$$x(\bar{t}) = x(0)\exp\left\{\int_0^{\bar{t}} [r_1 + ax(s) + \alpha x(s - \tau_1) - \beta y(s - \tau_2)]\,ds\right\} > 0.$$

This is a contradiction. For the same reason, we can also show that $y(t)$ is positive as long as it is defined.

Next, let us show that $(x(t), y(t))$ exists on $[0, \infty)$. If it is false, there exists a positive number T such that $\lim_{t \to T^-} x(t)$ or $\lim_{t \to T^-} y(t)$ does not exist. In the case $\tau_1 = 0$, we have

$$x'(t) = x(t)[r_1 + (a + \alpha)x(t) - \beta y(t - \tau_2)]$$
$$y'(t) = y(t)[r_2 + ay(t) + \beta x(t) + \alpha y(t - \tau_2)]$$

for $t \in [0, T)$. Since $x(t) > 0$ and $y(t) > 0$ on $[0, T)$, we have

$$x(t) < x(0)\exp\left\{\int_0^t r_1\, ds\right\} \le x(0)\exp\left\{\int_0^T |r_1|\, ds\right\}$$

for $t \in [0, T)$. Hence for $t \in [0, T)$, we obtain

$$y(t) < y(0)\exp\left\{\int_0^t [r_2 + \beta x(s)]\, ds\right\} \le y(0)\exp\left\{\int_0^T [|r_2| + \beta K_1]\, ds\right\}$$

if $\tau_2 = 0$ and

$$y(t) < y(0)\exp\left\{\int_0^T [|r_2| + \beta K_1 + |\alpha y(s - \tau_2)|]\, ds\right\}$$

if $\tau_2 > 0$, where $K_1 = x(0)\exp\{\int_0^T |r_1|\, ds\}$. Thus, there exist positive numbers K_2 and K_3 such that $|x'(t)| < K_2$ and $|y'(t)| < K_3$ for $t \in [0, T)$. Hence, we have

$$|x(t_1) - x(t_2)| \le \left|\int_{t_1}^{t_2} |x'(s)|\, ds\right| < K_2|t_1 - t_2|,$$

$$|y(t_1) - y(t_2)| \le \left|\int_{t_1}^{t_2} |y'(s)|\, ds\right| < K_3|t_1 - t_2|$$

for $t_1, t_2 \in [0, T)$. The well-known Cauchy theorem shows that $\lim_{t \to T^-} x(t)$ and $\lim_{t \to T^-} y(t)$ exist, which is a contradiction.

In the case $\tau_1 > 0$, for $t \in [0, T)$, we have

$$\begin{aligned} x(t) &= x(0)\exp\left\{\int_0^t [r_1 + ax(s) + \alpha x(s - \tau_1) - \beta y(s - \tau_2)]\, ds\right\} \\ &< x(0)\exp\left\{\int_0^t [|r_1| + |\alpha x(s - \tau_1)|]\, ds\right\} \\ &\le x(0)\exp\left\{\int_0^T [|r_1| + |\alpha x(s - \tau_1)|]\, ds\right\} \end{aligned}$$

and

$$\begin{aligned} y(t) &= y(0)\exp\left\{\int_0^t [r_2 + ay(s) + \beta x(s - \tau_1) + \alpha y(s - \tau_2)]\, ds\right\} \\ &< y(0)\exp\left\{\int_0^T [|r_2| + |\beta x(s - \tau_1)|]\, ds\right\} \end{aligned}$$

if $\tau_2 = 0$ and

$$y(t) < y(0)\exp\left\{\int_0^T \left[|r_2| + |\beta x(s-\tau_1)| + |\alpha y(s-\tau_2)|\right] ds\right\}$$

if $\tau_2 > 0$. This implies that $x(t)$ and $y(t)$ are bounded on $[0, T)$, from which we can also get a contradiction as above. The proof of Lemma 2.1 is complete.

To consider the permanence of (1.1), we next prove the following:

LEMMA 2.2. *If* $a + \alpha < 0$, *then any solution of* (1.1) *with the initial condition* (1.2) *is ultimately bounded, that is,*

$$\limsup_{t\to+\infty} x(t) \le B_1,$$

$$\limsup_{t\to+\infty} y(t) \le B_2,$$

where

$$B_1 = \max\left\{\frac{|r_1|}{|a|}, \frac{|r_1|}{|a+\alpha|}\right\}, \quad B_2 = \max\left\{\frac{|r_2 + \beta B_1|}{|a|}, \frac{|r_2 + \beta B_1|}{|a+\alpha|}\right\}.$$

Proof. In the case $\alpha \le 0$, it is easy to see that

$$\limsup_{t\to+\infty} x(t) \le \frac{|r_1|}{|a|} = B_1,$$

$$\limsup_{t\to+\infty} y(t) \le \frac{|r_2 + \beta B_1|}{|a|} = B_2,$$

by using the comparison theorem of ordinary differential equations.

Let us consider the case of $\alpha > 0$. From (1.1), for any sufficiently small $\varepsilon_1 > 0$, we have

$$x'(t) < x(t)\left[|r_1| + \varepsilon_1 + ax(t) + \alpha x(t-\tau_1)\right] \tag{2.1}$$

for $t \ge 0$. Now consider the following scalar delay differential equation:

$$u'(t) = u(t)\left[|r_1| + \varepsilon_1 + au(t) + \alpha u(t-\tau_1)\right] \tag{2.2}$$

for $t \ge 0$. Let $u(t)$ be the solution of (2.2) with the initial condition $u(\theta) = \phi(\theta) + 1$ $(-\tau_1 \le \theta \le 0)$. We will show that for $t \ge 0$,

$$x(t) \le u(t). \tag{2.3}$$

Otherwise, there exists some $t_1 > 0$ such that

$$t_1 = \inf\{t | x(t) > u(t), t \geq 0\}.$$

This implies that

$$x(t) \leq u(t), \qquad t \in [-\tau_1, t_1), \tag{2.4}$$

$$x(t_1) = u(t_1), \tag{2.5}$$

and there exists a decreasing sequence $\{t'_n\}$ such that $t'_n \to t_1$ as $n \to \infty$ and

$$x(t'_n) > u(t'_n). \tag{2.6}$$

(2.5) and (2.6) yield

$$\frac{x(t'_n) - x(t_1)}{t'_n - t_1} > \frac{u(t'_n) - u(t_1)}{t'_n - t_1}.$$

Letting $n \to \infty$, we have $x'(t_1) \geq u'(t_1)$, which, together with (2.1), (2.2), (2.5), and $\alpha > 0$, implies

$$x(t_1 - \tau_1) > u(t_1 - \tau_1).$$

This contradicts (2.4). Thus (2.3) is proved.

For (2.2), it is known from [5, pp. 218, 219] that

$$\lim_{t \to +\infty} u(t) = \frac{|r_1| + \varepsilon_1}{|a + \alpha|}$$

if $a + \alpha < 0$. Hence, it follows from (2.3) and the arbitrariness of ε_1 that

$$\limsup_{t \to +\infty} x(t) \leq \frac{|r_1|}{|a + \alpha|} = B_1. \tag{2.7}$$

For sufficiently small $\varepsilon_2 > 0$, there is some $t_2 > 0$ such that for $t \geq t_2$,

$$x(t) < B_1 + \varepsilon_2.$$

Then, it follows from (1.1) that for $t \geq t_2 + \tau_1$,

$$y'(t) \leq y(t)\left[|r_2 + \beta(B_1 + \varepsilon_2)| + ay(t) + \alpha y(t - \tau_2)\right].$$

By using the same argument as above, we can show that

$$\limsup_{t \to +\infty} y(t) \leq \frac{|r_2 + \beta B_1|}{|a + \alpha|} = B_2.$$

The proof of Lemma 2.2 is complete.

THEOREM 2.1. *The system* (1.1) *is permanent for all* $\tau_1 \geq 0$ *and* $\tau_2 \geq 0$ *if and only if*

$$a + \alpha < 0$$

holds.

Proof (*Sufficiency*). Let $z(t) = (x(t), y(t))$ be any solution of (1.1) with the initial condition (1.2). Construct the two continuous functions $V_1(t)$ and $V_2(t)$ for $t \geq 0$ as follows:

$$V_1(t) = (x(t))^{-(a+\alpha)}(y(t))^{-\beta} \times \exp\left[-\{(a+\alpha)\alpha + \beta^2\}\int_{t-\tau_1}^{t} x(s)\,ds + \beta a\int_{t-\tau_2}^{t} y(s)\,ds\right], \tag{2.8}$$

$$V_2(t) = (x(t))^{\beta}(y(t))^{-(a+\alpha)} \times \exp\left[-\beta a\int_{t-\tau_1}^{t} x(s)\,ds - \{(a+\alpha)\alpha + \beta^2\}\int_{t-\tau_2}^{t} y(s)\,ds\right]. \tag{2.9}$$

For any sufficiently small $\varepsilon > 0$, let

$$l_1 = \exp\left[-\left|(a+\alpha)\alpha + \beta^2\right|B_1'\tau_1 - |\beta a|B_2'\tau_2\right],$$
$$l_2 = \exp\left[-|\beta a|B_1'\tau_1 - \left|(a+\alpha)\alpha + \beta^2\right|B_2'\tau_2\right],$$
$$L_1 = \exp\left[\left|(a+\alpha)\alpha + \beta^2\right|B_1'\tau_1 + |\beta a|B_2'\tau_2\right],$$
$$L_2 = \exp\left[|\beta a|B_1'\tau_1 + \left|(a+\alpha)\alpha + \beta^2\right|B_2'\tau_2\right],$$

where $B_1' = B_1 + \varepsilon$ and $B_2' = B_2 + \varepsilon$. B_1 and B_2 are defined as in Lemma 2.2. Then, it follows from Lemmas 2.1 and 2.2, (2.8), and (2.9) that there exists some sufficiently large $T > 0$ such that for $t \geq T$,

$$0 < x(t) < B_1', \quad 0 < y(t) < B_2', \tag{2.10}$$

$$l_1(x(t))^{-(a+\alpha)}(y(t))^{-\beta} \leq V_1(t) \leq L_1(x(t))^{-(a+\alpha)}(y(t))^{-\beta} \tag{2.11}$$

$$l_2(x(t))^{\beta}(y(t))^{-(a+\alpha)} \leq V_2(t) \leq L_2(x(t))^{\beta}(y(t))^{-(a+\alpha)}. \tag{2.12}$$

Now calculating the derivative of V_1 with respect to t, we have that for $t \geq 0$,

$$V_1'(t) = \left[-(a+\alpha)r_1 - \beta r_2 - \left\{(a+\alpha)^2 + \beta^2\right\}x(t)\right]V_1(t).$$

Put

$$\eta_1 = -(a + \alpha) r_1 - \beta r_2 .$$

Then $\eta_1 > 0$ by our assumptions. Choose an h_1: $0 < h_1 < B_1$ small enough such that

$$V_1'(t) > (\eta_1/2) V_1(t) \tag{2.13}$$

for $0 < x(t) \le h_1$. By arguments similar to those above, there exists an h_2: $0 < h_2 < B_2$ such that

$$V_2'(t) > (\eta_2/2) V_2(t) \tag{2.14}$$

for $0 < y(t) \le h_2$, where

$$\eta_2 = \beta r_1 - (a + \alpha) r_2 > 0.$$

Now let us construct a region D as follows. First, define the curve Γ_1 by

$$\Gamma_1 \colon x^{-(a+\alpha)} y^{-\beta} = \frac{l_1 h_1^{-(a+\alpha)} (B_2')^{-\beta}}{L_1} .$$

Suppose that the intersection point of Γ_1 with $y = h_2$ is given by $(\bar{x}, h_2)$, and define the curve Γ_2 by

$$\Gamma_2 \colon x^{\beta} y^{-(a+\alpha)} = \frac{l_2 (\bar{x})^{\beta} h_2^{-(a+\alpha)}}{L_2} .$$

Let D denote the region enclosed by Γ_1, Γ_2, $x = B_1'$, and $y = B_2'$ (Fig. 1). In the following we prove that $z(t)$ eventually enters and remains in the region D. The proof is divided into four steps.

Step 1. We first show that, if there is a $t_0^* > T$ such that $z(t_0^*)$ lies in the right side of $x = h_1$, then the $z(t)$ will remain in the right side of Γ_1 for all $t \ge t_0^*$. In fact, if $z(t)$ meets Γ_1 at t_2: $t_2 > t_0^*$, then there exists a t_1: $t_0^* < t_1 < t_2$ such that $x(t_1) = h_1$ and $z(t)$ lies between $x = h_1$ and Γ_1 for all $t_1 < t < t_2$. By the inequality (2.13) we have

$$V_1(t_1) < V_1(t_2) . \tag{2.15}$$

On the other hand, from (2.10) and (2.11), we obtain

$$\begin{aligned} V_1(t_2) &\le L_1 (x(t_2))^{-(a+\alpha)} (y(t_2))^{-\beta} = l_1 h_1^{-(a+\alpha)} (B_2')^{-\beta} \\ &\le l_1 (x(t_1))^{-(a+\alpha)} (y(t_1))^{-\beta} \le V_1(t_1) . \end{aligned}$$

This contradicts (2.15).

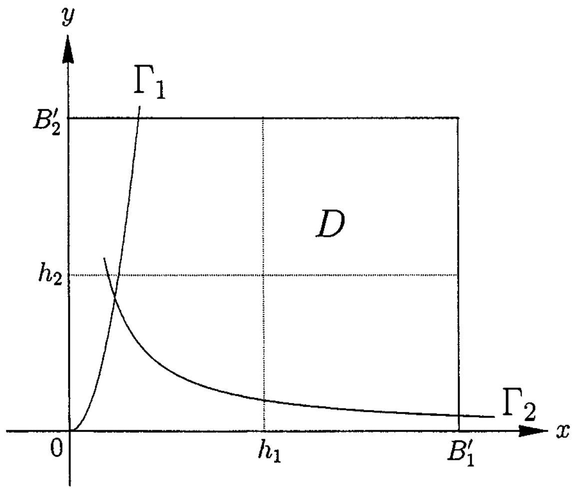

FIG. 1. The region D.

Step 2. In this step we show that if there is a $t_3 > T$ such that $y(t_3) > h_2$ and $z(t)$ lies in the right side of Γ_1 for all $t \geq t_3$, then $z(t)$ cannot meet Γ_2 for all $t \geq t_3$. In fact, if $z(t)$ meets Γ_2 at t_5, then there exists a t_4: $t_3 < t_4 < t_5$, such that $y(t_4) = h_2$ and $y(t) < h_2$ for $t_4 < t < t_5$. By (2.14) we have

$$V_2(t_4) < V_2(t_5) \leq L_2\big(x(t_5)\big)^{\beta}\big(y(t_5)\big)^{-(a+\alpha)} = l_2(\bar{x})^{\beta} h_2^{-(a+\alpha)}. \quad (2.16)$$

But since $z(t)$ lies in the right side of Γ_1 and $y(t_4) = h_2$, we have

$$V_2(t_4) \geq l_2\big(x(t_4)\big)^{\beta}\big(y(t_4)\big)^{-(a+\alpha)} \geq l_2(\bar{x})^{\beta} h_2^{-(a+\alpha)}.$$

This contradicts (2.16).

Step 3. In this step we show that if there is a $t_6 > T$ such that $y(t_6) \leq h_2$, then there exists a $t_7 > t_6$ such that $y(t_7) > h_2$. Otherwise, we have $y(t) \leq h_2$ for all $t \geq t_6$. Then (2.14) implies that $V_2(t)$ tends to infinity, but (2.10) and (2.12) imply that $V_2(t)$ is bounded, which is a contradiction.

Step 4. In this step we show that if there is a $t_8 > T$ such that $x(t_8) \leq h_1$, then there exists a $t_9 > t_8$ such that $x(t_9) > h_1$. Otherwise, we have $x(t) \leq h_1$ for all $t \geq t_8$. Then (2.13) implies that $V_1(t)$ tends to infinity. But, in the case $\beta = 0$, (2.10) and (2.11) imply that $V_1(t)$ is bounded, which is a contradiction. In the case $\beta > 0$, (2.10) and (2.11) imply that $y(t)$ tends to zero, which contradicts Step 3.

Now we are in a position to conclude the proof of sufficiency. First, if for some $T_1 > T$, $z(T_1)$ lies in the right side of $x = h_1$ and above $y = h_2$, then (2.10) and Steps 1 and 2 imply that $z(t)$ will remain in D for $t \geq T_1$. Next, if $z(T_2)$ lies in the right side of $x = h_1$ and below $y = h_2$ for some $T_2 > T$, then Step 1 implies that $z(t)$ will remain in the right side of Γ_1 for $t \geq T_2$. It follows from Step 3 that there exists a $T_3 > T_2$ such that $y(T_3) > h_2$. Hence, we can show that from (2.10) and Step 2, $z(t)$ will remain in D for $t \geq T_3$. Finally, if for some $t > T$, $z(t)$ lies in the left side of $x = h_1$, then Step 4 implies that $z(t)$ will enter in the right side of $x = h_1$ as t increases. By using the same arguments as above, we can show that $z(t)$ eventually enters and remains in D.

(*Necessity*). We can see easily if $a + \alpha \geq 0$ holds, then (1.1) is not permanent in the case $\tau_1 = \tau_2 = 0$ (cf. Appendix). This completes the proof.

Remark 2.1. In the proof of necessity above, we showed that (1.1) is not permanent in the case $\tau_1 = \tau_2 = 0$ if $a + \alpha \geq 0$. However, computer simulation seems to suggest that (1.1) is not permanent for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$ if $a + \alpha \geq 0$ holds.

3. GLOBAL STABILITY

To consider the global asymptotic stability of the positive equilibrium (x^*, y^*) of (1.1), we first introduce an extension of LaSalle's invariance principle.

For some constant $\Delta > 0$, let $C^n = C([-\Delta, 0], R^n)$. Consider the delay differential equations

$$z'(t) = f(z_t), \tag{3.1}$$

where $z_t \in C^n$ is defined as $z_t(\theta) = z(t+\theta)$ for $-\Delta \leq \theta \leq 0$, $f: C^n \to R^n$ is completely continuous, and solutions of (3.1) are continuously dependent on the initial data in C^n. The following lemma is actually a corollary of the LaSalle invariance principle and the proof is omitted (see, for example, [4, 5]).

LEMMA 3.1. *Assume that for a subset G of C^n such that $\overline{G}$ is positively invariant for* (3.1), *and V: $G \to R$,*

(i) *V is continuous on G.*

(ii) *For any $\phi \in \partial G$ (the boundary of G), the limit $l(\phi)$*

$$l(\phi) = \lim_{\substack{\psi \to \phi \\ \psi \in G}} V(\psi)$$

exists or is $+\infty$.

(iii) $\dot{V}_{(3.1)} \le 0$ *on* G, *where* $\dot{V}_{(3.1)}$ *is the upper right-hand derivative of* V *along the solution of* (3.1).

Let $E = \{\phi \in \overline{G} \mid l(\phi) < \infty$ *and* $\dot{V}_{(3.1)}(\phi) = 0\}$. *Here, for* $\phi \in \partial G$ *and* $l(\phi) < \infty$, *we define*

$$V(\phi) = l(\phi),$$

$$\dot{V}_{(3.1)}(\phi) = \limsup_{h \to 0^+} \frac{1}{h}\left[V(z_h(\phi)) - V(\phi)\right]$$

$$\left(\text{if } l(z_h(\phi)) < \infty\right).$$

Let M *denote the largest subset in* E *that is invariant with respect to* (3.1). *Then every bounded solution of* (3.1) *that remains in* G *approaches* M *as* $t \to +\infty$.

The following is our main result.

THEOREM 3.1. *The positive equilibrium* (x^*, y^*) *of* (1.1) *is globally asymptotically stable for all* $\tau_1 \ge 0$ *and* $\tau_2 \ge 0$ *if and only if*

$$\sqrt{\alpha^2 + \beta^2} \le -a$$

holds.

Proof (*Sufficiency*). By using the transformation

$$\bar{x} = x - x^*, \quad \bar{y} = y - y^*,$$

the system (1.1) is reduced to

$$\begin{aligned} x'(t) &= (x^* + x(t))\left[ax(t) + \alpha x(t - \tau_1) - \beta y(t - \tau_2)\right] \\ y'(t) &= (y^* + y(t))\left[ay(t) + \beta x(t - \tau_1) + \alpha y(t - \tau_2)\right], \end{aligned} \tag{3.2}$$

where we used $x(t)$ and $y(t)$ again instead of $\bar{x}(t)$ and $\bar{y}(t)$, respectively. Using Lemma 3.1 we now prove that the trivial solution of (3.2) is globally asymptotically stable for all $\tau_1 \ge 0$ and $\tau_2 \ge 0$. Define

$$G = \left\{\phi = (\phi_1, \phi_2) \in C^2 \mid \phi_i(s) + x_i^* \ge 0, \phi_i(0) + x_i^* > 0, i = 1, 2\right\},$$

where $C^2 = C([-\Delta, 0], R^2)$, $\Delta = \max\{\tau_1, \tau_2\}$, and $(x_1^*, x_2^*) = (x^*, y^*)$. Clearly, $\overline{G}$ is positively invariant for (3.2). We consider the functional V defined on G:

$$\begin{aligned} V(\phi) = &-2a \sum_{i=1}^{2} \left\{\phi_i(0) - x_i^* \log \frac{\phi_i(0) + x_i^*}{x_i^*}\right\} \\ &+ (\alpha^2 + \beta^2) \sum_{i=1}^{2} \int_{-\tau_i}^{0} \phi_i^2(\theta)\, d\theta. \end{aligned} \tag{3.3}$$

It is clear that V is continuous on G and that

$$\lim_{\substack{\psi \to \phi \in \partial G \\ \psi \in G}} V(\psi) = +\infty \quad \text{or} \quad \text{exists}.$$

Furthermore,

$$\begin{aligned}
\dot{V}_{(3.2)}(\phi) &= -2a\left[a\phi_1(0) + \alpha\phi_1(-\tau_1) - \beta\phi_2(-\tau_2)\right]\phi_1(0) \\
&\quad - 2a\left[a\phi_2(0) + \beta\phi_1(-\tau_1) + \alpha\phi_2(-\tau_2)\right]\phi_2(0) \\
&\quad + (\alpha^2 + \beta^2)\left\{\left[\phi_1^2(0) - \phi_1^2(-\tau_1)\right] + \left[\phi_2^2(0) - \phi_2^2(-\tau_2)\right]\right\} \\
&= -\left[a\phi_1(0) + \alpha\phi_1(-\tau_1) - \beta\phi_2(-\tau_2)\right]^2 \\
&\quad - \left[a\phi_2(0) + \beta\phi_1(-\tau_1) + \alpha\phi_2(-\tau_2)\right]^2 \\
&\quad - \left[a^2 - (\alpha^2 + \beta^2)\right]\left[\phi_1^2(0) + \phi_2^2(0)\right] \\
&\leq 0
\end{aligned} \tag{3.4}$$

on G. From (3.3) and (3.4), we see that the trivial solution of (3.2) is stable and that every solution is bounded.

Let

$$E = \left\{\phi \in \overline{G} \mid l(\phi) < \infty \text{ and } \dot{V}_{(3.2)}(\phi) = 0\right\},$$

M: the largest subset in E that is invariant with respect to (3.2).

For $\phi \in M$, the solution $z_t(\phi) = (x(t+\theta), y(t+\theta))$ $(-\Delta \leq \theta \leq 0)$ of (3.2) through $(0, \phi)$ remains in M for $t \geq 0$ and satisfies, for $t \geq 0$,

$$\dot{V}_{(3.2)}(z_t(\phi)) = 0.$$

Hence, for $t \geq 0$,

$$\begin{aligned}
ax(t) + \alpha x(t - \tau_1) - \beta y(t - \tau_2) &= 0 \\
ay(t) + \beta x(t - \tau_1) + \alpha y(t - \tau_2) &= 0,
\end{aligned} \tag{3.5}$$

which implies that for $t \geq 0$,

$$x'(t) = y'(t) = 0.$$

Thus, for $t \geq 0$,

$$x(t) = c_1, \quad y(t) = c_2 \tag{3.6}$$

for some constants c_1 and c_2. From (3.5) and (3.6), we have

$$\begin{bmatrix} a+\alpha & -\beta \\ \beta & a+\alpha \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $c_1 = c_2 = 0$ by our assumptions, and thus we have

$$x(t) = y(t) = 0 \qquad \text{for } t \geq 0.$$

Therefore, for any $\phi \in M$, we have

$$\phi(0) = (x(0), y(0)) = 0.$$

By Lemma 3.1, any solution $z_t = (x(t+\theta), y(t+\theta))$ tends to M. Thus

$$\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} y(t) = 0.$$

Hence, (x^*, y^*) is globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$.

(*Necessity*). The proof is by contradiction. Assume the assertion is false. That is, let (x^*, y^*) be globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$ and $\sqrt{\alpha^2 + \beta^2} > -a$.

Linearizing (3.2), we have

$$\begin{aligned} x'(t) &= x^*[ax(t) + \alpha x(t-\tau_1) - \beta y(t-\tau_2)] \\ y'(t) &= y^*[ay(t) + \beta x(t-\tau_1) + \alpha y(t-\tau_2)]. \end{aligned} \tag{3.7}$$

Now, we will show that there exists a characteristic root λ_0 of (3.7) such that

$$Re(\lambda_0) > 0 \tag{3.8}$$

for some τ_1 and τ_2, which implies that the trivial solution of (3.2) is not stable (see [1, pp. 160, 161]).

When $\alpha \geq -a$, it is clear that (x^*, y^*) is not globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$ by Theorem 2.1. Therefore, we have only to consider the case $\alpha < -a$.

(I) The case $0 < |\alpha| < -a$. Let $\tau_1 = \tau_2 = \tau$; then the characteristic equation of (3.7) takes the form

$$\lambda^2 + p\lambda + q + (r + s\lambda)e^{-\lambda\tau} + ve^{-2\lambda\tau} = 0, \tag{3.9}$$

where $p = -a(x^* + y^*)$, $q = a^2x^*y^*$, $r = 2a\alpha x^*y^*$, $s = -\alpha(x^* + y^*)$, and $v = (\alpha^2 + \beta^2)x^*y^*$.

When $x^* = y^*$, (3.9) can be factorized as

$$[\lambda - x^*\{a + (\alpha + i\beta)e^{-\lambda\tau}\}][\lambda - x^*\{a + (\alpha - i\beta)e^{-\lambda\tau}\}] = 0. \tag{3.10}$$

Let us consider the equation

$$\lambda - x^*\{a + (\alpha + i\beta)e^{-\lambda\tau}\} = 0. \tag{3.11}$$

Set $\alpha = b\cos\theta$ and $\beta = b\sin\theta$, where b and θ are constants with $b \geq 0$. Then, we note that $b > 0$ because of $a < 0$ and $\sqrt{\alpha^2 + \beta^2} > -a$. Substituting $\lambda = iy$ into (3.11), we have

$$iy - x^*[a + b\{\cos(y\tau - \theta) - i\sin(y\tau - \theta)\}] = 0. \tag{3.12}$$

By separating the real and imaginary parts of (3.12), we obtain

$$\begin{aligned} bx^*\cos(y\tau - \theta) &= -ax^* \\ bx^*\sin(y\tau - \theta) &= -y. \end{aligned} \tag{3.13}$$

From (3.13), we have

$$(bx^*)^2 = (ax^*)^2 + y^2.$$

To solve y in (3.13), define the following function:

$$f_1(Y) = Y + (ax^*)^2 - (bx^*)^2, \tag{3.14}$$

where $Y = y^2$. Then f_1 is an increasing linear function and

$$f_1(0) = x^{*2}\{a^2 - (\alpha^2 + \beta^2)\} < 0.$$

Thus, it follows that there exists a positive simple root Y_0 of $f_1(Y) = 0$. Substituting y_0, which satisfies $Y_0 = y_0^2$, into (3.13), we can get τ_0 such that (3.11) has a characteristic root iy_0 when $\tau = \tau_0$.

Furthermore, taking the derivative of λ with τ on (3.11), we have

$$\frac{d\lambda}{d\tau} = \frac{-x^* b e^{i\theta}\lambda e^{-\lambda\tau}}{1 + x^* b\tau e^{i\theta}e^{-\lambda\tau}}.$$

Using (3.11), we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{1}{-\lambda(\lambda - x^*a)} - \frac{\tau}{\lambda}.$$

Hence,

$$\operatorname{sign}\left[Re\left(\frac{d\lambda}{d\tau}\bigg|_{\lambda=iy_0,\ \tau=\tau_0}\right)\right] = \operatorname{sign}\left[Re\left(\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\lambda=iy_0,\ \tau=\tau_0}\right)\right]$$
$$= \operatorname{sign}\left[Re\left(\frac{1}{-iy_0(iy_0 - x^*a)} - \frac{\tau_0}{iy_0}\right)\right]$$
$$= \operatorname{sign}\left[Re\left(\frac{1}{y_0^2 + iy_0x^*a}\right)\right] > 0,$$

which implies that (3.8) holds. Therefore, the trivial solution of (3.2) is not stable, that is, (x^*, y^*) is not stable near τ_0, which is a contradiction.

When $x^* \neq y^*$, (3.9) cannot be factorized as (3.10). Substituting $\lambda = iy$ into (3.9), we have

$$(-y^2 + piy + q)e^{iy\tau} + r + siy + ve^{-iy\tau} = 0. \tag{3.15}$$

By separating the real and imaginary parts of (3.15), we have

$$\begin{aligned}\left[(-y^2+q)^2 - v^2 + p^2y^2\right]\cos(y\tau) &= (r-sp)y^2 - r(q-v)\\ \left[(-y^2+q)^2 - v^2 + p^2y^2\right]\sin(y\tau) &= sy^3 + [rp - s(q+v)]y,\end{aligned} \tag{3.16}$$

and thus

$$\left[(-y^2+q)^2 - v^2 + p^2y^2\right]^2$$
$$= \left[(r-sp)y^2 - r(q-v)\right]^2 + \left[sy^3 + [rp - s(q+v)]y\right]^2.$$

Define the following function:

$$f_2(Y) = \left[(-Y+q)^2 - v^2 + p^2Y\right]^2 - [(r-sp)Y - r(q-v)]^2$$
$$- Y[sY + rp - s(q+v)]^2, \tag{3.17}$$

where $Y = y^2$; then f_2 is a quartic function such that $f_2 \to +\infty$ as $|Y| \to +\infty$. Since

$$f_2(0) = \left[a^2 - (\alpha^2 + \beta^2)\right]^2\left[(a+\alpha)^2 + \beta^2\right]\left[(a-\alpha)^2 + \beta^2\right](x^*y^*)^4$$
$$> 0,$$

we cannot immediately find positive zeros of (3.17), and so we have to investigate f_2 in more detail. Define

$$F(Y) = \left[(-Y+q)^2 - v^2 + p^2 Y\right]^2$$

$$G(Y) = -\left[(r-sp)Y - r(q-v)\right]^2$$

$$H(Y) = -Y\left[sY + rp - s(q+v)\right]^2;$$

then $f_2 = F + G + H$. It is easy to see that positive zeroes of F, G, and H are mutually different as long as $x^* \neq y^*$. Hence, the value of f_2 at the positive zero of F is negative, which, together with $f_2(0) > 0$, implies that there exists a positive root of $f_2(Y) = 0$. It is also clear that there exists another positive root of $f_2(Y) = 0$ because $f_2 \to +\infty$ as $Y \to +\infty$. Thus, one of the two positive roots is a simple root at least.

Let Y_0 be such a simple root. Substituting y_0, which satisfies $Y_0 = y_0^2$, into (3.16), we can get some τ such that (3.9) has a characteristic root iy_0 at τ. We note that iy_0 is a simple root of (3.11) because Y_0 is a simple root of $f_2(Y) = 0$.

Furthermore, taking the derivative of λ with τ on (3.9), we have

$$\frac{d\lambda}{d\tau} = \frac{-2\lambda(\lambda^2 + p\lambda + q) - \lambda(r + s\lambda)e^{-\lambda\tau}}{2\lambda + p + 2\tau(\lambda^2 + p\lambda + q) + e^{-\lambda\tau}[s + \tau(r + s\lambda)]},$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + p + se^{-\lambda\tau}}{-2\lambda(\lambda^2 + p\lambda + q) - \lambda(r + s\lambda)e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

Hence, we have

$$\begin{aligned}
&\operatorname{sign}\left[Re\left(\left.\frac{d\lambda}{d\tau}\right|_{\lambda = iy_0}\right)\right] \\
&\quad = \operatorname{sign}\left[Re\left(\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda = iy_0}\right)\right] \\
&\quad = \operatorname{sign}\left[Re\left(\frac{2iy_0 + p + se^{-iy_0\tau}}{-2iy_0(-y_0^2 + piy_0 + q) - iy_0(r + siy_0)e^{-iy_0\tau}} - \frac{\tau}{iy_0}\right)\right] \\
&\quad = \operatorname{sign}\left[Re\left\{\left(\frac{2iy_0 + p + se^{-iy_0\tau}}{-2iy_0(-y_0^2 + piy_0 + q) - iy_0(r + siy_0)e^{-iy_0\tau}}\right)^{-1}\right\}\right] \\
&\quad = \operatorname{sign}\left[1 + \frac{(a^2 + a\alpha\cos(y_0\tau))(x^* - y^*)^2}{(p + s\cos(y_0\tau))^2 + (2y_0 - s\sin(y_0\tau))^2}\right]. \qquad (3.18)
\end{aligned}$$

Since

$$\left(a^2 + a\alpha\cos(y_0\tau)\right)(x^* - y^*)^2 \geq a(a + |\alpha|)(x^* - y^*)^2 > 0,$$

the last expression in (3.18) is positive. This implies that (3.8) holds, which is a contradiction.

(II) The case $\alpha = 0$. Let $\tau_1 = \tau_2 = \tau$; then the characteristic equation of (3.7) takes the form

$$\lambda^2 + p\lambda + q + ve^{-2\lambda\tau} = 0. \tag{3.19}$$

Substituting $\lambda = iy$ into (3.19), we have

$$-y^2 + piy + q + ve^{-2iy\tau} = 0. \tag{3.20}$$

By separating the real and imaginary parts of (3.20), we have

$$\begin{aligned} v\cos(2y\tau) &= y^2 - q \\ v\sin(2y\tau) &= py \end{aligned} \tag{3.21}$$

and

$$v^2 = \left(y^2 - q\right)^2 + (py)^2.$$

Define the following function:

$$f_3(Y) = (Y - q)^2 + p^2Y - v^2, \tag{3.22}$$

where $Y = y^2$; then f_3 is a downward convex quadratic function, and

$$f_3(0) = \left(a^4 - \beta^4\right)x^{*2}y^{*2} < 0.$$

Thus, it follows that there exists a positive simple root Y_0 of $f_3(Y) = 0$. Substituting y_0, which satisfies $Y_0 = y_0^2$, into (3.21), we can get some τ such that (3.19) has a characteristic root iy_0 at τ. Here iy_0 is a simple root of (3.19), by the same reasoning as above.

Taking the derivative of λ with τ on (3.19), we have

$$\frac{d\lambda}{d\tau} = \frac{2v\lambda e^{-2\lambda\tau}}{2\lambda + p - 2v\tau e^{-2\lambda\tau}},$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + p}{2\lambda\left(-\lambda^2 - p\lambda - q\right)} - \frac{\tau}{\lambda}.$$

Hence,

$$\begin{aligned}\operatorname{sign}\left[Re\left(\frac{d\lambda}{d\tau}\bigg|_{\lambda=iy_0}\right)\right] &= \operatorname{sign}\left[Re\left(\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\lambda=iy_0}\right)\right] \\ &= \operatorname{sign}\left[Re\left(\frac{2iy_0+p}{2iy_0\left(y_0^2-piy_0-q\right)}-\frac{\tau}{iy_0}\right)\right] \\ &= \operatorname{sign}\left[Re\left(\frac{2iy_0+p}{2y_0\left[py_0+i\left(y_0^2-q\right)\right]}\right)\right] \\ &= \operatorname{sign}\left[2y_0^2+a^2\left(x^{*2}+y^{*2}\right)\right]>0.\end{aligned}$$

This implies that (3.8) holds, which is a contradiction.

(III) The case $\alpha \le a$. Let $\tau_1 = \tau$ and $\tau_2 = 0$; then the characteristic equation of (3.7) takes the form

$$\lambda^2 + \tilde{p}\lambda + \tilde{q} + (\tilde{r} + \tilde{s}\lambda)e^{-\lambda\tau} = 0, \tag{3.23}$$

where $\tilde{p} = -ax^* - (a+\alpha)y^*$, $\tilde{q} = a(a+\alpha)x^*y^*$, $\bar{r} = [\alpha(a+\alpha) + \beta^2]x^*y^*$, and $\tilde{s} = -\alpha x^*$. Let us use p, q, r, and s again instead of $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, and $\tilde{s}$, respectively. Substituting $\lambda = iy$ into (3.23), we have

$$-y^2 + piy + q + (r + siy)e^{-iy\tau} = 0. \tag{3.24}$$

By separating the real and imaginary parts of (3.24), we have

$$\begin{aligned}(r^2+s^2y^2)\cos(y\tau) &= r\left(y^2-q\right) - spy^2 \\ (r^2+s^2y^2)\sin(y\tau) &= sy\left(y^2-q\right) + pry\end{aligned} \tag{3.25}$$

and

$$\left[r^2+s^2y^2\right]^2 = \left[r\left(y^2-q\right)-spy^2\right]^2 + \left[sy\left(y^2-q\right)+pry\right]^2.$$

Define the following function:

$$f_4(Y) = Y\left[s(Y-q)+pr\right]^2 + \left[r(Y-q)-spY\right]^2 - \left[r^2+s^2Y\right]^2, \tag{3.26}$$

where $Y = y^2$; then f_4 is an upward cubic function to the right, and

$$\begin{aligned}f_4(0) &= \left[\alpha(a+\alpha)+\beta^2\right]^2\left[(a+\alpha)^2+\beta^2\right]\left[a^2-\left(\alpha^2+\beta^2\right)\right](x^*y^*)^4 \\ &< 0.\end{aligned}$$

Thus, there can exist some positive roots of $f_4(Y) = 0$. Now, let us show that there exists a simple root in such positive roots. We see that

$$f_4'(Y) = 3s^2Y^2 + 2\left[s^2(p^2 - 2q - s^2) + r^2\right]Y + s^2(q^2 - 2r^2) + r^2(p^2 - 2q)$$

and

$$f_4''(Y) = 6s^2Y + 2\left[s^2(p^2 - 2q - s^2) + r^2\right].$$

Let $f_4''(Y) = 0$; then

$$3s^2Y + \left[s^2(p^2 - 2q - s^2) + r^2\right] = 0,$$

and thus we have

$$\begin{aligned}-3s^2 f_4'(Y) &= \left[s^2(p^2 - 2q - s^2) + r^2\right]^2 \\ &\quad - 3s^2\left[s^2(q^2 - 2r^2) + r^2(p^2 - 2q)\right] \\ &= x^{*4}y^{*2}\left[\alpha^2(4\alpha^2 - a^2)x^{*2} + \{\alpha(a+\alpha) + \beta^2\}^2 y^{*2}\right] \\ &\quad \times\left[\{\alpha(a+\alpha) + \beta^2\}^2 - \alpha^2(a+\alpha)^2\right] \\ &\quad + \alpha^4 x^{*4}\left[(a^2 - \alpha^2)x^{*2} - (a+\alpha)^2 y^{*2}\right]^2. \qquad (3.27)\end{aligned}$$

Since $\alpha \le a < 0$, (3.27) is positive. This proves that there exists no triple root of $f_4(Y) = 0$, which implies that there exists at least a positive simple root Y_0 of $f_4(Y) = 0$.

Substituting y_0, which satisfies $Y_0 = y_0^2$, into (3.25), we can get some τ such that (3.23) has a characteristic root iy_0 at τ. Here again iy_0 is a simple root of (3.23).

Taking the derivative of λ with τ on (3.23), we have

$$\frac{d\lambda}{d\tau} = \frac{\lambda(r + s\lambda)e^{-\lambda\tau}}{2\lambda + p + e^{-\lambda\tau}[s - \tau(r + s\lambda)]},$$

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{2\lambda + p + se^{-\lambda\tau}}{\lambda(r + s\lambda)e^{-\lambda\tau}} - \frac{\tau}{\lambda} \\ &= \frac{2\lambda + p}{-\lambda(\lambda^2 + p\lambda + q)} + \frac{s}{\lambda(r + s\lambda)} - \frac{\tau}{\lambda}.\end{aligned}$$

Hence, we have

$$
\begin{aligned}
&\operatorname{sign}\left[Re\left(\left.\frac{d\lambda}{d\tau}\right|_{\lambda=iy_0}\right)\right]\\
&\quad=\operatorname{sign}\left[Re\left(\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda=iy_0}\right)\right]\\
&\quad=\operatorname{sign}\left[Re\left(\frac{2iy_0+p}{-iy_0\left(-y_0^2+piy_0+q\right)}+\frac{s}{iy_0(r+siy_0)}-\frac{\tau}{iy_0}\right)\right]\\
&\quad=\operatorname{sign}\left[\frac{s^2y_0^4+2r^2y_0^2-s^2q^2-2r^2q+p^2r^2}{\left[(py_0)^2+\left(y_0^2-q\right)^2\right]\left[r^2+(sy_0)^2\right]}\right].
\end{aligned}
\tag{3.28}
$$

Since

$$
\begin{aligned}
&-s^2q^2-2r^2q+p^2r^2\\
&\qquad=\left[a^2x^{*2}+(a+\alpha)^2y^{*2}\right]\left[\alpha(a+\alpha)+\beta^2\right]^2x^{*2}y^{*2}\\
&\qquad\quad-a^2\alpha^2(a+\alpha)^2x^{*4}y^{*2}\\
&\qquad\geq\left[a^2x^{*2}+(a+\alpha)^2y^{*2}\right]\alpha^2(a+\alpha)^2x^{*2}y^{*2}\\
&\qquad\quad-a^2\alpha^2(a+\alpha)^2x^{*4}y^{*2}\\
&\qquad=\alpha^2(a+\alpha)^4x^{*2}y^{*4}>0,
\end{aligned}
$$

the last expression in (3.28) is positive. This implies that (3.8) holds, which is a contradiction. This completes the proof.

Here, we give three portraits of the trajectory of (1.1) with (1.2), drawn by a computer using the Runge–Kutta method, to illustrate Theorem 3.1 ($r_1 = 10$, $r_2 = -10$) (Figs. 2–4).

4. APPENDIX

When $\tau_1 = \tau_2 = 0$, the system (1.1) becomes

$$
\begin{aligned}
x'(t) &= x(t)\left[r_1+(a+\alpha)x(t)-\beta y(t)\right]\\
y'(t) &= y(t)\left[r_2+\beta x(t)+(a+\alpha)y(t)\right].
\end{aligned}
\tag{4.1}
$$

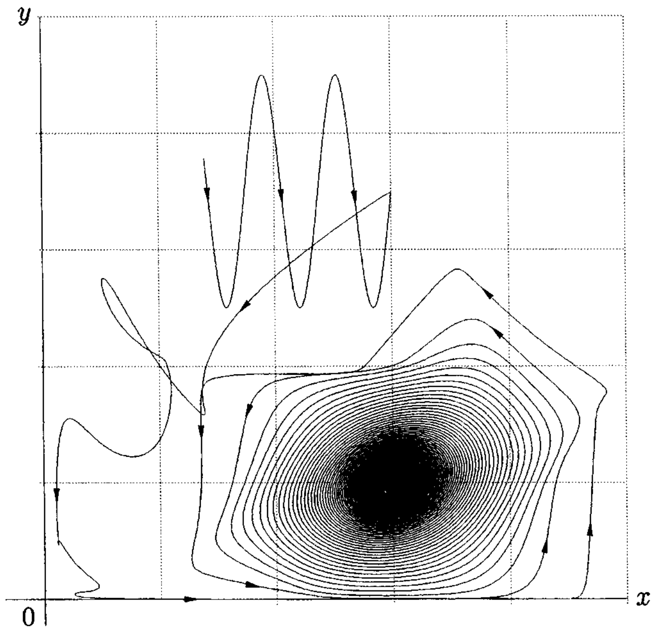

FIG. 2. $a = -5$, $\alpha = 3$, $\beta = 3.99$ ($\sqrt{\alpha^2 + \beta^2} < -a$), $\tau_1 = 1$, $\tau_2 = 2$, $(\phi, \psi) = (3 + 0.8t, 3.5 + \sin(8t))$.

By using the transformation

$$\bar{x} = x - x^*, \quad \bar{y} = y - y^*,$$

we have from (4.1) that

$$\begin{aligned} x'(t) &= (x^* + x(t))[(a + \alpha)x(t) - \beta y(t)] \\ y'(t) &= (y^* + y(t))[\beta x(t) + (a + \alpha)y(t)], \end{aligned} \tag{4.2}$$

where we used $x(t)$ and $y(t)$ again instead of $\bar{x}(t)$ and $\bar{y}(t)$, respectively. Consider the following Liapunov function:

$$V(x, y) = \left(x - x^* \log \frac{x + x^*}{x^*}\right) + \left(y - y^* \log \frac{y + y^*}{y^*}\right), \tag{4.3}$$

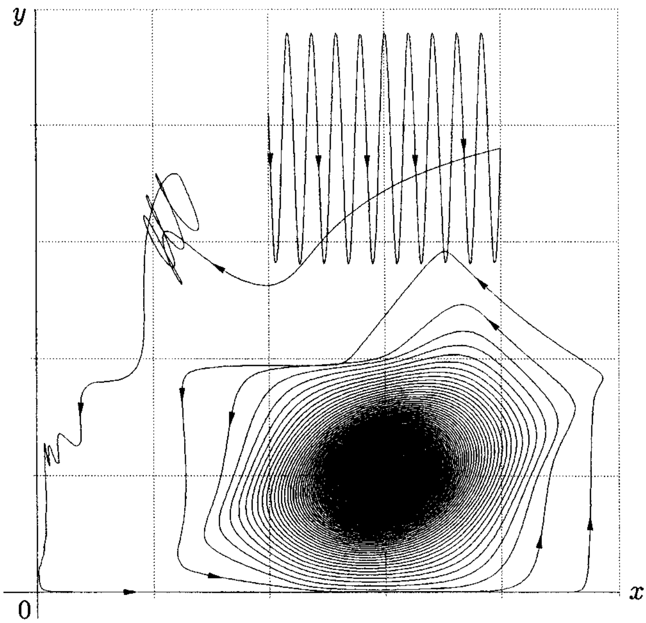

FIG. 3. $a = -5$, $\alpha = 3$, $\beta = 4$ ($\sqrt{\alpha^2 + \beta^2} = -a$), $\tau_1 = 1$, $\tau_2 = 2$, $(\phi, \psi) = (4 + t, 3.8 + \sin(30t))$.

for $x > -x^*$ and $y > -y^*$, then V is positive definite. Calculating the derivative of V along the solution of (4.2), we have that

$$\dot{V}_{(4.2)}(x, y) = (a + \alpha)(x^2 + y^2).$$

Clearly, $\dot{V}_{(4.2)}$ is negative definite if and only if $a + \alpha < 0$ holds. The well-known Liapunov theorem shows that the origin $(0, 0)$ is globally asymptotically stable if and only if $a + \alpha < 0$ holds.

If $a + \alpha = 0$ holds, $\dot{V}_{(4.2)}$ vanishes identically. So all solutions are periodic solutions. Thus, (4.1) is not permanent. If $a + \alpha > 0$ holds, (4.1) is also not permanent. In fact, otherwise, it follows that there exists some compact set D_0 in the interior of the region $\{(x, y) \in R^2 \mid x + x^* > 0, y + y^* > 0\}$ such that any solution of (4.2) will ultimately stay in D_0. From (4.3), there exists some positive number k such that the closed curve $V(x, y) = k$ covers D_0. A solution through a point on the closed curve does not enter that curve because $\dot{V}_{(4.2)} > 0$ there, which is a contradiction.

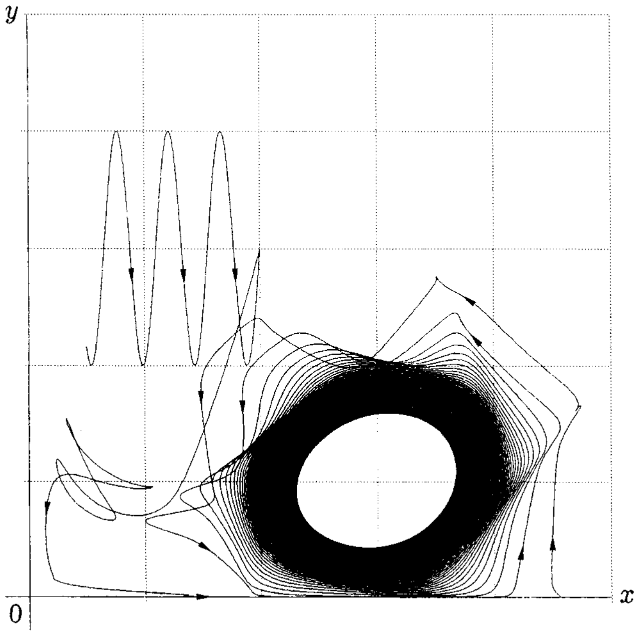

FIG. 4. $a = -5$, $\alpha = 3$, $\beta = 4.01$ ($\sqrt{\alpha^2 + \beta^2} > -a$), $\tau_1 = 2$, $\tau_2 = 3$, $(\phi, \psi) = (2 + 0.5t, 3 + \sin(7t))$.

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