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Journal of Algebra

www.elsevier.com/locate/jalgebra

Powerful p -groups have non-inner automorphisms of order p and some cohomology[☆]

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ARTICLE INFO

Article history:

Received 20 January 2009

Available online 18 November 2009

Communicated by Gernot Stroth

Keywords:

Automorphisms of p -groupsFinite p -groups

Non-inner automorphisms

Powerful p -groups p -Central groups

ABSTRACT

In this paper we study the longstanding conjecture of whether there exists a non-inner automorphism of order p for a finite non-abelian p -group. We prove that if G is a finite non-abelian p -group such that $G/Z(G)$ is powerful then G has a non-inner automorphism of order p leaving either $\Phi(G)$ or $\Omega_1(Z(G))$ elementwise fixed. We also recall a connection between the conjecture and a cohomological problem and we give an alternative proof of the latter result for odd p , by showing that the Tate cohomology $H^n(G/N, Z(N)) \neq 0$ for all $n \geq 0$, where G is a finite p -group, p is odd, $G/Z(G)$ is p -central (i.e., elements of order p are central) and $N \triangleleft G$ with G/N non-cyclic.

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1. Introduction and results

Let p be a prime number and let G be a non-abelian finite p -group. A longstanding conjecture asserts that G admits a non-inner automorphism of order p (see also Problem 4.13 of [13]). By a famous result of W. Gaschütz [3], non-inner automorphisms of G of p -power order exist. M. Deaconescu and G. Silberberg [2] reduced the verification of the conjecture to the case in which $C_G(Z(\Phi(G))) = \Phi(G)$. H. Liebeck [8] has shown that finite p -groups of class 2 with $p > 2$ must have a non-inner automorphism of order p fixing the Frattini subgroup elementwise. In [1] we showed the validity of the conjecture when G is nilpotent of class 2. In fact we proved that for any prime number p , every finite non-abelian p -group G of class 2 has a non-inner automorphism of order p leaving either the Frattini subgroup $\Phi(G)$ or $\Omega_1(Z(G))$ elementwise fixed.

[☆] This research was in part supported by a grant from IPM (No. 87200118).

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In Section 2 of this paper, we give some classes of p -groups G for which the conjecture holds. We prove the validity of the conjecture whenever $G/Z(G)$ is powerful, or of coclass 1 or G is 2-generated and nilpotent of class 2 with non-cyclic center (see Theorems 2.9 and 2.6 and Corollary 2.4, below).

Therefore, on the negative side, if the conjecture had a counter-example, it would not be a p -group of types above. On the positive side, we may mention nothing expect of only stating our intuition which seem very optimistic: from a result of Mann and Lubotzky [10] one knows that any finite p -group G has a characteristic powerful subgroup M whose index is bounded by a function of the rank of G and p ; so maybe, one can prove the validity of the conjecture by an appropriate induction argument on the rank of G and/or finding a way to lift a non-inner automorphism of order p of M to one of G .

The proof of Gaschütz's result [3] relies on a cohomological property of finite p -groups. This may suggest that the cohomological tools may be still useful to attack on the conjecture. On the other hand, by using Deaconescu and Silberberg's result and a cohomological property of regular p -groups proved by P. Schmid [12], the validity of the conjecture is shown for regular p -groups. The question of which other classes of finite p -groups have the same cohomological property not only has its own interest and is asked in [12] but also having proved the cohomological property like regular p -groups, it may be hoped (by the following means) to prove the conjecture. So we are also motivated to study the latter question in Section 3. We explain the cohomological property of regular p -groups and its connection with the conjecture.

We first recall some definitions and results concerning Tate cohomology of groups. Let Q and A be finite groups where A is abelian. If Q acts on A (from the right) as a group, then A can be viewed as a (right) Q -module. We denote by A_Q the submodule $\{a \in A \mid a^x = a \text{ for all } x \in Q\}$ of fixed points under Q . The trace map $a \mapsto a \sum_{x \in Q} x$ of A is written $\tau = \tau_Q$, and its image will be denoted by A^τ . In dealing with Tate cohomology, by dimension-shifting it is often enough to consider the situation in dimension 0. Recall that $H^0(Q, A) = A_Q/A^\tau$. If Q and A are p -groups, by a theorem of Gaschütz and Uchida A is cohomologically trivial provided the Tate cohomology $H^n(Q, A) = 0$ for just one integer $n \geq 0$ (cf. [4, p. 110]). Let G be a group and N a normal subgroup of G . Then G/N may act on $Z(N)$ as follows: $a^{gN} = a^g$ for all $a \in Z(N)$ and $g \in G$. Thus $Z(N)$ is a G/N -module via this action. The group of all crossed homomorphisms of G/N to $Z(N)$ is denoted by $Z^1(G/N, Z(N))$ and $B^1(G/N, Z(N))$ is the subgroup of all principal crossed homomorphisms.

In [12], P. Schmid proved that if G is a regular p -group and $N \triangleleft G$ such that G/N is not cyclic then the Tate cohomology $H^n(G/N, Z(N)) \neq 0$ for all n . He then conjectured that

Conjecture 1.1. *Let G be a finite non-regular p -group. Then $H^n(\frac{G}{\Phi(G)}, Z(\Phi(G))) \neq 0$ for all integer n .*

The following question which naturally arises from the work of Schmid will be studied in Section 3.

Question 1.2. *For which finite p -groups G and which normal subgroups N of G we have $H^n(\frac{G}{N}, Z(N)) \neq 0$ for all integers n .*

A relation between non-triviality of Tate cohomology $H^n(G/N, Z(N))$ and the existence of non-inner automorphisms of order p in $Aut(G)$ is behind the using of the following well-known result and its corollary.

Proposition 1.3. (See e.g., [11, Result 1.1].) *Suppose that N is a normal subgroup of a group G . Then there is a natural isomorphism $\varphi : Z^1(\frac{G}{N}, Z(N)) \rightarrow C_{Aut(G)}(N; G/N)$ given by $g^{\varphi(f)} = g(gN)^f$ for $g \in G$, $f \in Z^1(\frac{G}{N}, Z(N))$. The image of $B^1(\frac{G}{N}, Z(N))$ under φ is the group of inner automorphisms of G induced by $Z(N)$.*

Here $C_{Aut(G)}(N; G/N)$ denotes all automorphisms α of G such that $x^\alpha = x$ for all $x \in N$ and $g^{-1}g^\alpha \in N$ for all $g \in G$.

Proposition 1.4. (See Corollary 1.2 of [11].) Assume that N is a normal subgroup of a group G such that $C_G(N) = Z(N)$ and $H^1(G/N, Z(N)) \neq 0$. Then $C_{\text{Aut}(G)}(N; G/N)$ is not contained in $\text{Inn}(G)$

So, for applying Proposition 1.4, we need to have a normal subgroup $N \triangleleft G$ such that

$$(1) \quad H^1(G/N, Z(N)) \neq 0 \quad \text{and} \quad (2) \quad C_G(N) = Z(N).$$

By Deaconescu and Silberberg’s result [2], $\Phi(G)$ satisfies the condition (2) and so we should verify (1). Of course non-triviality of $H^1(G/N, Z(N))$ is only a sufficient condition to have a non-inner p -automorphism (not necessarily of order p) and it is not sufficient for our purpose. Therefore according to Propositions 1.3 and 1.4, the condition

$$(3) \quad \Omega_1\left(Z^1\left(\frac{G}{N}, Z(N)\right)\right) \not\subseteq B^1\left(\frac{G}{N}, Z(N)\right)$$

with together conditions (1) and (2) are sufficient to have a non-inner automorphism of order p leaving both N and G/N elementwise fixed. A condition which implies (3) is the being elementary abelian of $Z^1(G/N, Z(N))$. This is proved for regular p -groups G in [12] whenever $N = \Phi(G)$.

In Section 3, we give classes of p -groups satisfying Schmid’s cohomological conclusion requested in Question 1.2. In particular we prove p -groups of class 2 and for p -groups of class 3 whenever $p > 2$ satisfy this cohomological property (see Theorem 3.6, below). By using this result, we give an alternative proof for the validity of the conjecture for p -groups (p odd) with a powerful central factor.

Throughout p always denotes a prime number. For a finite group G , we denote by $d(G)$, $Z(G)$, G' , $\Phi(G)$, $\text{Aut}(G)$ and $\text{Inn}(G)$, the minimum number of generators, the center, the derived subgroup, the Frattini subgroup, the automorphism group, the inner automorphism group of G , respectively. If G is a p -group, $\Omega_1(G)$ denotes the subgroup generated by elements of order p . For two groups G and H , $\text{Hom}(G, H)$ denotes the set of group homomorphisms from G to H . If H is abelian $\text{Hom}(G, H)$ has a group structure with pointwise multiplication. The unexplained notation is standard and follows that of Gorenstein [5].

2. Finite p -groups without non-inner automorphism of order p and the existence of non-inner automorphism of order p in powerful p -groups

Let G be a group and let A be a normal abelian subgroup of G . Then it is easy to see that the set $[A, x] = \{[a, x] \mid a \in A\}$ is a subgroup of A for any element $x \in G$. Lemmas 2.1, 2.2 and Corollary 2.3 may be well known, but we could not find them as the following forms in the published literatures.

Lemma 2.1. Let G be a finite p -group such that G has no non-inner automorphism of order p leaving $\Phi(G)$ elementwise fixed. Then $\Omega_1(Z(G)) \leq [Z(M), g]$ for every maximal subgroup M and every $g \in G \setminus M$. In particular $\Omega_1(Z(G)) \leq G'$.

Proof. By the main result of [2], we have $\Phi(G) = C_G(Z(\Phi(G)))$ and so in particular $Z(G) \leq \Phi(G)$. Take a maximal subgroup M of G and $g \in G \setminus M$. Suppose, for a contradiction, that there exists an element $z \in \Omega_1(Z(G)) \setminus [Z(M), g]$. Note that $z \in M$, as $Z(G) \leq \Phi(G)$. It is now easy to see that the map α on G defined by $(mg^i)^\alpha = mg^i z^i$ for all $m \in M$ and integers i , is an automorphism of order p leaving M elementwise fixed. If α is inner, then there exists $h \in G$ such that $m^h = m$ for all $m \in M$ and $z = [h, g]$. Since $G = M \langle g \rangle$, $h = m' g^i$ for some $i \in \{0, \dots, p - 1\}$. As $h \in C_G(M)$, we have $[m', m' g^i] = 1$ which implies that $[m', g^i] = 1$. On the other hand, $[m' g^i, g] = z$ and so $[m', g] = z$, as z is central. Now it follows from $[m', g^i] = 1$ that $[m', g]^i = 1$ and so $z^i = 1$. Thus $i = 0$ as $o(z) = p$. Therefore $h = m' \in Z(M)$ and so $z \in [Z(M), g]$, a contradiction. Therefore α is non-inner, again a contradiction. This completes the proof. \square

Lemma 2.2. *Let G be a finite p -group such that G has no non-inner automorphism of order p leaving $\Phi(G)$ elementwise fixed. Then*

$$\Omega_1(Z(\text{Inn}(G))) \cong \Omega_1\left(\frac{Z_2(G)}{Z(G)}\right) \cong \underbrace{\Omega_1(Z(G)) \times \cdots \times \Omega_1(Z(G))}_{d(G)\text{-times}}.$$

Proof. Let C be the group of all automorphisms ϕ of G such that $x^{-1}x^\phi \in \Omega_1(Z(G))$ and $t^\phi = t$ for all $x \in G$ and $t \in \Omega_1(Z(G))$. As $\Omega_1(Z(G))$ is a characteristic subgroup of G , C is a normal subgroup of $\text{Aut}(G)$. Note that every such automorphism ϕ leaves $\Phi(G)$ elementwise fixed: for, as ϕ is a central automorphism of G , it fixes G' elementwise; and since $x^{-1}x^\phi$ is a central element of order at most p , $1 = (x^{-1}x^\phi)^p = x^{-p}(x^p)^\phi$. Thus ϕ fixes G^p elementwise. Hence ϕ leaves $\Phi(G) = G'G^p$ elementwise fixed. Now the map ψ from

$$T := \text{Hom}\left(\frac{G}{\Omega_1(Z(G))}, \Omega_1(Z(G))\right)$$

to C defined by $g^{\psi(f)} = g(g\Omega_1(Z(G)))^f$ for all $f \in T$ and $g \in G$ is a group isomorphism. The abelian group T is isomorphic to

$$\underbrace{\Omega_1(Z(G)) \times \cdots \times \Omega_1(Z(G))}_{d'\text{-times}},$$

where d' is the rank of $\frac{G}{\Omega_1(Z(G))G'}$. By Lemma 2.1, $\Omega_1(Z(G)) \leq G' \leq \Phi(G)$, and from which it follows that $d' = d(G)$. Thus C is an elementary abelian p -group of $\text{Aut}(G)$ and so by hypothesis, $C \leq \text{Inn}(G)$ which implies that $C \leq \Omega_1(Z(\text{Inn}(G)))$. Now let ρ_g be the inner automorphism of G induced by g such that $\rho_g \in \Omega_1(Z(\text{Inn}(G)))$. Thus $g^p \in Z(G)$ and $x^{g^h} = x^{hg}$ for all $x, h \in G$. It follows easily from these relations that $\rho_g \in C$. Hence $C = \Omega_1(Z(\text{Inn}(G)))$. This completes the proof. \square

The contents of the following result must be well known.

Corollary 2.3. *Let G be a finite p -group such that G has no non-inner automorphism of order p leaving $\Phi(G)$ elementwise fixed. Then $d\left(\frac{Z_2(G)}{Z(G)}\right) = d(Z(G)) \cdot d(G)$.*

Proof. Note that for a finite abelian p -group A , we have $d(A) = d(\Omega_1(A))$. Now the proof follows from Lemma 2.2. \square

A finite p -group G of order p^n , $n > 2$, is called of coclass c whenever G is nilpotent of class $n - c$.

Corollary 2.4. *Let G be a finite non-abelian p -group of coclass 1. Then G has a non-inner automorphism of order p leaving $\Phi(G)$ elementwise fixed.*

Proof. Since G is of maximal class, we have that $|Z_2(G)/Z(G)| = p$ and $d(G) = 2$. Now Corollary 2.3 completes the result. \square

Theorem 2.5. *Let G be a finite non-abelian p -group of coclass c . If G has no non-inner automorphism of order p leaving $\Phi(G)$ elementwise fixed, then*

$$d(Z(G))(d(G) + 1) \leq c + 1.$$

Proof. Let $d = d(G)$, $\ell = d(Z(G))$ and $|G| = p^n$. By Lemma 2.2, $|Z_2(G)/Z(G)| \geq p^{\ell d}$. Since G is of coclass c and $|G/Z_{n-c-1}(G)| \geq p^2$,

$$p^\ell \cdot p^{\ell d} \cdot p^{n-c-3} \cdot p^2 \leq |Z(G)| |Z_2(G)/Z(G)| \cdots |G/Z_{n-c-1}(G)| = p^n.$$

Thus $p^{\ell+d\ell+n-c-3+2} \leq p^n$ and so $\ell(d+1) \leq c+1$. This completes the proof. \square

A p -group G is called p -central whenever $\Omega_1(G) = \Omega_1(Z(G))$.

Theorem 2.6. *Let G be a non-abelian finite p -group such that $G/Z(G)$ is powerful. If either $p > 2$ or $Z(G)$ is not cyclic then $\text{Aut}(G)$ contains a non-inner automorphism of order p leaving $\Phi(G)$ elementwise fixed; and if $p = 2$, $\text{Aut}(G)$ contains a non-inner automorphism of order 2 leaving either $\Phi(G)$ or $\Omega_1(Z(G))$ elementwise fixed. There is a powerful 2-group G of class 2 such that the only automorphisms of order 2 leaving $\Phi(G)$ elementwise fixed are inner automorphisms of G .*

Proof. Suppose first that $\text{Aut}(G)$ contains no non-inner automorphism of order p leaving $\Phi(G)$ elementwise fixed. By [2, Theorem], we may assume that $\Phi(G) = C_G(Z(\Phi(G)))$. If $d(G) = d$, then $d(G/Z(G)) \leq d$ and it follows from [10, 1.12 and 4.2.2] that $d(Z(G/Z(G))) \leq d$. Now Corollary 2.3 implies that $d(Z(\frac{G}{Z(G)})) \geq d(Z(G))d$. This completes the proof whenever $Z(G)$ is non-cyclic. Thus, from now on, we further assume that $Z(G)$ is cyclic and $d(Z(G/Z(G))) = d$. Now we show that $\bar{G} = G/Z(G)$ is p -central. Let x be an element of order p in \bar{G} . If $x \notin Z(\bar{G})$, then $d(\langle x, Z(\bar{G}) \rangle) = d+1$, a contradiction, as $d(\langle x, Z(\bar{G}) \rangle) \leq d$ by [10, 1.12 and 4.2.2]. It follows that $\Omega_1(\bar{G}) = \Omega_1(Z(\bar{G}))$.

Let H be the subgroup of G such that $H/Z(G) = \Omega_1(Z_2(G)/Z(G))$. Clearly $H \leq Z_2(G)$ and $H/Z(G)$ is an elementary abelian group of rank d and since G is non-abelian, $d \geq 2$. Also $[H, G] \leq \Omega_1(Z(G))$ and $C_G(h)$ is a maximal subgroup of G for all $h \in H \setminus Z(G)$: for the map $x \mapsto [h, x]$ is a group homomorphism from G onto the cyclic group $\Omega_1(Z(G))$ of order p with the kernel $C_G(h)$.

We now show the conclusion for $p > 2$. We need to prove the following:

(*) There exist $h \in H \setminus Z(G)$ and $x \in G \setminus C_G(h)$ such that $h^p = 1$ and $(xh)^p = x^p$.

Since $d \geq 2$, there are $a, b \in H \setminus Z(G)$ such that $ab^k, a^k b \in H \setminus Z(G)$ for all integers k . As $a^p, b^p \in Z(G)$ and $Z(G)$ is cyclic, $a^p = b^{ps}$ or $a^{ps} = b^p$ for some integer s . Assume, without loss of generality, that $a^p = b^{ps}$. Now as $[a, b^{-s}] \in Z(G)$, $a^p \in Z(G)$ and p is assumed to be odd, we may write

$$(ab^{-s})^p = a^p b^{-ps} [b^{-s}, a]^{\binom{p}{2}} = [b^{-s}, a^p]^{\frac{p-1}{2}} = 1.$$

Thus we have proved that if $p > 2$, there is $h \in H \setminus Z(G)$ such that $h^p = 1$. Let x be any element of $G \setminus C_G(h)$. Since $[h, x] \in Z(G)$ and $h^p = 1$ we have

$$(xh)^p = x^p h^p [h, x]^{\binom{p}{2}} = x^p [h^p, x]^{\frac{(p-1)}{2}} = x^p.$$

This completes the proof of (*).

Now by (*), it is easy to check that the map β on G defined by $(ux^i)^\beta = u(xh)^i$ for all $u \in C_G(h)$ and all integers i , defines an automorphism of order p which leaves $\Phi(G)$ elementwise fixed. If β were inner, then β would be conjugation by some element $y \in G \setminus Z_2(G)$ with $y^p \in Z(G)$. Since $G/Z(G)$ is p -central, it follows that $y \in Z_2(G)$, which is impossible. This completes the proof for the case $p > 2$.

From now on, we assume that $p = 2$ and we suppose, for a contradiction, that $\text{Aut}(G)$ contains no non-inner automorphism of order p leaving either $\Phi(G)$ or $\Omega_1(Z(G))$ elementwise fixed. Note that $\Omega_1(Z(G))$ has only one non-trivial element z .

First suppose that H is non-abelian. Then there are $a, b \in H \setminus Z(G)$ such that $z = [a, b]$. Let $K = \langle a, b \rangle$. Note that $[H, G] = \langle [a, b] \rangle = K'$. Now by an argument similar to [1, Remark 2.2], we have that $G = KC_G(K)$. We give it here for the reader's convenience: for any $x \in G$, we have $[a, x] = [a, b]^s$ and $[b, x] = [a, b]^t$ for some integers s, t . Then $[a, b^{-s}a^fx] = 1$ and $[b, b^{-s}a^fx] = 1$. Hence $b^{-s}a^fx \in C_G(\langle a, b \rangle)$ and so $G = \langle a, b \rangle C_G(\langle a, b \rangle)$. Note that $Z(K) = \langle [a, b], a^2, b^2 \rangle = \Phi(K)$. Now it follows from [2, Theorem] that K has a non-inner automorphism of order 2 leaving $Z(K)$ elementwise fixed. Now by [1, Remark 2.5], we have that G has a non-inner automorphism of order 2 leaving $Z(G)$ elementwise fixed, a contradiction.

Hence, we may assume that H is abelian. Since $Z(G)$ is cyclic and $H/Z(G)$ is elementary abelian of rank d , one of the following may happen:

- (1) $H = \langle h_1 \rangle \times \cdots \times \langle h_d \rangle \times \langle h_{d+1} \rangle$, where h_1, \dots, h_d are all of order 2 and $Z(G) = \langle h_{d+1} \rangle$.
- (2) $H = \langle h_1 \rangle \times \cdots \times \langle h_{d-1} \rangle \times \langle h_d \rangle$, where h_1, \dots, h_{d-1} are all of order 2 and $Z(G) = \langle h_d^2 \rangle$.

First suppose that $C_G(h_i) \neq C_G(h_j)$ for some i and j such that $i, j \leq d$ in the case (1) and $d \geq 3$ and $i, j \leq d - 1$ in the case (2). Since $C_G(h_i)$ and $C_G(h_j)$ are maximal subgroups of G , there exist elements $x_i \in C_G(h_i) \setminus C_G(h_j)$ and $x_j \in C_G(h_j) \setminus C_G(h_i)$. Thus $[x_i, h_j] = [x_j, h_i] = z$ and $G/C_G(h_i, h_j) \cong C_2 \times C_2$. Now it is easy to see that the map ϕ on G defined by $(ux_i^\ell x_j^k)^\phi = u(x_i h_i)^\ell (x_j h_j)^k$ for all $u \in C_G(h_i, h_j)$ and all integers ℓ, k is an automorphism of order 2 leaving $\Phi(G)$ elementwise fixed. If ϕ were inner, we would have an element $g \in G \setminus H$ with $g^2 \in Z(G)$, which is impossible as $G/Z(G)$ is 2-central.

Therefore, we may further assume that in the case (1): $C_G(h_1) = \cdots = C_G(h_d)$ and in the case (2): $C_G(h_1) = \cdots = C_G(h_{d-1})$. Now we prove that the case (1) does not happen and in the case (2) we should have $d = 2$. Suppose not. Then $M = C_G(h_1) = C_G(h_2)$ is a maximal subgroup of G and $h_1, h_2, h_1 h_2 \in H \setminus Z(G)$ are all of order 2. Let x be any element of $G \setminus M$. Then $[x, h_1] = [x, h_2] = z$ and so $[x, h_1 h_2] = 1$ and so $(x h_1 h_2)^2 = x^2$. It follows that the map α on G defined by $(ux^i)^\alpha = u(x h_1 h_2)^i$ for all $u \in M$ and all integers i , is an automorphism of G of order 2 leaving $\Phi(G)$ elementwise fixed. If α were inner, we would have an element $g \in G \setminus H$ with $g^2 \in Z(G)$, a contradiction.

Thus it remains to settle the case (2) for $d = 2$; i.e., $H = \langle h_1 \rangle \times \langle h_2 \rangle$, where $h_1^2 = 1$ and $Z(G) = \langle h_2^2 \rangle$. Now we prove that G is also powerful. For this, it is sufficient to show $Z(G) \leq G^4$, since $G'/Z(G) \leq G^4 Z(G)$ as $G/Z(G)$ is powerful. Note that since $[h_2, x^2] = 1$ for all $x \in G$, we have $h_2 \in C_G(\Phi(G)) = Z(\Phi(G))$. Hence $h_2 Z(G) \in \Phi(G)/Z(G) = \{a^2 Z(G) \mid a \in G\}$, since $G/Z(G)$ is powerful. Thus $h_2 = a^2 h_2^{2^s}$ for some integer s and so $h_2 = a^{2^k}$ for some integer k . Therefore $Z(G) = \langle h_2^2 \rangle = \langle a^{4^k} \rangle \leq G^4$. Hence G is a non-abelian powerful 2-group of rank 2. Hence G is a non-abelian ordinary metacyclic 2-group [10] and so it follows from ([7] or [6]) that G has a presentation as following:

$$\langle a, b \mid a^{2^r} = b^{2^s}, b^{2^{s+t}} = 1, b^a = b^{2^u+1} \rangle,$$

for some integers $r \geq s \geq u \geq 2$ and $u \geq t \geq 0$. It follows from [6, Corollary 4.5 and Lemma 2.1] that $Z(G) = \langle a^{2^{s+t-u}}, b^{2^{s+t-u}} \rangle$, $G/G' \cong C_{2^r} \times C_{2^u}$, $|G| = 2^{r+s+t}$, $\exp(G) = 2^{r+t} = o(a)$ and $o(b) = 2^{s+t}$. Since $Z(G)$ is cyclic and $o(a) \geq o(b)$, it follows that $b^{2^{s+t-u}} \in \langle a \rangle \cap \langle b \rangle$. If $u > t$, then $|\langle a \rangle \cap \langle b \rangle| \geq 2^{t+1}$ and so

$$2^{r+s+t} = |G| = \frac{|\langle a \rangle| |\langle b \rangle|}{|\langle a \rangle \cap \langle b \rangle|} \leq 2^{r+s+t-1},$$

a contradiction. Thus $u = t \geq 2$ and G has the following presentation:

$$\langle a, b \mid a^{2^r} = b^{2^s}, b^{2^{s+t}} = 1, b^a = b^{2^t+1} \rangle \text{ for some integers } r \geq s \geq t \geq 2.$$

If $t = s$, then $G' \leq Z(G)$ and it follows from [1] that G has a non-inner automorphism of order 2 leaving $\Omega_1(Z(G))$ elementwise fixed.

Now suppose that $s > t$. Thus $b^{2^{s-1}} \in G'$ and so $h = b^{2^{s-1}} a^{-2^{r-1}} \notin G'$ since $\exp(G/G') = 2^r$, and by $a^{2^r} = b^{2^s}$ we have that $h = b^{2^{s-1}} a^{-2^{r-1}}$ is of order 2 so that $h \in H = \langle g \in Z_2(G) \mid g^2 \in Z(G) \rangle$.

If $r > s$, then we further have $[b, h] = 1$. Now it is easy to check (by using the latter presentation of G) that the map α on G defined by $(a^i b^j)^\alpha = (ah)^i b^j$ for all integers i, j , is a non-inner automorphism of order 2 leaving elementwise fixed $\langle b \rangle \cong \Omega_1(Z(G))$.

If $r = s$, then $[a, h] = [b, h]$ is of order 2. It is now easy to check that the map δ on G defined by $(a^i b^j)^\delta = (ah)^i (bh)^j$ for all integers i, j , is a non-inner automorphism of order 2 leaving elementwise fixed $\Omega_1(Z(G))$. This completes the proof for the case $p = 2$.

H. Liebeck [8, Example, p. 272] considered the group G with the following presentation

$$\langle a, b \mid a^4 = [a, b, a] = 1, b^8 = [a, b] \rangle.$$

The group G is of class 2 and since $G' = \langle [a, b] \rangle \leq \langle b \rangle^8 \leq G^4$, G is a powerful 2-group. As Liebeck observed the only automorphisms of G leaving $\Phi(G)$ elementwise fixed and having order 2 are of the form σ , where $a^\sigma = av^{2r}$, $b^\sigma = bv^{2s}$ ($r = 0, 1$; $s = 0, 1$), $v = [a, b]$. These are all inner automorphisms. The group constructed by Liebeck is of order 128. The following is a powerful 2-group of order 64 and class 2 such that every automorphism of order 2 leaving the Frattini subgroup elementwise fixed is an inner automorphism.

$$\langle a, b \mid a^4 = b^4, b^{16} = 1, b^4 = [b, a] \rangle.$$

We leave the proof of the latter assertion to the reader. \square

Remark 2.7. Regarding the proof of Theorem 2.6, case $p = 2$, one of the referees, who I am really grateful to him/her for his/her ideas, has given the following argument to clarify some points of the proof.

The subgroup H is the inverse image in G of $\Omega_1(Z_2(G)/Z(G))$. At this point in the proof, we have shown that $Z(G)$ is cyclic and that, letting $z \in Z(G)$ of order 2, $H/Z(G)$ is isomorphic with $\text{Hom}(G/\Phi(G), \langle z \rangle)$; if $G/\Phi(G)$ is regarded as a $GF(p)$ -vector space V , then $H/Z(G)$ is isomorphic to V^* (the dual space of V). Having shown that H is abelian, one might let $\Omega_1(H) = \langle z \rangle \times D$, to see that in our case (1), $C_G(D) = \Phi(G)$, and in the case (2), $|C_G(D) : \Phi(G)| = 2$ and that, indeed, $C_G(h_i) \neq C_G(h_j)$ whenever $1 \leq i \leq j \leq d$. Hence (arguing as we did), case (1) is out and, in the case (2), we know that $d = 2$. Having shown that G must be itself powerful, hence ordinary metacyclic, we now invest some effort into proving the existence of an involution contained in $Z_2(G) \setminus Z(G)$; this seems to be a bit roundabout, since we already have got such an involution in the shape of h_1 . One might argue like the following (this is much the same thing we did only shorter, and, in some sense, more transparent).

As G is known to be ordinary metacyclic, so G has a normal subgroup $\langle b \rangle$ such that $G = \langle b \rangle \langle a \rangle$ for some $a \in G$ and $G' = \langle [a, b] \rangle \leq \langle b^4 \rangle$; in particular, $o(\langle b \rangle G') \geq 4$. Furthermore, z is the only involution in G' , so $h_1 \notin G'$. If $G/G' = \langle c_1 G' \rangle \times \langle c_2 G' \rangle$ with $o(c_1 G') \geq 4 \leq o(c_2 G')$, then, letting $[c_i, h_1] = z^{\epsilon_i}$, $i = 1, 2$, $\epsilon_i \in \{0, 1\}$, there is $\alpha \in \text{Aut}(G)$ of order 2 given by $c_i^\alpha = c_i h_1^{\epsilon_i}$, $i = 1, 2$. As $h_1 \notin G'$, α is not inner. The only remaining possibility is that, say, $o(c_2 G') = 2$. Now $b \notin \Phi(G)$ and $o(\langle b \rangle G') \geq 4$, so, without loss of generality, $c_1 = b$ and $c_2 = a$. In particular, $a^2 \in G' \leq \langle b^4 \rangle$, and, as G is ordinary metacyclic, we may take $o(a) = 2$; i.e. it turns out that, in this case, actually $a = h_1$ (and, of course, $cl(G) = 2$). One might use [1], or else point out that the automorphism α mapping b to $ba (= bh_1)$ and a to a , will do.

Lemma 2.8. *Let G be a finite nilpotent 2-generated group of class 2. Then $d(Z(G)) \leq 3$.*

Proof. Suppose that $G = \langle a, b \rangle$ and let $x \in Z(G)$. Since G is nilpotent of class 2, $x = a^i b^j [a, b]^k$ for some integers i, j, k . As $G' \leq Z(G)$, $[a^i b^j, a] = [a^i b^j, b] = 1$. It follows that $[a, b]^i = [a, b]^j = 1$ and so $[a, b] = k$ divides both i and j . This implies that $Z(G) = \langle a^k, b^k, [a, b] \rangle$ and so $d(Z(G)) \leq 3$. \square

Theorem 2.9. *Let G be a finite p -group of class 3. If $G/Z(G)$ is 2-generated and $Z(G)$ is not cyclic, then $\text{Aut}(G)$ contains a non-inner automorphism of order p leaving $\Phi(G)$ elementwise fixed.*

Proof. Suppose, for a contradiction that, the conclusion is false. Then by Corollary 2.3, the minimum number of generators of $Z := Z(\frac{G}{Z(G)})$ is 4. On the other hand, by Lemma 2.8, $d(Z) \leq 3$, a contradiction. This completes the proof. \square

3. Cohomologically trivial modules and non-inner p -automorphisms of finite p -groups

In this section we prove non-triviality of Tate cohomology

$$H^n(G/N, Z(N)) \quad \text{for all } n,$$

for certain p -groups G and normal subgroups $N \triangleleft G$.

As we mentioned in Section 1, in [11], Schmid proved that if G is a regular p -group and N a non-trivial normal subgroup of G , then $H^n(Q, Z(N)) \neq 0$ for all n and all non-cyclic $Q = G/N$. Here we prove the same cohomological property for certain classes of groups. We first need the following result concerning cohomologically trivial groups.

Lemma 3.1. *Let p be an odd prime and let G be a finite p -group such that $G/Z(G)$ is p -central. Suppose that A is a normal abelian subgroup of G , $a \in A$ and $g \in G$ such that $g^p \in C_G(A)$. If $p > 3$ then $a^{g^{p-1}+\dots+g+1} = a^p$ and if $p = 3$ then $a^{g^2+g+1} = a^3z$ for some central element z of G .*

Proof. Let $H = \langle A, g \rangle$. We have $[a, g^p] = 1$ and so $(g^a Z(G))^p = (gZ(G))^p$ for all $a \in A$. Now as $G/Z(G)$ is p -central, we have $[g^a, g] \in Z(G)$ (see e.g. [9, Theorem 5]). Now it follows from the Hall–Petrescu formula that

$$(g^{-1}g^a)^p = g^{-p}(g^p)^a [g^a, g^{-1}]^{\binom{p}{2}} [g^a, g^{-1}, g^{-1}]^{\binom{p}{3}} \dots [g^a, g^{-1}, g^{-1}]^{\binom{p}{p-1}}.$$

Since $p \geq 3$, $g^p = (g^p)^a$ and $[g^a, g^{-1}] \in Z(G)$, we can write

$$(g^{-1}g^a)^p = ([g^a, g^{-1}]^p)^{(p-1)/2} = [(g^p)^a, g^{-1}]^{(p-1)/2} = [g^p, g^{-1}]^{(p-1)/2} = 1.$$

Therefore $[g, a]^p = 1$ for all $a \in A$ and so the derived subgroup H' of H is of exponent dividing p . Now by the Hall–Petrescu formula we have

$$a^{g^{p-1}+\dots+g+1} = a^p [a, g]^{\binom{p}{2}} [a, g]^{\binom{p}{3}} \dots [a, g]^{\binom{p}{p-1}}.$$

Since $\exp(H')$ divides p , $[a, g]^{\binom{p}{i+1}} = 1$ for all $i \in \{1, \dots, p-2\}$ and as $[g^a, g] \in Z(G)$, we have $[a, g] \in Z(G)$ and if $p > 3$, $[a, g] = 1$. This completes the proof. \square

Lemma 3.2. *Let p be an odd prime and let G be a finite p -group of class at most 3. Suppose that A is a normal abelian subgroup of G , $a \in A$ and $g \in G$ such that $g^p \in C_G(A)$. Then $a^{g^{p-1}+\dots+g+1} = a^p z$ for some central element z of G .*

Proof. Let $H = \langle A, g \rangle$. We have $[a, g^p] = 1$ for all $a \in A$. Now by the Hall–Petrescu formula we have

$$(g^{-1}g^a)^p = g^{-p}(g^p)^a [g^a, g^{-1}]^{\binom{p}{2}} [g^a, g^{-1}, g^{-1}]^{\binom{p}{3}} \dots [g^a, g^{-1}, g^{-1}]^{\binom{p}{p-1}}.$$

Since $g^p = (g^p)^a$, $p \geq 3$ and $[g^a, g^{-1}] \in Z(G)$, we can write

$$(g^{-1}g^a)^p = ([g^a, g^{-1}]^p)^{(p-1)/2} = [(g^p)^a, g^{-1}]^{(p-1)/2} = [g^p, g^{-1}]^{(p-1)/2} = 1.$$

Therefore $[g, a]^p = 1$ for all $a \in A$ and so the derived subgroup H' of H is of exponent dividing p . Now by the Hall–Petrescu formula we have

$$a^{g^{p-1} + \dots + g + 1} = a^p [a, g]^{p \binom{p}{2}} [a, {}_2g]^{p \binom{p}{3}} \dots [a, {}_{p-1}g].$$

Since $\exp(H')$ divides p , $[a, {}_i g]^{\binom{p}{i+1}} = 1$ for all $i \in \{1, \dots, p-2\}$ and as $[g^a, g] \in Z(G)$, we have $[a, {}_2g] \in Z(G)$. This completes the proof. \square

Lemma 3.3. *Let G be a finite 2-group of class at most 3. Suppose that A is a normal abelian subgroup of G , $a \in A$ and $x, y \in G$ such that $x^2, y^2, (xy)^2 \in C_G(A)$. Then $a^{xy+y+x+1} = a^4z$ for some central element z of G .*

Proof. We may write

$$\begin{aligned} a^{xy+x+y+1} &= a^4 [a, xy][a, x][a, y] \\ &= a^4 [a, y][a, x]^y [a, x][a, y] \\ &= a^4 [a, x]^2 [a, y]^2 [a, x, y]. \end{aligned}$$

On the other hand, for an element $g \in G$ such that $g^2 \in C_G(A)$, we have

$$1 = [a, g^2] = [a, g]^g [a, g] = [a, g]^2 [a, g, g],$$

and so $[a, g]^2 = [a, g, g]^{-1} \in Z(G)$ as G is of class 3. Hence, it follows from (*) that $a^{xy+x+y+1} = a^4z$ for the central element $z = [a, x]^2 [a, y]^2 [a, x, y]$. This completes the proof. \square

Lemma 3.4. *Let G be a finite p -group of class at most 2. Then for all $x, y \in G$ and $n \in \mathbb{N}$, there exists $z \in Z(G)$ such that $y^{x^{n-1} + \dots + x + 1} = y^n z$.*

Proof. It is straightforward as we have the identity

$$(xy)^n = x^n y^n [y, x]^{n(n-1)/2}$$

in a nilpotent group of class at most 2. \square

Proposition 3.5. *(See Proposition 1 of [12].) Suppose that $A \neq 0$ is a cohomologically trivial Q -module where A and Q are finite p -groups. Then for every subgroup H of Q , the centralizer $C_Q(A_H) = H$.*

Theorem 3.6. *Let G be a finite p -group and let N be a non-trivial normal subgroup of G such that G/N is not cyclic. Suppose that one of the following holds:*

- (1) $p > 2$ and $G/Z(G)$ is either nilpotent of class at most 2 or p -central;
- (2) G is nilpotent of class at most 2.

Then, in any case we have $H^n(\frac{G}{N}, Z(N)) \neq 0$ for all $n \geq 0$.

Proof. The proof follows from the lines of the proof of the theorem in [12] but instead of using [12, Proposition 2] in the proof, one may use Lemmas 3.1, 3.2, 3.4. We give the proof for the reader's convenience.

Suppose, for a contradiction, that $H^n(G/N, Z(N)) = 0$ for some $n \geq 0$. Let H/N be a subgroup of G/N of order p . By Gaschütz and Uchida's result, we have $H^0(H/N, A) = 0$ where $A = Z(N)$. Thus

$A_{\frac{H}{N}} = A^{\tau_{\frac{H}{N}}}$. Now it follows from Lemmas 3.1, 3.2 or 3.4 that there exist elements $z_a \in Z(G)$ ($a \in A$) such that $A^{\tau_{\frac{H}{N}}} = \{a^p z_a \mid a \in A\}$. Now since $z_a \in Z(G)$, we have that

$$C_{\frac{G}{N}}(A^{\tau_{\frac{H}{N}}}) = C_{\frac{G}{N}}(A^p).$$

Thus by Proposition 3.5 we have that $\frac{H}{N} = C_{\frac{G}{N}}(A^p)$. As the right-hand side of the latter equality is independent from the choice of H/N , we have that G/N has a unique subgroup of order p . Therefore G/N is cyclic or generalized quaternion and so in the case (1), we are done. Thus we are left with the case (2) and we may assume further that G/N is a generalized quaternion group. In this case as G is nilpotent of class at most 2, we have that G/N is the quaternion group of order 8. Now it follows from Lemma 3.4 and a similar argument as above that G/N has only one cyclic subgroup of order 4, a contradiction. This completes the proof. \square

Let us finish by an alternative proof of Theorem 2.6 for the case $p > 2$ in which we use Theorem 3.6. Before that, we need the following lemma, however it has its own interest, one can see the extra amounts of works in respect to the more quick and straightforward proof of the first part of Theorem 2.6.

Lemma 3.7. *Let $p > 2$ and let G be a finite p -group such that $G/Z(G)$ is p -central. Then $Z^1(\frac{G}{\Phi(G)}, Z(\Phi(G)))$ is an elementary abelian p -group.*

Proof. Let $f \in Z^1(\frac{G}{\Phi(G)}, Z(\Phi(G)))$ and $\bar{x} = x\Phi(G)$ for $x \in G$. We have to prove that $\bar{x}^{f^p} = 1$. We have $a = \bar{x}^f \in Z(\Phi(G))$. Since $x^p \in \Phi(G)$, $(\bar{x}^p)^f = 1$ and so $a^{x^{p-1} + \dots + x + 1} = 1$ which is equivalent to the equality $(xa)^p = x^p$. Thus $[xa, x] \in Z(G)$ by [9, Theorem 5]. Hence $[x, a] \in Z(G)$ and so $(xa)^p = x^p a^p [a, x]^{p(p-1)/2}$. On the other hand $[(xa)^p, x] = [x^p, x] = 1$ and as $[xa, x] \in Z(G)$, it follows that $[xa, x]^p = 1$. Thus $(xa)^p = x^p a^p$ and so $a^{x^{p-1} + \dots + x + 1} = a^p = 1$. This completes the proof. \square

Second Proof of part 1 of Theorem 2.6 for odd p . Suppose, for a contradiction, that G has no non-inner automorphism of order p leaving $\Phi(G)$. By an easy argument given in the first part of the proof of Theorem 2.6, we have that $G/Z(G)$ is p -central. By [2] we may further assume that $\Phi(G) = C_G(Z(\Phi(G)))$ which implies $Z(\Phi(G)) = C_G(\Phi(G))$. Now taking $N = \Phi(G)$ in Theorem 3.6, we find that $H^1(\frac{G}{\Phi(G)}, Z(\Phi(G))) \neq 0$. By Lemma 3.7, $Z^1(\frac{G}{\Phi(G)}, Z(\Phi(G)))$ is an elementary abelian p -group and so it follows from [11, Result 1.1 and Corollary 1.2] that G has a non-inner automorphism of order p leaving $\Phi(G)$ elementwise fixed, a contradiction. This completes the proof. \square

Acknowledgments

The author is grateful to the referees for their valuable comments and careful readings. The ‘‘cohomology free’’ proof of the first part of Theorem 2.6 is due to one of the referees; who I am indebted to him/her, as I was inspired and encouraged by his/her argument to solve the second part of Theorem 2.6 for $p = 2$. Some parts of this work was done during author’s visit from University of Bath in Summer 2009. The author thanks the Department of Mathematical Sciences of University of Bath for its hospitality and especially he wishes to thank Gunnar Traustason. This research was partially supported by the Center of Excellence for Mathematics, University of Isfahan and the author gratefully acknowledges the financial support of University of Isfahan for the sabbatical leave study in University of Bath.

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