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A class of nonhomogeneous singular integrals in $\mathbf{R}^n$

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Abstract

Let $\{\delta_t\}_{t>0}$ be a nonisotropic dilation group on $\mathbf{R}^n$, let ρ be a distance function on $\mathbf{R}^n$ which is homogeneous with respect to $\{\delta_t\}_{t>0}$, and for $f \in C_0^\infty(\mathbf{R}^n)$ define $Tf = [\text{p.v.}\,\rho(\cdot)^{-\alpha} e^{i/|\cdot|^\beta}] * f$, where α and β are positive parameters. We give necessary and sufficient conditions on p, α and β for which T extends to a bounded linear operator on $L^p(\mathbf{R}^n)$.

1. Introduction

Consider the dilation group $\{\delta_t\}_{t>0}$ on $\mathbf{R}^n$ given by

$$\delta_t : x = (x_1, x_2, \ldots, x_n) \mapsto (t^{b_1} x_1, t^{b_2} x_2, \ldots, t^{b_n} x_n),$$

where $1 = b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$ are fixed numbers. A $\{\delta_t\}$-homogeneous distance function is defined to be a continuous nonnegative function on $\mathbf{R}^n$ which is homogeneous with respect to $\{\delta_t\}_{t>0}$ and vanishes only at the origin. A standard example is given by

$$\tau(x) = \left(|x_1|^{2b_2b_3\ldots b_n} + |x_2|^{2b_1b_3\ldots b_n} + \cdots + |x_n|^{2b_1b_2\ldots b_{n-1}}\right)^{1/(2b_1b_2\ldots b_n)}.$$

Any $\{\delta_t\}$-homogeneous distance function defines a quasi-norm on $\mathbf{R}^n$ (see (3)).

Let $\alpha, \beta > 0$, let ρ be a $\{\delta_t\}$-homogeneous distance function which is C^1 in $\mathbf{R}^n - 0$, and for $f \in C_0^\infty(\mathbf{R}^n)$ define

$$Tf(x) = \text{p.v.} \int_{|y|\leqslant 1} \frac{e^{i/|y|^\beta}}{\rho(y)^\alpha} f(x-y)\,dy. \tag{1}$$

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As we shall see below, the principal value integral in (1) exists whenever $\alpha < b$ or $\beta > \alpha - b \geqslant 0$, where

$$b = b_1 + \cdots + b_n.$$

In fact, as can be seen from Lemma 1 below, if $\alpha < b$, then the kernel of T is an L^1 function.

The purpose of this paper is to study the mapping properties of T on $L^p(\mathbf{R}^n)$ when $\alpha \geqslant b$. As always, the major problem in dealing with this kind of singular integrals is how to translate the cancellation properties of the kernel into L^p boundedness of the operator. The fact that the singularity of the kernel of T and the cancellation conditions it satisfies are given in terms of two distance functions on $\mathbf{R}^n$ having different types of homogeneity, puts this problem into a geometric setting where the incompatibility between the Euclidean balls $B(x,r) = \{y \in \mathbf{R}^n\colon |y-x| < r\}$ and the ρ-balls $B_\rho(x,r) = \{y \in \mathbf{R}^n\colon \rho(y-x) < r\}$ reflects the incompatibility between the radial cancellations of the kernel and the nonisotropic singularity it has at the origin.

In the case $n = 2$, it was shown in [3] that T is bounded on $L^p(\mathbf{R}^2)$ whenever

$$\left|\frac{1}{p} - \frac{1}{2}\right| < Q(\alpha,\beta) = \frac{2 + 2b_2 + \beta - 2\alpha}{2\beta},$$

and that T fails to be bounded on $L^p(\mathbf{R}^2)$ if

$$\left|\frac{1}{p} - \frac{1}{2}\right| > \max\bigl[Q(\alpha,\beta), Q(\alpha, 1 + 2\beta - b_2)\bigr].$$

In the present paper we generalize these results to dimensions $n \geqslant 2$:

Theorem 1. *Suppose $\beta > \alpha - b \geqslant 0$. Let*

$$Q(\alpha,\beta) = \frac{2b + \beta - 2\alpha}{2\beta}$$

and

$$Q_j = Q\bigl(\alpha, j(1+\beta) - b_1 - \cdots - b_j\bigr),$$

$j = 1, \ldots, n$. Then:

(i) *The principal value integral in* (1) *exists for every $x \in \mathbf{R}^n$.*

(ii) *T extends to a bounded linear operator on $L^p(\mathbf{R}^n)$ whenever*

$$\left|\frac{1}{p} - \frac{1}{2}\right| < Q(\alpha,\beta).$$

(iii) *If*

$$\left|\frac{1}{p} - \frac{1}{2}\right| > \max[Q_1, \ldots, Q_n],$$

then T is not bounded on $L^p(\mathbf{R}^n)$.

Notice that, putting $b_{n+1} = \infty$,

$$b_j \leqslant 1 + \beta \leqslant b_{j+1} \quad \Rightarrow \quad \max[Q_1, \dots, Q_n] = Q_j.$$

In particular, if $1 + \beta \leqslant b_2$, $\max[Q_1, \dots, Q_n] = Q_1 = Q(\alpha, \beta)$. So we have a sharp result (up to the end-points) in the case $1 + \beta \leqslant b_2$. When $b = n$ and $\rho(y) = |y|$, it is known that T extends to a bounded operator on $L^p(\mathbf{R}^n)$ whenever $1 < p < \infty$ and

$$\left| \frac{1}{p} - \frac{1}{2} \right| \leqslant Q(\alpha, n\beta),$$

and that T fails to be bounded on $L^p(\mathbf{R}^n)$ in the complementary range. For these results we refer the reader to [1,2,4,8].

Throughout this paper, the letter C denotes a constant that may change from line to line. In Section 2, constants are only allowed to depend on $b_1, \dots, b_n$, α, and ρ. In Section 3, constants are only allowed to depend on $b_1, \dots, b_n$, α, ρ, β, p, u, and γ. Also, constants are always positive numbers.

2. Properties of ρ

From the definition of $\{\delta_t\}$-homogeneous distance functions, it is easy to see that there are constants c_1 and c_2 such that

$$c_1^{-1}\tau(x) \leqslant \rho(x) \leqslant c_1\tau(x) \quad (x \in \mathbf{R}^n) \tag{2}$$

and

$$\rho(x + y) \leqslant c_2\big(\rho(x) + \rho(y)\big) \quad (x, y \in \mathbf{R}^n). \tag{3}$$

Also, for later reference, let us note that (2) implies

$$c_3|x| \leqslant \rho(x) \leqslant c_4|x|^{1/b_n} \quad \big(|x| \leqslant 1\big) \tag{4}$$

and

$$c_3|x|^{1/b_n} \leqslant \rho(x) \leqslant c_4|x| \quad \big(|x| \geqslant 1\big), \tag{5}$$

where $c_3 = (c_1^{-1})(\inf_{\mathbf{S}^{n-1}} \tau)$ and $c_4 = (c_1)(\sup_{\mathbf{S}^{n-1}} \tau)$.

The generalization of the results of [3] to higher dimensions is made possible mainly because of the following lemma that generalizes Lemma 2.1 in [3].

Lemma 1. *Suppose* $\alpha > b - 1$. *Then*[1]

$$\int_{\mathbf{S}^{n-1}} \frac{d\sigma(\theta)}{\rho(r\theta)^\alpha} \approx r^{b-n-\alpha}$$

and

[1] The notation $A \approx B$ means $C^{-1}B \leqslant A \leqslant CB$ for some constant C.

$$\int_{\mathbf{S}^{n-1}} \left| \frac{d}{dr} \rho(r\theta)^{-\alpha} \right| d\sigma(\theta) \leqslant C r^{b-n-\alpha-1}$$

for $0 < r \leqslant 1$, *where* σ *is surface measure on the sphere* $\mathbf{S}^{n-1}$.

Proof. The observation

$$\int_{\mathbf{S}^{n-1}} \frac{d\sigma(\theta)}{\rho(r\theta)^{\alpha}} = \int_{\mathbf{S}^{n-1}} \int_{\rho(r\theta)}^{\infty} \alpha \lambda^{-\alpha-1} \, d\lambda \, d\sigma(\theta) = \int_{0}^{\infty} \alpha \lambda^{-\alpha-1} \sigma(\Omega_\lambda) \, d\lambda,$$

where $\Omega_\lambda = \{\theta \in \mathbf{S}^{n-1}\colon\ \rho(r\theta) \leqslant \lambda\}$, suggests that we start by estimating $\sigma(\Omega_\lambda)$. Of course, $\sigma(\Omega_\lambda) = n|A_\lambda|$, where $A_\lambda = \{s\theta\colon\ 0 \leqslant s \leqslant 1,\ \theta \in \Omega_\lambda\}$ and $|A_\lambda|$ is the n-dimensional Lebesgue measure of A_λ.

The inequalities in (4) tell us that $c_3 r \leqslant \rho(r\theta) \leqslant c_4 r^{1/b_n}$. So, $\Omega_\lambda = \emptyset$ if $\lambda < c_3 r$ and

$$\int_{\mathbf{S}^{n-1}} \frac{d\sigma(\theta)}{\rho(r\theta)^{\alpha}} = \int_{c_3 r}^{\infty} \alpha \lambda^{-\alpha-1} \sigma(\Omega_\lambda) \, d\lambda.$$

It is easy to see (e.g., using (5)) that there is a constant c such that $\rho(x) \leqslant 1 \Rightarrow |x| \leqslant c$. If $\theta \in \Omega_\lambda$, then $\rho(\delta_{1/\lambda} r\theta) = \rho(r\theta)/\lambda \leqslant 1$, so that $|\delta_{1/\lambda} r\theta| \leqslant c$. Thus Ω_λ and A_λ are contained in the solid ellipsoid

$$\frac{x_1^2}{\lambda^{2b_1}} + \frac{x_2^2}{\lambda^{2b_2}} + \cdots + \frac{x_n^2}{\lambda^{2b_n}} \leqslant \left(\frac{c}{r}\right)^2.$$

Now if we simply estimate $|A_\lambda|$ by the measure of the ellipsoid, we obtain

$$\sigma(\Omega_\lambda) = n|A_\lambda| \leqslant C \frac{\lambda^b}{r^n},$$

and consequently

$$\int_{\mathbf{S}^{n-1}} \frac{d\sigma(\theta)}{\rho(r\theta)^{\alpha}} \leqslant \frac{C}{r^n} \int_{c_3 r}^{\infty} \lambda^{b-\alpha-1} \, d\lambda \leqslant C r^{b-n-\alpha}$$

provided $\alpha > b$, which falls short of the needed $\alpha > b - 1$. So we go back and observe that in fact $\Omega_\lambda \subset \mathbf{S}^{n-1} \cap B_\lambda$, where B_λ is the box centered at the origin with dimensions $2 \times (2c\lambda^{b_2}/r) \times \cdots \times (2c\lambda^{b_n}/r)$. We then get the better estimate

$$\sigma(\Omega_\lambda) = n|A_\lambda| \leqslant n|B_\lambda| \leqslant C \frac{\lambda^{b-1}}{r^{n-1}},$$

which leads to

$$\int_{\mathbf{S}^{n-1}} \frac{d\sigma(\theta)}{\rho(r\theta)^{\alpha}} \leqslant \frac{C}{r^{n-1}} \int_{c_3 r}^{\infty} \lambda^{b-\alpha-2} \, d\lambda \leqslant C r^{b-n-\alpha}$$

provided $\alpha > b - 1$.

In order to get the inequality in the other direction, we proceed as follows. Let

$$E=\left\{\theta=(\theta_1,\dots,\theta_n)\in\mathbf{S}^{n-1}\colon\ |\theta_i|\leqslant\frac{r^{b_i-1}}{\sqrt{n2^{b_i}}},\ i=2,3,\dots,n\right\}.$$

Then $\sigma(E)\approx r^{b-n}$. We shall prove that $\int_E\rho(r\theta)^{-\alpha}\,d\sigma(\theta)\geqslant Cr^{b-n-\alpha}$. In fact, we shall show that $\rho(r\theta)\approx r$ on E. Since, by (4), $\rho(r\theta)\geqslant Cr$, it will suffice to show that $\rho(r\theta)\leqslant Cr$ on E. Let $\theta\in E$. Then for $i=2,3,\dots,n$,

$$\theta_1^2=1-\sum_{k=2}^n\theta_k^2\geqslant 1-\sum_{k=2}^n\frac{1}{n}=\frac{1}{n}\geqslant\theta_i^2,$$

and using $0\leqslant r^2\theta_i^2\leqslant 1$, $0<1/b_i\leqslant 1$, we get

$$\big(r^2\theta_i^2\big)^{1/b_i}-r^2\big(1-\theta_i^2\big)\leqslant 2\big(r^2\theta_i^2\big)^{1/b_i}-r^2\leqslant 0,$$

so that

$$\big(r^2\theta_i^2\big)^{1/b_i}\leqslant r^2\big(1-\theta_i^2\big)=r^2\sum_{k\neq i}\theta_k^2\leqslant(n-1)r^2\theta_1^2.$$

It follows that $\tau(r\theta)\leqslant Cr$, which together with (2) gives $\rho(r\theta)\leqslant Cr$.

To estimate

$$\int_{\mathbf{S}^{n-1}}\left|\frac{d}{dr}\rho(r\theta)^{-\alpha}\right|d\sigma(\theta),$$

we now proceed as in [3]. First, the homogeneity of ρ,

$$\rho(\delta_t x)=t\rho(x)\quad(t>0,\ x\in\mathbf{R}^n),$$

tells us that

$$t^{b_j}\frac{\partial\rho}{\partial x_j}(\delta_t x)=t\frac{\partial\rho}{\partial x_j}(x)\quad(t>0,\ x\in\mathbf{R}^n-0)$$

for $j=1,\dots,n$. So, putting $t=1/\rho(x)$, we get

$$\left|\frac{\partial\rho}{\partial x_j}(x)\right|\leqslant\frac{C}{\rho(x)^{b_j-1}}\quad(x\in\mathbf{R}^n-0)$$

(here $C=\sup_{\rho(y)=1}|\partial\rho/\partial x_j(y)|$). Second, by (2),

$$|x_j|\leqslant\tau(x)^{b_j}\leqslant C\rho(x)^{b_j}\quad(x\in\mathbf{R}^n),$$

so that

$$\big|x\cdot\nabla\rho(x)\big|\leqslant C\rho(x)\quad(x\in\mathbf{R}^n-0).$$

Hence

$$\left|\frac{d}{dr}\rho(r\theta)\right|\leqslant C\frac{\rho(r\theta)}{r}\quad(r>0,\ \theta\in\mathbf{S}^{n-1}),\tag{6}$$

and it follows that

$$\int_{\mathbf{S}^{n-1}} \left| \frac{d}{dr} \rho(r\theta)^{-\alpha} \right| d\sigma(\theta) \leqslant C r^{b-n-\alpha-1} \quad (0 < r \leqslant 1)$$

by our previous estimate on

$$\int_{\mathbf{S}^{n-1}} \rho(r\theta)^{-\alpha}\, d\sigma(\theta).$$

3. Proof of Theorem 1

For $z = u + iv \in \mathbf{C}$, define

$$K_z(y) = \begin{cases} \rho(y)^{-z} e^{i/|y|^\beta} & \text{if } 0 < |y| \leqslant 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$L_z = \text{p.v.}\, K_z.$$

We need three facts about L_z. The first is that L_z exists and defines a tempered distribution when $u < \beta + b$, i.e.,

$$L_z \in \mathcal{S}' \quad \text{if } u < \beta + b, \tag{7}$$

the second is that

$$L_z = K_z \in L^1(\mathbf{R}^n) \quad \text{if } u < b, \tag{8}$$

and the third is that

$$\|\hat{L}_z\|_{L^\infty} \leqslant C\big(1 + |v|\big) \quad \text{if } u < \frac{\beta}{2} + b. \tag{9}$$

Clearly, Lemma 1 implies (8). Before we prove (7) and (9), however, let us notice that since $Tf = L_\alpha * f$ for $f \in \mathcal{S}$, part (i) of Theorem 1 follows immediately from (7). Also, part (ii) of Theorem 1 follows from (7)–(9) by using complex interpolation as in [3]. The argument goes as follows. Let $\gamma > 0$ and consider the family $\{R_z\}_{0 \leqslant u \leqslant 1}$ of analytic linear operators defined on $\mathcal{S}$ by

$$R_z f = M_z * f \quad \text{with } M_z = L_{(\beta/2)z + b - \gamma}.$$

Then (7) becomes

$$M_z \in \mathcal{S}' \quad \text{if } u < 2 + \frac{2\gamma}{\beta}.$$

Also, (8) and (9) become

$$M_z \in L^1(\mathbf{R}^n) \quad \text{if } u < \frac{2\gamma}{\beta}$$

and

$$\|\hat{M}_z\|_{L^\infty} \leqslant C\big(1+|v|\big) \quad \text{if } u < 1+\frac{2\gamma}{\beta}.$$

Thus R_{iv} is bounded on $L^1(\mathbf{R}^n)$, and R_{1+iv} is bounded on $L^2(\mathbf{R}^n)$ with operator norm that grows polynomially in v. Complex interpolation now tells us that R_u is bounded on $L^p(\mathbf{R}^n)$ whenever

$$0 \leqslant u \leqslant 1 \quad \text{and} \quad \frac{1}{p} = 1 - \frac{u}{2}.$$

Since $T = R_{(2/\beta)(\alpha-b+\gamma)}$, it follows that T is bounded on $L^p(\mathbf{R}^n)$ whenever

$$b-\alpha \leqslant \gamma \leqslant \frac{\beta}{2} + b - \alpha \quad \text{and} \quad \frac{1}{p} - \frac{1}{2} = \frac{2b+\beta-2\alpha-2\gamma}{2\beta}.$$

But $b-\alpha \leqslant 0$, so T is bounded on $L^p(\mathbf{R}^n)$ whenever

$$0 < \gamma \leqslant \frac{\beta}{2} + b - \alpha \quad \text{and} \quad \frac{1}{p} - \frac{1}{2} = \frac{2b+\beta-2\alpha-2\gamma}{2\beta}.$$

Hence T is bounded on $L^p(\mathbf{R}^n)$ whenever

$$0 \leqslant \frac{1}{p} - \frac{1}{2} < \frac{2b+\beta-2\alpha}{2\beta}.$$

Finally, duality implies that T is bounded on $L^p(\mathbf{R}^n)$ whenever

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{2b+\beta-2\alpha}{2\beta}.$$

It is now time to prove (7). If $u < b$, there is nothing to prove by (8). Suppose $u \geqslant b$ and $f \in \mathcal{S}$. Let $0 < \epsilon < 1$. Changing

$$I_\epsilon = \int\limits_{\epsilon \leqslant |y| \leqslant 1} K_z(y) f(y)\, dy$$

into polar coordinates, we can write

$$I_\epsilon = \int\limits_{\mathbf{S}^{n-1}} \int\limits_{\epsilon}^{1} \psi_\theta(r) F'(r)\, dr\, d\sigma(\theta),$$

where

$$\psi_\theta(r) = r^{n-1} f(r\theta)\rho(r\theta)^{-z} \quad \text{and} \quad F(r) = \int\limits_{\epsilon}^{r} e^{i/s^\beta}\, ds.$$

A simple computation (together with (6)) shows that

$$\left|\psi_\theta'(r)\right| \leqslant C\left(|v| r^{n-2}\rho(r\theta)^{-u} + r^{n-1}\left|\frac{d}{dr}\rho(r\theta)^{-u}\right|\right)\|f\|_{C^1},$$

where $\|f\|_{C^1}$ is the C^1 norm of f. Lemma 1 then tells us that

$$\int_{\mathbf{S}^{n-1}} \left|\psi_\theta'(r)\right| dr \leqslant C\bigl(1+|v|\bigr) r^{b-u-2} \|f\|_{C^1}$$

and

$$\int_{\mathbf{S}^{n-1}} \left|\psi_\theta(r)\right| dr \leqslant C r^{b-u-1} \|f\|_{C^1}.$$

Also, by Van der Corput's lemma (see [6]),

$$\left|F(r)\right| \leqslant C r^{1+\beta},$$

and so integrating by parts we get

$$\left| \int_\epsilon^1 \psi_\theta(r) F'(r)\, dr \right| \leqslant C\left(\left|\psi_\theta(1)\right| + \left|\psi_\theta(\epsilon)\right| \epsilon^{1+\beta} + \int_\epsilon^1 \left|\psi_\theta'(r)\right| r^{1+\beta}\, dr \right),$$

which in turn leads to

$$|I_\epsilon| \leqslant C\left(1 + \epsilon^{\beta+b-u} + \bigl(1+|v|\bigr) \int_\epsilon^1 r^{\beta+b-u-1}\, dr \right) \|f\|_{C^1}.$$

Thus $L_z(f) = \lim_{\epsilon\to 0} I_\epsilon$ exists and satisfies

$$\left|L_z(f)\right| \leqslant C\bigl(1+|v|\bigr)\|f\|_{C^1}$$

if $\beta + b - u > 0$.

The proof of (9) is exactly the same, except that now

$$\psi_\theta(r) = r^{n-1}\rho(r\theta)^{-z} \quad \text{and} \quad F(r) = \int_\epsilon^r e^{i(s^{-\beta} - 2\pi s\, \xi\cdot\theta)}\, ds,$$

and one has to apply Van der Corput's lemma using the second derivative of the phase rather than the first to get $|F(r)| \leqslant C r^{1+\beta/2}$.

Our task for the remaining part of this paper is to prove part (iii) of Theorem 1. Our plan is to localize T, as in [5], to subsets of $\mathbf{R}^n$ where the phase function $1/|y|^\beta$ is almost stationary. To do this we need the dilation methods of [3].

Let $1 \leqslant j \leqslant n$ and suppose $b_j \leqslant 1+\beta < b_{j+1}$ ($b_{n+1} = \infty$). We define a dilation group $\{d_t^{(j)}\}_{t>0}$ by

$$\begin{aligned} d_t^{(j)} : x &= (x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) \\ &\mapsto (t^{1+\beta}x_1, \ldots, t^{1+\beta}x_j, t^{b_{j+1}}x_{j+1}, \ldots, t^{b_n}x_n). \end{aligned}$$

To simplify the notation we are going to write d_t for $d_t^{(j)}$. Following Stein and Wainger [7], we define a $\{d_t\}$-homogeneous distance function ρ_j as follows. If $x \neq 0$, $|d_{1/t}x|$ as a function of t is strictly decreasing and is therefore equal to 1 for a unique $t \in (0, \infty)$. Define

$\rho_j(x)$ to be this unique t. If $x=0$, set $\rho_j(x)=0$. Notice that if $0<|x|\leqslant 1$, $t_0=\rho_j(x)^{-1}$, $t_1=|x|^{-1/(\beta+1)}$, and $t_2=|x|^{-1/b_n}$, then

$$|d_{t_1}x|\geqslant 1,\qquad |d_{t_2}x|\leqslant 1,\quad \text{and}\quad |\delta_{t_0}x|\leqslant |d_{t_0}x|=1.$$

Hence

$$|x|^{1/(\beta+1)}\leqslant \rho_j(x)\leqslant |x|^{1/b_n}\quad \bigl(|x|\leqslant 1\bigr) \tag{10}$$

and

$$\rho(x)=t_0^{-1}\rho(\delta_{t_0}x)\leqslant t_0^{-1}c_4=c_4\rho_j(x)\quad \bigl(|x|\leqslant 1\bigr), \tag{11}$$

where we have also used (4).

Fix a nonnegative C_0^∞ function ϕ such that $\phi(y)=1$ if $\rho_j(y)\leqslant 1/2$, and $\phi(y)=0$ if $\rho_j(y)\geqslant 1$. Set

$$\phi_\epsilon(y)=\phi(d_{1/\epsilon}y).$$

Now suppose $0<\epsilon<1$ and the positive integer k are such that the interval

$$I_k=\left[\frac{1}{(2\pi k+\pi/3)^{1/\beta}}+\epsilon^{1+\beta},\ \frac{1}{(2\pi k-\pi/3)^{1/\beta}}-\epsilon^{1+\beta}\right]$$

has positive length and is contained in the interval $(\epsilon,1)$. We are going to estimate $T\phi_\epsilon$ from below on the ring $|x|\in I_k$.

Suppose $|x|\in I_k$. Then

$$\begin{aligned}\bigl|T\phi_\epsilon(x)\bigr| &= \left|\int\limits_{\rho_j(x-y)\leqslant\epsilon}\frac{e^{i/|y|^\beta}}{\rho(y)^\alpha}\phi_\epsilon(x-y)\,dy\right|\\ &\geqslant \frac{1}{2}\int\limits_{\rho_j(x-y)\leqslant\epsilon}\frac{1}{\rho(y)^\alpha}\phi_\epsilon(x-y)\,dy\geqslant \frac{C}{\rho(x)^\alpha}\int\phi_\epsilon(y)\,dy.\end{aligned}$$

The first inequality holds because, by (10),

$$\begin{aligned}\rho_j(x-y)\leqslant\epsilon\quad &\Rightarrow\quad |x-y|\leqslant\epsilon^{1+\beta}\quad\Rightarrow\quad |y|\in I_k\\ &\Rightarrow\quad \cos\frac{1}{|y|^\beta}\geqslant\frac{1}{2}.\end{aligned}$$

The second inequality holds because

$$\begin{aligned}\rho(y) &\leqslant C\rho(y-x)+C\rho(x) && \bigl(\text{by (3)}\bigr)\\ &\leqslant C\rho_j(y-x)+C\rho(x) && \bigl(\text{by (11)}\bigr)\\ &\leqslant C\epsilon+C\rho(x)\\ &\leqslant C|x|+C\rho(x) && \bigl(\text{because } |x|\in I_k\subset(\epsilon,1)\bigr)\\ &\leqslant C\rho(x) && \bigl(\text{by (4)}\bigr).\end{aligned}$$

Hence

$$\bigl|T\phi_\epsilon(x)\bigr|\geqslant C\frac{\epsilon^{j(1+\beta)+b_{j+1}+\cdots+b_n}}{\rho(x)^\alpha}$$

for $|x|\in I_k$.

Now as was observed in [3] (Lemma 3.1), there are constants A and B, with $B < A^{1/\beta} < 1$, such that if

$$J_k = \left[\frac{1}{(2\pi(k+1) - \pi/3)^{1/\beta}} - \epsilon^{1+\beta}, \frac{1}{(2\pi k + \pi/3)^{1/\beta}} + \epsilon^{1+\beta} \right],$$

and if $0 < \epsilon < B$, then

$$[A^{-1/\beta}\epsilon, 2A^{-1/\beta}\epsilon] \subset [A^{-1/\beta}\epsilon, 7^{-1/\beta}] \subset I_{k'} \cup \left[\bigcup_{k=1}^{k'-1} (I_k \cup J_k) \right] \tag{12}$$

and

$$|J_k| \leqslant C|I_{k+1}| \quad \text{for } 1 \leqslant k \leqslant k' - 1, \tag{13}$$

where k' is the largest integer k such that $1 \leqslant k \leqslant A\epsilon^{-\beta}$. From this it follows that if $0 < \epsilon < B$ and $p > 1$, then

$$\begin{aligned}
\|T\phi_\epsilon\|_{L^p}^p &\geqslant C\epsilon^{p(j(1+\beta)+b_{j+1}+\cdots+b_n)} \sum_{k=1}^{k'} \int_{|x|\in I_k} \frac{dx}{\rho(x)^{\alpha p}} \\
&= C\epsilon^{p(j(1+\beta)+b_{j+1}+\cdots+b_n)} \sum_{k=1}^{k'} \int_{\mathbf{S}^{n-1}} \int_{r\in I_k} \frac{r^{n-1}}{\rho(r\theta)^{\alpha p}} \, dr \, d\sigma(\theta) \\
&= C\epsilon^{p(j(1+\beta)+b_{j+1}+\cdots+b_n)} \sum_{k=1}^{k'} \int_{r\in I_k} r^{n-1} \int_{\mathbf{S}^{n-1}} \frac{d\sigma(\theta)}{\rho(r\theta)^{\alpha p}} \, dr \\
&\geqslant C\epsilon^{p(j(1+\beta)+b_{j+1}+\cdots+b_n)} \sum_{k=1}^{k'} \int_{r\in I_k} r^{b-\alpha p-1} \, dr \\
&\geqslant C\epsilon^{p(j(1+\beta)+b_{j+1}+\cdots+b_n)} \int_{A^{-1/\beta}\epsilon}^{7^{-1/\beta}} r^{b-\alpha p-1} \, dr \\
&\geqslant C\epsilon^{p(j(1+\beta)+b_{j+1}+\cdots+b_n)} \epsilon^{b-\alpha p},
\end{aligned}$$

where in the line before the last we used (12) and (13), and in the line before that we used Lemma 1.

Finally, suppose the estimate

$$\|T\phi_\epsilon\|_{L^p} \leqslant C\|\phi_\epsilon\|_{L^p}$$

holds for some $1 < p \leqslant 2$. Then

$$\epsilon^{p(j(1+\beta)+b_{j+1}+\cdots+b_n)} \epsilon^{b-\alpha p} \leqslant C\epsilon^{j(1+\beta)+b_{j+1}+\cdots+b_n}$$

for $0 < \epsilon < B$. This implies

$$\frac{1}{p} - \frac{1}{2} \leqslant Q_j.$$

Thus T is not bounded on $L^p(\mathbf{R}^n)$ if $1 < p \leqslant 2$ and

$$\frac{1}{p} - \frac{1}{2} > \max[Q_1, \ldots, Q_n].$$

The corresponding inequality for $p > 2$ is then obtained by using duality. □

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