Affine Planes with Primitive Collineation Groups

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1. INTRODUCTION

In his paper [14], A. Wagner showed that every finite affine plane with a line transitive collineation group is a translation plane. On the other hand there are many examples of finite affine planes with point transitive collineation groups which are not translation planes. Hence the hypothesis of line transitivity cannot be replaced by point transitivity. However, all the known finite affine planes with point primitive collineation groups are translation planes. In fact, it is known that point primitivity is sufficient for an affine plane of even order to be a translation plane. (See [8, Lemmas 15.1 and 15.3].) Furthermore a result of Keiser shows that every affine plane admitting a point primitive collineation group which contains a nontrivial perspectivity is a translation plane. (See [11; 8, Corollary 15.1.1].)

The purpose of this paper is to settle this problem and prove the following.

THEOREM 1. A finite affine plane with a primitive collineation group is a translation plane.

We note that the problem was proposed by W. M. Kantor in [9, p. 180, "Open problems"]. At the same time he pointed out that his theorem on primitive permutation groups [10] might be important to solving it.

In Section 3 we reduce the proof of our theorem to a situation, where we can assume that the order $n$ is odd and that the primitive collineation group is a normal extension of a nonabelian simple group, say $G$. From the classification of the finite simple groups, $G$ is isomorphic to an alternating group $\text{Alt}(k)$, a group of Lie type $G(q)$ of characteristic $p$, or a sporadic simple group (cf. [6]).
In Section 4 we deal with the case where \( G \simeq G(q) \). By Ostrom-Wagner's result on transitive affine planes (cf. [8, Theorem 4.3]), \( |G : G_X| = n^2 \) and \( n \mid |G_Q| \) for each affine point \( X \) and each point \( Q \) on the line at infinity. Then Kantor's results [10], together with "\( p \)-primitive divisor" arguments give information about maximal subgroups containing \( G_X \). This rules out most possibilities for \( G \).

In Section 5 we consider the case \( G \simeq \text{Alt}(k) \). Applying Kantor's results again, we obtain a contradiction.

In Section 6 the case where \( G \) is a sporadic group is eliminated by a standard argument of permutation groups.

2. PRELIMINARIES

Let \( \pi = \pi(\mathcal{P}, \mathcal{L}) \) be an affine plane of order \( n \) and denote by \( l_\infty \) the line at infinity. The unique line joining two different points \( P, Q \in \mathcal{P} \) is denoted by \( PQ \) and the unique point incident with two different lines \( l, g \in \mathcal{L} \) by \( l \cap g \). A line \( l \) is sometimes identified with the set of points on \( l \). Let \( S \) be a nonempty set of collineations of \( \pi \). Then any element of \( S \) is often viewed as a collineation of the projective plane \( \pi^* \) associated with \( \pi \) (cf. [7]). Define \( \mathcal{F}(S) \) to be the fixed structure consisting of all elements of \( \pi^* \) fixed by \( S \). Let \( G \) be a collineation group of \( \pi \) and \( \mathcal{A} \) a subset of \( \mathcal{P} \) or \( \mathcal{L} \). Then \( G_\mathcal{A} = \{ g \in G \mid X^g = X \ \text{for each} \ X \in \mathcal{A} \} \) is the elementwise stabilizer and \( G(\mathcal{A}) = \{ g \in G \mid \mathcal{A}^g = \mathcal{A} \} \) is the global stabilizer. Denote by \( S^G \) the set of all \( G \)-conjugates of \( S \) and by \( S^G \cap H \) the set of all elements of \( S^G \) contained in a subgroup \( H \) of \( G \). Further \( N_G(H) \) or \( C_G(H) \) is the normalizer or centralizer of \( H \) in \( G \), respectively. For subgroups \( X \) and \( Y \) of \( G \), set \( [X,Y] = \langle x^{-1}y^{-1}xy \mid x \in X, y \in Y \rangle \).

To prove our theorem we use some results on affine planes, permutation groups, or simple groups of Lie type, which are contained in [8], [15], or [3], respectively. Throughout the paper all sets are assumed to be finite.

We now prove some preliminary lemmas to be used in later sections.

**Lemma 2.1.** Let \( G \) be a transitive collineation group of a finite affine plane \( \pi(\mathcal{P}, \mathcal{L}) \) of order \( n \) and set \( H = G_P \) for some \( P \in \mathcal{P} \). Then the following hold.

(i) For any nonempty subset \( X \) of \( H \),

\[
|N_G(X)| \times |X^G \cap H| = |H| \times |\mathcal{F}(X) \cap \mathcal{P}|.
\]

(ii) For any element \( z \) of \( G \), define \( I(z) = |G| \times |H|/|C_G(z)|^2 \). If \( z \) is either a Baer collineation or a nontrivial perspectivity which has more than one fixed point, then \( \sqrt{I(z)} \) is a positive integer.
Proof. Set $\mathcal{I} = \{(Q, Y) | Y \in X^G, \ Q \in \mathcal{F}(Y) \cap \mathcal{P}\}$. By counting the number of elements of $\mathcal{I}$ in two ways we have $|G : H| \times |X^G \cap H| = |G : N_G(X)| \times |\mathcal{F}(X) \cap \mathcal{P}|$. Hence (i) holds.

Under the hypothesis of (ii) the number of fixed points of $z$ on $\mathcal{P}$ is exactly $n$. Applying (i), $|C_\alpha(z)|| \times |z^G \cap H| = |H| \times \sqrt{|G : H|}$. Hence $|z^G \cap H|^2 = |G| \times |H|/|C_\alpha(z)|^2 \in \mathbb{N}$. Here $\mathbb{N}$ is the set of all positive integers.

Lemma 2.2. Let $\pi(\mathcal{P}, \mathcal{L})$ be an affine plane of odd order $n$ and $S$ its collineation group of order $2^m$ for some $m \in \mathbb{N}$. Suppose $4 \sqrt{n} \notin \mathbb{N}$. Then, either $S$ contains a nontrivial perspectivity or $|S| = 2(\sqrt{n} - 1)^3$.

Proof. Since $n$ is odd, $\mathcal{F}(S) \cap \mathcal{P} \neq \emptyset$. Let $P \in \mathcal{F}(S) \cap \mathcal{P}$ and assume that $S$ contains no perspectivities. Then the fixed structure of each involution of $S$ is a subplane of $\pi^*$ of order $\sqrt{n}$ by Baer's theorem (cf. [8, Theorem 4.4]). In particular $|l_\alpha| = n + 1 = (\sqrt{n})^2 + 1 \equiv 2 \pmod{4}$. Hence, there is an $S$-invariant subset $A$ of $l_\alpha$ such that $|A| = 2$. Put $T = S_A$. Then $[S : T] \leq 2$ and so $S \supset T$. If $T = 1$, the assertion is obvious. Assume $T \neq 1$ and let $z$ be an involution in $Z(S) \cap T$, where $Z(S)$ is the center of $S$. Put $\Gamma = \mathcal{F}(z)$, $K \in S_{\Gamma}(<T)$ and denote by $S^\Gamma$ the restriction of $S$ on $\Gamma$. Then $S^\Gamma \simeq S/K$. As $K$ acts semiregularly on the points $l_\alpha - l_\alpha \cap \Gamma$, $|K||n - \sqrt{n} = \sqrt{n}(\sqrt{n} - 1)$. Hence $|K| \sqrt{n} - 1$. If $S^\Gamma = 1$, then the assertion is obvious and so assume $S^\Gamma \neq 1$. As $4 \sqrt{n} \notin \mathbb{N}$, each involution in $S^\Gamma$ is a perspectivity of $\Gamma$. Therefore $T^\Gamma (\simeq T/K)$ acts semiregularly on $\Gamma \cap \mathcal{P} - (PQ \cup QR \cup RP)$, where $A = \{Q, R\}$, and hence $|T^\Gamma| = (\sqrt{n} - 1)^2$. Thus $|S| = |S : T| \times |T : K| \times |K| = 2(\sqrt{n} - 1)^3$.

Let $G$ be a simple group of Lie type associated with a root system $A$ and $B$ a Borel subgroup of $G$. Denote by $\Sigma$ the set of fundamental roots of $A$. Throughout the paper the Dynkin diagram of $A$ will be labeled as follows:

\[
\begin{align*}
A_1 & \quad - \quad \ldots \quad - \quad l - 1 \quad - \quad l \\
B_1 & \quad - \quad \ldots \quad - \quad l - 1 \quad - \quad l \\
C_1 & \quad - \quad \ldots \quad - \quad l - 1 \quad - \quad l \\
D_1 & \quad - \quad \ldots \quad - \quad l - 1 \quad - \quad l \\
E_6 & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \\
E_7 & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \\
F_4 & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\
G_2 & \quad 1 \quad 2
\end{align*}
\]
For each subset $J$ of $\Sigma$ denote by $G_J$ ($\supseteq B$) the parabolic subgroup of $G$ corresponding to $J$ [3]. View $G$ as a normal subgroup of $\text{Aut}(G)$ and let $G_0$ be the normal subgroup of $\text{Aut}(G)$ generated by $G$ together with all the diagonal and field automorphisms of $G$. Each element $\theta$ of $\text{Aut}(G)$ can be expressed in the form

$$\theta = \text{idfg}$$

with $i$, $d$, $f$, and $g$ being inner, diagonal, field, and graph automorphisms, respectively. A mapping $\psi$ from $\text{Aut}(G)$ into the symmetric group on $\Sigma$ defined by $\theta \mapsto g^\Sigma$ is a homomorphism having the kernel $G_0$.

**Lemma 2.5.** Let $G$, $A$, and $\Sigma$ be as stated above. Let $F$ be a subgroup of $\text{Aut}(G)$ containing $G$ and assume that $F$ acts faithfully and primitively on a finite set $\Omega$. If the stabilizer $G_x$ of a point $x \in G$ is a parabolic subgroup of $G$ which contains $B$, then $G_x = G_J$ for some maximal $\psi(F)$-invariant subset $J$ of $\Sigma$.

**Proof.** By the assumption, $G_x = G_J$ for some subset $J$ of $\Sigma$. Let $\theta = \text{idfg}$ be any element of $F_x$ expressed in the form as stated in (2.4). Since $F_x \supseteq G_x = G_J \supseteq B$, $(G_J)^i = (G_J)^{(dfg)^{-1}} \supseteq B^{(dfg)^{-1}} = B$ and hence $(G_J)^i = (G_J)^{(dfg)^{-1}} = G_J$ by [3, Theorem 8.3.3]. It follows that $i \in G_J$ and $dfg \in F_x$. Hence $F_x = G_J A$, where $A = \langle dfg \mid dfg \in F_x \rangle$. As $F_x$ is a maximal subgroup of $F$, $G_J$ is a maximal $A$-invariant subgroup of $G$. Since all diagonal and field automorphisms leave each parabolic subgroup which contains $B$ invariant, $G_J$ must be a maximal $\psi(A)$-invariant parabolic subgroup of $G$. As $\psi(A) = \psi(F)$, the lemma holds.

Let $p$ be a prime and $s$ a positive integer. A prime $u$ is said to be a prime $p$-primitive divisor of $p^s - 1$ if $u \mid p^s - 1$ but $u \nmid p^i - 1$ for all integers $i$ with $1 \leq i < s$. In [16] Zsigmondy proved that a prime $p$-primitive divisor exists unless either (i) $s = 6$ and $p = 2$ or (ii) $n = 2$ and $p$ is a Mersenne prime.

**Definition 2.6.** Let $G = G(q)$ be a group of Lie type of characteristic $p$ associated with a root system $\Delta$. We define a positive integer $v = v(\Delta)$ as follows:

<table>
<thead>
<tr>
<th>Type of $\Delta$</th>
<th>$v(\Delta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$l + 1$</td>
</tr>
<tr>
<td>$B_l$</td>
<td>$2l$</td>
</tr>
<tr>
<td>$C_l$</td>
<td>$2l - 2$</td>
</tr>
<tr>
<td>$D_l$</td>
<td>$4l + 2$</td>
</tr>
<tr>
<td>$A_{2l}$</td>
<td>$4l + 2$</td>
</tr>
<tr>
<td>$A_{2l+1}$</td>
<td>$4l + 2$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$6$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$12$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$18$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$30$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$4$</td>
</tr>
</tbody>
</table>
Furthermore denote by $A_{G(q)}$ the set of all prime $p$-primitive divisors of
$p^v - 1$, where $v = v(d)$ and $p^c = q$. Note that $u| |G(q)|$ for each $u \in A_G$ and
that $w| |G(q)|$ for any prime $p$-primitive divisor $w$ of $p^v - 1$, where $\mu$ is an
integer greater than $cv$. (See [6, Table 2.4].)

**Lemma 2.7.** Let $G, p, v$ and $A_G$ be as stated above. If $M$ is an arbitrary
proper subgroup of $G$ which contains a Sylow $p$-subgroup of $G$, then the
following hold.

(i) If $A_G \neq \emptyset$, then $u|M|$ for any $u \in A_G$.

(ii) If $A_G = \emptyset$, then either $q^v = 2^6$ or $G = A_1(p)$, where $p$ is a Mersenne
prime.

**Proof.** By (1.6) of [13, 1.63 it suffices to consider the case that $M$ is a
maximal parabolic subgroup of $G$. If $A_G = \emptyset$, then by Zsigmondy's
theorem [16] we have (ii). If $A_G \neq \emptyset$, then it is easy to check that $u|M|
for any $u \in A_G$ (cf. [3, Theorem 8.5.2]).

In Section 4 we will use Kantor's result [10] on primitive permutation
groups in the following special form.

**Lemma 2.8 (W. M. Kantor [10]).** Let $F$ be a primitive permutation
group on a set $\Omega$ of odd degree having a normal subgroup $G$ isomorphic to
a simple group of Lie type of odd characteristic $p$. Set $M = G_x$ for some
$x \in \Omega$. Then $A_G \neq \emptyset$ and $u|M|$ for all $u \in A_G$ except in the following cases:

| $\ G$   | $|\Omega|$         | $q$               |
|---------|-------------------|-------------------|
| $A_1(q)$| $q(q^2 - 1)/283$  | $q \equiv \pm 3(\text{mod } 8)$ |
| $A_2(q)$| $q(q^2 - 1)/243$  | $q \equiv \pm 7(\text{mod } 16)$ |
| $A_1(q)$| $q(q - 1)/2$      | $q \equiv -1(\text{mod } 4)$    |
| $B_3(p)$| $p^8(p^2 - 1)(p^4 - 1)(p^6 - 1)/2103457$ | $q = p \equiv \pm 3(\text{mod } 8)$ |
| $D_4(p)$| $p^{12}(p^2 - 1)(p^4 - 1)^2(p^6 - 1)/2^{12}3^{10}5^7(4, p - 1)$ | $q = p \equiv \pm 3(\text{mod } 8)$ |
| $D_4(p)$| $p^{12}(p^2 - 1)(p^4 - 1)^2(p^6 - 1)/2^{12}3^{10}5^7(4, p - 1)$ | $q = p \equiv \pm 3(\text{mod } 8)$ |
| $G_2(q)$| $q^8(q^8 - 1)/2$  | $q \equiv -1(\text{mod } 4)$    |
| $G_2(p)$| $p^6(p^2 - 1)(p^6 - 1)/2^{63}7$ | $q = p \equiv \pm 3(\text{mod } 8)$ |
| $G_2(p)$| $p^6(p^2 - 1)(p^6 - 1)/2^{63}7$ | $q = p \equiv \pm 3(\text{mod } 8)$ |

**Remark.** In the case $G = D_4(p)$ we have $v = v(D_4) = 6$ and
$|M|/|G|/|\Omega| = 2^{12}3^{10}5^7$ or $2^{12}3 \cdot 7$. Hence, if $u|M|$ for some $u \in A_G$, then
$u \neq 3, 5$ by the definition of $A_G$. Therefore $A_G = \{7\}$ in this case.

**Lemma 2.10 (W. M. Kantor [10]).** Let $F$ be a primitive permutation
group on a set $\Omega$ of odd degree having a normal subgroup $G$. Assume that $G$
is isomorphic to a group of Lie type of odd characteristic $p$ associated with a root system $\Delta$. Set $M = G_x$ for some $x \in \Omega$ and let $\Sigma = \{r_1, ..., r_l\}$ be the set of all fundamental roots for $\Delta$. If $M$ contains a Sylow $p$-subgroup of $G$, then one of the following holds. (See (2.3).)

(i) $G = A_2(q)$ and $M = G_J$, where $J = \Sigma \setminus \{r_i\}$, $1 \leq i \leq l$, and $(i + 1) \equiv 1$ (mod 2).

(ii) $G = A_2(q)$ and $M = G_{J}$, where $J = \Sigma - \{r_i, r_j\}$ ($i \neq j$) and $i + j = l + 1$.

(iii) $G = E_6(q)$ and $M = G_J$, where $J = \Sigma - \{i\}$ and $i \in \{1, 6\}$.

**Proof.** By using a result of Kantor [10, Lemma 2.3] together with Lemma 2.5, we can verify the assertion.

**LEMMA 2.11.** Let $p$ be a prime and let $n$ and $c$ be integers such that $n \in \mathbb{N}$ and $|c| < p^n$. If $\sqrt{p^{2n} + cp^n + 1} \in \mathbb{N}$, then (i) $c = \pm 2$, (ii) $p = 2$, $c = -3 \cdot 2^{n-2} \pm 1$, and $n \geq 3$, or (iii) $p = 2$, $c = 7$, and $n = 3$. 

**Proof.** Set $z = \sqrt{p^{2n} + cp^n + 1}$. Clearly $z \geq 2$. Since $z^2 = p^{2n} + cp^n + 1 \leq 2p^{2n} - p^n + 1 < 2p^{2n}$, we have $2 \leq z < \sqrt{2} p^n$.

First suppose $p > 2$. As $p^n(p^n + c) = (z - 1)(z + 1)$ and $p | (z - 1, z + 1) = (z - 1, 2)$, $z - \varepsilon = tp^n$, where $\varepsilon = 1$ or $-1$ and $t \in \mathbb{N}$. By the first paragraph, $t = 1$. Hence $p^n(p^n + c) = (p^n + \varepsilon + 1)$ and so $c = 2\varepsilon \in \{\pm 2\}$.

Next suppose $p = 2$. As $z$ is odd, $(z - 1, z + 1) = 2$. Hence $2(z - \varepsilon) = t2^n$ for some $t \in \mathbb{N}$ and $\varepsilon \in \{\pm 1\}$. As $z < \sqrt{2} p^n$, $t \leq 3$. If $t = 3$, then $(2 \cdot 2^n)(2(2^n + c)) = (2z - 2)(2z + 2) = (3 \cdot 2^n + 2z + 1)$ and so $c = 5 \cdot 2^{n-2} + 3 \varepsilon \leq 2^n - 1$. Hence (iii) holds unless $c \neq -2$. If $t = 2$, then $(2 \cdot 2^n)(2(2^n + c)) = (2 \cdot 2^n + 2\varepsilon - 2)(2 \cdot 2^n + 2\varepsilon + 2)$. Hence $c = 2\varepsilon$ and (i) follows. If $t = 1$, then $(2 \cdot 2^n)(2(2^n + c)) = (2^n + 2\varepsilon - 2)(2^n + 2\varepsilon + 2)$. Hence $c = -3 \cdot 2^{n-2} + \varepsilon$. Assume $c \neq -2$. Then $n \geq 3$ and $|c| < 2^n$ and (ii) follows.

3. A Minimal Normal Subgroup

In the rest of the paper $\pi = \pi(\mathcal{P}, \mathcal{L})$ is an affine plane of order $n$ which admits a collineation group $F$ primitive on the set of affine points $\mathcal{P}$. Further, assume the following.

**HYPOTHESIS 3.0.** $F$ contains no nontrivial perspectivities.

In this section we consider a minimal normal subgroup of $F$ and show that it must be a nonabelian simple group.

**LEMMA 3.1.** Let $G$ be a minimal normal subgroup of $F$. Then
(i) \(|F(G) \cap l_\infty| \neq 1\) and \(G\) is transitive on \(\mathcal{P}\).

(ii) \(G = M_1 \times \cdots \times M_r\) for some isomorphic nonabelian simple groups \(M_i\)'s.

(iii) \(n = m^2\) for some odd integer \(m \geq 1\) and \(F(t)\) is a subplane of \(\pi^*\) of order \(m\) for any involution \(t \in G\).

(iv) There exists a point \(P \in l_\infty\) such that \(G_P\) is a proper subgroup of \(G\) of index less than \(n\).

Proof. By [15, Theorem 8.8], \(G\) is transitive on \(\mathcal{P}\). Furthermore, we have \(|F(G) \cap l_\infty| \neq 1\) for otherwise \(F(G) \cap l_\infty \supseteq F(F) \cap l_\infty \neq \emptyset\) as \(G \preceq F\), contrary to the primitivity of \(F\) on \(\mathcal{P}\). From this, together with [8, Theorem 9.1 and Lemmas 15.1, 15.3], (ii) follows. By Baer's theorem and Hypothesis 3.0, (iii) holds.

Since \(G\) contains no perspectivities, there exists a point \(P\) on \(l_\infty\), not fixed by \(G\). Set \(N = G_P\). Clearly \(|G : N| \leq |l_\infty| = n + 1\). As \(|F(G) \cap l_\infty| \neq 1\), \(|G : N| \neq n\). If \(|G : N| = n + 1\), then \(G\) is transitive on \(\mathcal{P}\) and so contains a nontrivial perspectivity by [14], contrary to Hypothesis 3.0. Thus (iv) holds.

**Lemma 3.2.** Assume \(r \geq 2\) and set \(M = M_1\) and \(N = M_2\). Let \(P \in \mathcal{P}\). Then the following hold.

(i) \(M_P\) is \(F\)-conjugate to \(N_Q\) for any \(P\) and \(Q\) in \(\mathcal{P}\).

(ii) Define \(\Psi_i = F((M_i)_P)\) for \(i = 1\) or \(2\). Then \(\Psi_i\) is an \(L_i\)-invariant subplane of \(\pi^*\), where \(L_i = \langle M_j \rangle | 1 \leq j \leq r, j \neq i \rangle\).

Proof. Since \(F = G \cdot F_P\) and \(G \triangleright M\), there is an element \(g \in F_P\) such that \(M^g = N\). Then \((M_P)^g = (M \cap F_P)^g = M^g \cap F_P = N_P\). As \(G\) is transitive on \(\mathcal{P}\), we can choose an element \(h \in G\) so that \(P^h = Q\). Hence \((M_P)^g = (N_P)^h = (N \cap G_P)^h = N \cap G_Q = N_Q\) and (i) holds.

To show (ii) it suffices to consider the case \(i = 1\). Set \(L = L_1\) and \(\Psi = \Psi_1\). Since \(L\) centralizes \(M_P\), \(L\) leaves \(\Psi\) invariant. Set \(\Gamma = \Psi \cap \mathcal{P}\). Then \(L \geq N_Q\) and \(L\) leaves \(\Gamma\) invariant.

If \(|\Gamma| \leq 4\), \(F(N_Q) \cap \mathcal{P} \supseteq F(L) \cap \mathcal{P} \supseteq \Gamma\) since \(L \subseteq \text{Sym}(\Gamma) \leq \text{Sym}(4)\). Hence, by (i), \(F(N_Q) \cap \mathcal{P} = F(L) \cap \mathcal{P} = \Gamma\). As \([L_2, N_Q] = 1\) and \(F(N_Q) \cap \mathcal{P} = \Gamma\), similarly we have \(F(L_2) \cap \mathcal{P} = \Gamma\) and so \(F(G) \cap \mathcal{P} = F(L_2) \cap \mathcal{P} = \Gamma\), contrary to the transitivity of \(G\) on \(\mathcal{P}\). Therefore \(|\Gamma| \geq 5\).

Deny (ii). Then, as \(|\Gamma| \geq 5\), \(\Gamma \subset L\) for some affine line \(L\) and \(|\Psi \cap l_\infty| \leq 2\). Hence \(F(L) \supseteq \{l_\infty, L\}\) and \(L\) fixes \(\Psi \cap l_\infty\) pointwise. By (i), \(F(L) \cap l_\infty = F(N_Q) \cap l_\infty = \Psi \cap l_\infty\). If \(|F(L) \cap l_\infty| = 1\), then \(|F(G) \cap l_\infty| = 1\), contrary to Lemma 3.1(i). By the transitivity of \(G\) on \(\mathcal{P}\), \(F(L) \cap \mathcal{P} = \emptyset\). Hence
Lemma 3.3. Assume \( r \geq 2 \) and let notations be as defined in Lemma 3.2. For \( i \in \{1, 2\} \), set \( \mathcal{A}_i = \{(\Gamma_i)^g \mid g \in G\} \), where \( \Gamma_i = \mathcal{P}_i \cap \mathcal{P} \). Then,

(i) \( \mathcal{A}_i = \{(\Gamma_i)^g \mid g \in M_i\} \) and \( \bigcup_{\Gamma_0 \in \mathcal{A}_i} \Gamma_0 = \mathcal{P} \).

(ii) If \( g, h \in G \) and \( (\Gamma_i)^g \neq (\Gamma_i)^h \), then \( (\Gamma_i)^g \cap (\Gamma_i)^h = \emptyset \).

Proof. In order to show the lemma it suffices to consider the case \( i = 1 \).

Set \( \Gamma = \Gamma_1 \), \( \mathcal{A} = \mathcal{A}_1 \), \( M = M_1 \), and \( L = \langle \mathcal{M}_i \rangle_2 \leq \Gamma \).

Since \( G \) is transitive on \( \mathcal{P} \), \( \bigcup_{\Gamma_0 \in \mathcal{A}_1} \Gamma_0 = \mathcal{P} \). Any element \( g \) of \( G \) can be expressed in the form \( g = xy \), where \( x \in L \) and \( y \in M \). By Lemma 3.2(ii), \( \Gamma^g = (\mathcal{P}_1 \cap \mathcal{P})^y = (\mathcal{P}_1 \cap \mathcal{P})^y = \Gamma^y \) and hence (i) holds.

In order to show (ii), we may assume \( g = 1 \) and \( h \in M \). Suppose \( \Gamma \cap \Gamma^h \neq \emptyset \) and let \( Q \in \Gamma \cap \Gamma^h \). Then \( M_Q \supseteq M_P \) and \( M_Q \supseteq (M_P)^h = M_R \), where \( R = P^h \). By Lemma 3.2(i), \( M_P \), \( M_Q \), and \( M_R \) are \( F \)-conjugate. Thus \( M_Q = M_P = M_R \) and \( F(M_P) = F(M_Q) = F(M_R) \), so we have \( \Gamma = \Gamma^h \).

Lemma 3.4. \( G \) is a nonabelian simple group and \( C_F(G) = 1 \).

Proof. Assume \( r \geq 2 \) and let notations be as defined in Lemmas 3.1-3.3. By Lemma 3.3, \( \mathcal{A} \) gives a partition of \( \mathcal{P} \) for each \( i \in \{1, 2\} \). Furthermore \( |\mathcal{A}_i| = |M_i : M_i(\Gamma_i)| \) and \( |\mathcal{A}_i| \times |\Gamma_i| = |\mathcal{P}| = n^3 \). Since \( [M_2, (M_1)_P] = 1 \), \( M_2 \) leaves \( \Gamma_1 \) invariant. Hence, by Lemma 3.3(i), \( \Gamma_1 \cap \Gamma_0 \neq \emptyset \) for any \( \Gamma_0 \in \mathcal{A}_2 \).

It follows that \( |\Gamma_1| \geq |\mathcal{A}_2| = n^2/|\Gamma_2| \) and so \( |\Gamma_1|/|\Gamma_2| \geq n^2 \). On the other hand \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are subplanes of \( \pi^* \) and therefore \( |\Gamma_1| \leq n \) and \( |\Gamma_2| \leq n \) by Bruck's theorem [7, Theorem 3.7]. Thus \( |\Gamma_1| = |\Gamma_2| \) and \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are subplanes of \( \pi^* \) of order \( m (= \sqrt{n}) \). In particular, \( (M_2)_P \) acts semiregularly on \( \mathcal{P} - \mathcal{P}_2 \cap \mathcal{P} \). Further \( |\Gamma_1 \cap \Gamma_0| = 1 \) for any \( \Gamma_0 \in \mathcal{A}_2 \). Therefore \( M_2 \) acts on \( \Gamma_1 \) as a Frobenius group. This implies that \( M_2 \) is not a simple group (cf. [5, Theorem 7.5]), a contradiction. If \( C_F(G) \neq 1 \), then \( C_F(G) \) is transitive on \( \mathcal{P} \) and hence \( C_F(C_F(G)) = \langle G \rangle \) is regular on \( \mathcal{P} \), a contradiction.

By Lemma 3.4, \( G \) is a nonabelian simple group and \( F \) is isomorphic to a subgroup of \( \text{Aut}(G) \). From the classification of the finite simple groups, \( G \) is isomorphic to one of the following groups: (i) alternating groups \( \text{Alt}(k) \), (ii) groups of Lie type \( G(q) \), (iii) the 26 sporadic simple groups. We will deal with these cases separately in Sections 4-6.

4. Groups of Lie Type

Let \( \pi = \pi(\mathcal{P}, \mathcal{L}) \), \( F \), \( G \) be as defined in Section 3. Let \( M \) be the stabilizer of an affine point of \( \pi \) in \( G \). In this section we assume that \( G \) is isomorphic
to $G(q)$, a simple group of Lie type associated with a root system $\mathcal{A}$ of rank $l$. Let $\Sigma$ and the Dynkin diagram of $\mathcal{A}$ be as defined in (2.3). In Lemmas 4.1–4.4 we consider the case $2 \nmid q$.

**Lemma 4.1.** Assume that $G$ is isomorphic to a simple group of Lie type of odd characteristic $p$. Then $G^*$ is isomorphic to one of the groups listed in (2.9) or Lemma 2.10.

**Proof.** Assume $p \nmid [G : M]$. Then $G^*$ is isomorphic to one of the groups listed in Lemma 2.10. Assume $p \nmid [G : M] = n^2$ and $A_\mathcal{G} \neq \emptyset$. Then $p \mid n$ and $|J_G| \equiv 1 \pmod{p}$. Hence $p \nmid [G : G_Q]$ for some point $Q \in \mathcal{I}_G$.

If $G \neq G_Q$, then by Lemma 2.7, $u \nmid [G_Q]$ for any $u \in A_\mathcal{G}$. Moreover $n \mid [G_Q]$ by Ostrom–Wagner’s result [8, Theorem 4.3]. Therefore, for each $u \in A_\mathcal{G}$, $u \nmid n^2$ and so $u \nmid n^2 = [G : M]$. Hence one of the cases in (2.9) occurs.

Now suppose $G = G_Q$. Then, by Ostrom–Wagner’s result, $|G : G_h| = n$ for an affine line $h$ through $Q$. Let $X$ be a maximal subgroup of $G$ containing $G_h$. We regard $G$ as a primitive permutation group on the cosets $G/X$ in the usual manner. By Lemma 2.8 we may assume that $A \mathcal{G} \neq \emptyset$ and $u \mid |G_P|$ for every $u \in A_\mathcal{G}$, where $P \in \mathcal{P}$. Hence $u \nmid [G : G_P] = n^2$ and so $u \mid n$. Since $|G : G_h|^2 = n^2 \mid |G|$ and $X \supseteq G_h$, $|G : X| \mid n$ and $|G : X|^2 \mid |G|$. On the other hand $u \mid |X|$ as $n \mid |X|$. Therefore we have $p \cdot u \mid |X|$, $u^2 \mid |G|$ and $|G : X|^2 \mid |G|$. Applying these facts to Lemma 2.8, we obtain a contradiction.

**Lemma 4.2.** $(G, M) \neq (A_i(q), A_j(q), J)$, where $2 \nmid q$ and $J \subseteq \Sigma$.

**Proof.** Assume $G = A_i(q)$ and $M = G_j$ for some $J \subseteq \Sigma$. First we show that $J = \Sigma - J_0$, where $J_0 = \{r_i\} \ (i \in \{1, 2, l-1, l\})$ or $J_0 = \{r_i, r_j\}$. By Lemma 2.10, this is clear when $l \leq 3$. So assume $l \geq 4$. There is an involution $z$ such that $C_G(z)$ involves $A_{l-2}(z)$ (cf. [2, Section 14]). As $l \geq 4$, $A_{l-2}(z) \neq \emptyset$. Let $u \in A_{l-2}(z)$. Note that $|A_i(q)_u| = |A_{l-2}(q)_z|$ as $l \geq 4$. (Here $|X|_u$ denotes the maximal power of $u$ dividing the order of $X$.) By Lemma 2.1(ii), we have $u \mid |M|$. Hence, by Lemma 2.10, $J$ can be expressed in the form stated above.

Next we consider the index of $M$ in $G$. As above $n^2 = |G : M| = |G : G_j| = (q^{l+1} - 1)/(q - 1)$, $(q - 1)(q^{l+1} - 1)/(q - 1)(q^2 - 1)$, $(l \geq 2)$ or $(q - 1)(q^{l+1} - 1)/(q - 1)^2$, $(l \geq 2)$ according as $J_0 = \{r_i\}$, $(i \in \{1, l\})$, $\{r_j\}$, $(j \in \{2, l-1\})$, or $\{r_i, r_j\}$. Hence $n < (q^{l+1} - 1)/(q - 1)$ if $l \geq 2$ and $n \leq \sqrt{q} + 1$ if $l = 1$. By Lemma 3.1(iii), $|G : G_P| < n$ for any $P \in \mathcal{I}_G$. By [4], $G_P = G$ for any $P \in \mathcal{I}_G$, a contradiction.

**Lemma 4.3.** $(G, M) \neq (E_6(q), (E_6(q))_j)$, where $2 \nmid q$ and $J = \{r_i\}$, $i \in \{1, 6\}$ (See (2.3).)
Proof. Suppose false. Then \( m^4 = |G : M| = (q^9 - 1)(q^{12} - 1)/(q - 1) \)
\((q^4 - 1) = ((q^2 + q + 1)^2(q^6 + q^3 + 1)(q^4 - q^2 + 1)(q^2 - q + 1).\)
Since \( q^6 + q^3 + 1 = (q^2 - q + 1)(q^4 + q^3 + 1) \) and \( q^2 - q + 1 = (q^2 - q + 1) \)
\((q^2 + q - 1) - 2(q - 1), \) we have \((q^6 + q^3 + 1, q^2 - q + 1) = (q^6 + q^3 + 1, q^2 - q + 1) = 1. \) Hence \( q^2 - q = b^2 \) for some \( b \in \mathbb{N}. \) However, this is a contradiction by Lemma 2.11.

**Lemma 4.4.** Assume \( |G : M| = q^b \) for some \( b \in \mathbb{N}. \) Then the following hold.

(i) If \( 2 | b, \) then \( q \equiv 1 \pmod{16}. \)

(ii) If \( 2 \nmid b \) but \( 4 | b, \) then \( q \equiv 1 \pmod{8}. \)

Proof. Set \( q = p^\ell. \) By Lemma 3.1(iii), \( 4 \sqrt{|G : M|} \in \mathbb{N}. \) Hence \( bc/4 \in \mathbb{N}. \) If \( 4 | b, \) then \( 2 | c \) and so \( q \equiv 1 \pmod{8}. \) Thus (ii) holds. Moreover, if \( 2 \nmid b, \) then \( 4 | c \) and (i) holds.

**Lemma 4.5.** \( G^\sigma \) is isomorphic to none of the groups listed in (2.9).

Proof. By Lemma 4.4, it suffices to consider the case that \( G = D_4(p). \) As \( 4 \sqrt{|G|} \in \mathbb{N}, \) \( p \neq 3. \) We note that \( (p-1)(p+1)(p^2-p+1)(p^2+p+1) = p^6 - 1 \) \(|G| \) and that \( |M| = |G|/|\mathcal{P}| = 2^{12}3^55^7 \) or \( 2^{12}3 \cdot 7. \) Clearly \( (p^2 + ep + 1, p + e) = (p^2 + ep + 1, 2) = 1, \) \((p^2 + ep + 1, p^2 + 1) = 1 \) and \((p^2 + ep + 1, p + e) = (3, p + e) \) for any \( e \in \{ \pm 1 \}. \) Since \( A_G = \{ 7 \} \) (See Remark after (2.9).), one of the following holds: (a) \( 3 | p + 1 \) and \( (p^2 + p + 1, 2 \cdot 3 \cdot 5 \cdot 7) = 1. \) (b) \( 3 | p - 1 \) and \( p^2 - p + 1 = 7^t \) for some \( t \in \mathbb{N}. \) If (a) occurs, then \( 4 \sqrt{p^2 + p + 1} \in \mathbb{N}, \) contrary to Lemma 2.11. By [12], (b) does not occur.

By Lemmas 4.1 4.5, we have

**Lemma 4.6.** \( G \) is not isomorphic to any simple group of Lie type of odd characteristic.

We now consider the case that \( p = 2. \) By [10, Lemma 2.3], together with Lemma 2.5, we have the following result.

**Lemma 4.7.** Assume that \( G \) is isomorphic to a simple group of Lie type of characteristic 2 and let \( M \) be the stabilizer of a point \( P \in \mathcal{P} \) in \( G. \) Then \( M = G_J \) is isomorphic to a maximal parabolic subgroup except in the following cases.

(i) \( G = A_1(q), J = \Sigma - \{ r_i, r_j \}, (i < j, i + j = l + 1). \)

(ii) \( G = C_2(q), J = \emptyset. \)

(iii) \( G = D_4(q), J = \Sigma - \{ r_{i-1}, r_i \}. \)
(iv) $G = D_4(q)$, $J = \{r_2\}, \{r_1, r_3\}, \{r_1, r_4\}$, or $\{r_1, r_3, r_4\}$.
(v) $G = E_6(q)$, $J = \{r_1, r_4\}$ or $\{r_2, r_3\}$.
(vi) $G = F_4(q)$.

**Lemma 4.8.** Under the hypothesis in Lemma 4.7, one of the following holds.

(i) $G = A_1(q)$ and $J = \Sigma - \{r_i\}$, $i \in \{1, 2, l - 1, l\}$ or $J = \Sigma - \{r_1, r_{11}\}$.
(ii) $G = C_4(q)$, $(l \geq 2)$ and $J = \Sigma - \{r_1\}$ or $G = C_3(q)$ and $J = \emptyset$.
(iii) $G = D_4(q)$, $(l \geq 4)$ and $J = \Sigma - \{r_1\}$ or $G = D_4(q)$ and $J = \Sigma - \{r_i\}$ $(i \in \{3, 4\})$.
(iv) $G = 2A_1(q)$ and $J = \Sigma - \{r_1\}$ or $G = 2A_3(q)$ and $J = \Sigma - \{r_2\}$.
(v) $G = 2D_4(q)$ and $J = \Sigma - \{r_1\}$.
(vi) $G = F_4(q)$.
(vii) $G = 2E_6(q)$ and $J = \Sigma - \{r_1\}$.
(viii) $G = E_6(q)$ and $J = \Sigma - \{r_i\}$, $1 \leq i \leq 6$, $i \neq 4$.
(ix) $G = E_7(q)$ and $J = \Sigma - \{r_1\}$.
(x) $G = E_8(q)$ and $J = \Sigma - \{r_8\}$.
(xi) $G = G_2(q)$ and $J = \Sigma - \{r_i\}$, $1 \leq i \leq 2$.
(xii) $G = 3D_4(q)$, $2F_4(q)$ or $2B_2(q)$.

**Proof.** Deny (xi) and (xii). Choosing a suitable involution $z$ of $G$, $C_G(z)$ involves a group $X$ isomorphic to a group of Lie type. The possibilities are listed below. (See [1].)

<table>
<thead>
<tr>
<th>$G$</th>
<th>$X$</th>
<th>$G$</th>
<th>$X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1(q)$</td>
<td>$A_{l-2}(q)$</td>
<td>$E_6(q)$</td>
<td>$A_5(q)$</td>
</tr>
<tr>
<td>$2A_1(q)$</td>
<td>$2A_{l-2}(q)$</td>
<td>$E_7(q)$</td>
<td>$D_6(q)$</td>
</tr>
<tr>
<td>$C_1(q)$</td>
<td>$C_{l-2}(q)$</td>
<td>$E_8(q)$</td>
<td>$E_7(q)$</td>
</tr>
<tr>
<td>$D_1(q)$</td>
<td>$D_{l-1}(q)$</td>
<td>$2E_6(q)$</td>
<td>$2D_4(q)$</td>
</tr>
<tr>
<td>$2D_1(q)$</td>
<td>$2D_{l-1}(q)$</td>
<td>$F_4(q)$</td>
<td>$C_3(q)$</td>
</tr>
</tbody>
</table>

Let $G$ be one of the groups above except in the following:

(a) $A_4(4)$, (b) $A_7(2)$, (c) $A_1(q)$ for $l \leq 3$, (d) $2A_4(2)$, (e) $2A_5(2)$,
(f) $2A_1(q)$ for $l \leq 3$, (g) $C_4(2)$, (h) $C_1(q)$ for $l \leq 3$, (i) $D_3(2)$,
(j) $2D_4(2)$, (k) $E_6(2)$, (l) $F_4(2)$.

Applying Lemma 2.7(ii) to $X$, we can verify that $A_X \neq \emptyset$ and $|X|_u = |G|_u$ for any $u \in A_G$. By Lemma 2.1(ii), $|G| \cdot |M|/|X| \in \mathbb{N}$. Hence $u |M|$ and so the assertion follows from Lemma 4.7.
On the other hand if (c), (f), or (h) occurs, we have (i), (iv), or (ii), respectively. Each of the remaining cases we can verify the following:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$A_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{d}(4)$, $2A_{d}(2)$, $2A_{2}(2)$, $C_{3}(2)$</td>
<td>$A_G = {11}$, $11^{2}\mid</td>
</tr>
<tr>
<td>$A_{3}(2)$, $D_{3}(2)$, $2D_{4}(2)$</td>
<td>$A_G = {17}$, $17^{2}\mid</td>
</tr>
<tr>
<td>$E_{6}(2)$, $F_{4}(2)$</td>
<td>$A_G = {13}$, $13^{2}\mid</td>
</tr>
</tbody>
</table>

Let $G$ be one of these and put $A_G = \{u\}$. By Lemma 2.7(i), $u \mid |G : M|$. Hence $u^4 \mid |G : M|$ by Lemma 3.1(iii). In particular $u^4 \mid |G|$, a contradiction.

**Lemma 4.9.** $G$ is not isomorphic to any simple group of Lie type of characteristic 2.

**Proof.** In each case in Lemma 4.8 we can calculate $|G : M|$. If $G$ is isomorphic to a classical group, then the possibilities for $G$ and $|G : M|$ are as follows:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$[G : M]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{l}(q)$</td>
<td>$q^{l+1} - 1$ $q - 1$ $q - 1$ $l = 2$, $q(q^4 - 1)(q^4 - 1)$ $(l = 2)$</td>
</tr>
<tr>
<td>$C_{l}(q)$</td>
<td>$(q^l - 1)(q^{l+1} - 1)$ $q - 1$ $l &gt; 2$ $q - 1$ $l = 2$</td>
</tr>
<tr>
<td>$D_{l}(q)$</td>
<td>$(q - 1)(q^l - 1)$ $q - 1$ $q - 1$ $l = 2$</td>
</tr>
<tr>
<td>$2A_{l}(q)$</td>
<td>$(q^{l+1} - 1)(q^l - 1)$ $q - 1$ $q - 1$ $l = 2$ $l!$ $(2l!)$</td>
</tr>
<tr>
<td>$2A_{2}(q)$</td>
<td>$(q^3 + 1)(q^2 - 1)$ $q + 1$ $q + 1$</td>
</tr>
<tr>
<td>$2D_{l}(q)$</td>
<td>$(q^{l+1} - 1)(q^l - 1)$ $q - 1$ $q - 1$ $l!$ $(2l!)$</td>
</tr>
</tbody>
</table>

It is easy to check that in each case $\sqrt{[G : M]}$ is less than the minimal degree of $G$ by the results of [4]. Since $\sqrt{[G : M]} = n < |I_x|$, $G = G_P$ for any $P \in I_\infty$, a contradiction.

In each of the remaining cases we can check by Lemma 4.7 that $[G : M] = (q^{2t} + cq^{t} + 1)k$ for suitable $t$, $c$, and $k$ such that $t, k \in \mathbb{N}$, $c \in \{\pm 1, 0\}$ and $(q^{2t} + cq^{t} + 1, k) = 1$:  

481 128 28
\[
\begin{array}{ccc}
G & [G : M] & q^{2j} + cq' + 1 \\
F_4(q) & \frac{(q^4 + 1)(q^{12} - 1)}{q - 1}, \frac{(q^6 - 1)(q^4 + 1)(q^{12} - 1)}{(q - 1)^2} & q^4 + 1 \\
 & \frac{(q^6 - 1)(q^8 - 1)(q^{12} - 1)}{(q^2 - 1)^2 (q - 1)} & \\
^2E_6(q) & \frac{(q^5 + 1)(q^9 + 1)(q^{12} - 1)}{(q - 1)(q + 1)} & q^6 + 1 \\
E_6(q) & \frac{(q^9 - 1)(q^{12} - 1)}{(q - 1)(q^4 - 1)}, \frac{(q^6 - 1)(q^8 - 1)(q^9 - 1)(q^{12} - 1)}{(q - 1)(q^2 - 1)(q^3 - 1)(q^4 - 1)}, & q^6 + 1 \\
 & \frac{(q^8 - 1)(q^9 - 1)(q^{12} - 1)}{(q - 1)(q^3 - 1)(q^4 - 1)} & \\
E_7(q) & \frac{(q^{12} - 1)(q^{14} - 1)(q^{18} - 1)}{(q - 1)(q^4 - 1)(q^6 - 1)} & q^6 + 1 \\
F_6(q) & \frac{(q^{20} - 1)(q^{24} - 1)(q^{30} - 1)}{(q - 1)(q^6 - 1)(q^{10} - 1)} & q^4 + 1 \\
G_5(q) & \frac{q^6 - 1}{q - 1} & q^2 + q + 1 \\
^3D_4(q) & (q + 1)(q^{12} + q^6 + 1) & q^6 + q^3 + 1 \\
^2F_4(q) & (q + 1)^2 (q^2 + 1)(q^2 - q + 1)(q^4 - q^3 + 1) & q^4 - q^2 + 1 \\
 & (q + 1)(q^2 + 1)^2 (q^2 - q + 1)(q^4 - q^2 + 1) & \\
^2B_2(q) & q^2 + 1 & q^2 + 1 \\
\end{array}
\]

Since \((q^{2j} + cq' + 1)k = [G : M] = n^2\) and \((q^{2j} + cq' + 1, k) = 1, \sqrt{q^{2j} + cq' + t} \) must be an integer. By Lemma 2.11, this is a contradiction. Thus we have the lemma.

5. ALTERNATING GROUPS

In this section we assume that \(G\) is isomorphic to \(\text{Alt}(\Omega)\), the alternating group on a finite set \(\Omega\) with \(d\) elements. By Lemma 3.1(iii)(iv), \(d \geq 18\).

Let \(M\) be the stabilizer of an affine point of \(\pi(\mathcal{P}, \mathcal{L})\). By [10, Theorem C], one of the following occurs:

(i) \(M\) is the global stabilizer of a subset \(A\) of \(\Omega\) such that \(1 \leq r \leq |A| \leq |\Omega|/2\).
(ii) $M$ is the global stabilizer of a partition $\mathcal{B} = \{A_i | 1 \leq i \leq l\}$ of $\Omega$ into subsets $A_i$ of size $k$, where $2 \leq k < d = kl$.

**Lemma 5.1.** Assume (i). Then $n^2 = \binom{d}{r} \cdot 8 \cdot \binom{\binom{4}{4} \cdot 3 + \binom{r}{2} \binom{z}{2} + \binom{r}{4} \cdot 3}{r! \cdot s!}$.

**Proof.** If $r = 1$, then $G^p = \text{Alt}(\mathcal{P})$ and $d = n^2$. Hence $G$ contains an involution $t$ such that $|\mathcal{F}(t) \cap \mathcal{P}| = n^2 - 4 > n$, a contradiction. Thus $r \geq 2$.

Set $\Gamma = \Omega - A$ and $s = |\Gamma|$. As $d \geq 18$, $s \geq 9$. Since $M^r = (G(A))^r = \text{Sym}(\Gamma)$ and $(M^r)^d = \text{Alt}(\Delta)$, we have $|M| = |M|^r \times |M^r| = s! \cdot r!/2$. From this $n^2 = |G : M| = (d!/2)((r!/s!/2) = \binom{d}{r} \cdot 8 \cdot \binom{\binom{4}{4} \cdot 3 + \binom{r}{2} \binom{z}{2} + \binom{r}{4} \cdot 3}{r! \cdot s!}$. Let $z$ be an involution of $M$ such that $z^q = (z^\beta)(\gamma^\delta)$ for some $z$, $\beta$, $\gamma$, $\delta \in \Gamma$. Set $\Delta_0 = \{z, \beta, \gamma, \delta\}$, $\tau_0 = \Omega - \Delta_0$ and $H = C_0(z)$. Then $H^r_0 = \text{Sym}(\tau_0)$, $(H^r_0)^d_0 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence $|C_0(z)| = (d - 4)! \times 4$. On the other hand $|z^Q \cap M| = \binom{z}{2} \cdot 3 + \binom{z}{2} \binom{z}{2} + \binom{z}{4} \cdot 3$. Here $\binom{z}{4} = 0$ when $r \leq 3$. Hence, by Lemma 2.1(i), $n = |\mathcal{F}(z) \cap \mathcal{P}| = 4(d - 4)! \binom{\binom{4}{4} \cdot 3 + \binom{z}{2} \binom{z}{2} + \binom{z}{4} \cdot 3}{r! \cdot s!/1}$. Thus the lemma holds.

**Lemma 5.2.** The case (i) does not occur.

**Proof.** Suppose false. First we show that $r \geq 4$. If $r = 2$, then $n^2 = d(d-1)/2$ and $n = 2 + (d-4)(d-5)/2$ by Lemma 5.1. Hence $(d-5)/2 < n < n^2 < d^2/2$ and so $(d-5)/2 < d/\sqrt{2}$, contrary to $d \geq 18$. If $r = 3$, similarly we have $(d-6)/3 < n < n^2 < d^3/6$, a contradiction. Thus $r \geq 4$.

Since $\binom{d}{r} = n^2 > \{(d-4)! \cdot 3 \binom{z}{2} \binom{r}{s} \}/2$ we have $\binom{d}{r} \cdot 8 \cdot \binom{\binom{4}{4} \cdot 3 + \binom{r}{2} \binom{z}{2} + \binom{r}{4} \cdot 3}{r! \cdot s!} = (d-4)! \cdot 8 < \binom{\binom{4}{4} \cdot 3 + \binom{r}{2} \binom{z}{2} + \binom{r}{4} \cdot 3}{r! \cdot s!}$. Let $z$ be an element of $M$ such that $z^Q = (z^\beta)(\gamma^\delta)$ for some $z$, $\beta$, $\gamma$, $\delta \in \Gamma$. We have $|z^Q \cap M| = 4 \times \binom{d}{r} \cdot 8 \cdot \binom{\binom{4}{4} \cdot 3 + \binom{r}{2} \binom{z}{2} + \binom{r}{4} \cdot 3}{r! \cdot s!} \binom{d}{r} \cdot 8 = \binom{d}{r} \cdot 8 \cdot \binom{\binom{4}{4} \cdot 3 + \binom{r}{2} \binom{z}{2} + \binom{r}{4} \cdot 3}{r! \cdot s!}$. On the other hand $n^2 = \binom{d}{r} \cdot 8 \cdot \binom{\binom{4}{4} \cdot 3 + \binom{r}{2} \binom{z}{2} + \binom{r}{4} \cdot 3}{r! \cdot s!}$. Thus the lemma holds.

**Lemma 5.3.** Assume (ii). Then $n^2 = (lk)!/[l! \cdot (k!)^l]$, $n = [(d-1)(d-2)(d-3)]/[((k-1)(k!^l - 2)^k + 6)]$ and $k \geq 4$, $l \geq 5$.

**Proof.** Set $H = G_{\mathcal{A}}$ and $K_i = M_{A_i \cup \ldots \cup A_i} (1 \leq i \leq l)$, the pointwise stabilizer of $A_i \cup \ldots \cup A_i$. Clearly $K_i \simeq 1$. Since $M/H \simeq \text{Sym}(\mathcal{A})$ and $H/K_i \simeq \text{Sym}(A_i)$, $K_i/K_i \simeq \text{Sym}(A_i)$ (2 $\leq i \leq l-1$), $K_{l-1}/K_i \simeq \text{Alt}(A_i)$, we have $|M| = l!(k!)^{l-1} k!/2$ and so $n^2 = (lk)!/[l! \cdot (k!^l - 2)^k + 6]$. This is a contradiction.

**Lemma 5.4.** Assume (iii). Then $n^2 = (l(k!^l - 2)^k + 6)$.

**Proof.** Set $H = G_{\mathcal{A}}$ and $K_i = M_{A_i \cup \ldots \cup A_i} (1 \leq i \leq l)$, the pointwise stabilizer of $A_i \cup \ldots \cup A_i$. Clearly $K_i \simeq 1$. Since $M/H \simeq \text{Sym}(\mathcal{A})$ and $H/K_i \simeq \text{Sym}(A_i)$, $K_i/K_i \simeq \text{Sym}(A_i)$ (2 $\leq i \leq l-1$), $K_{l-1}/K_i \simeq \text{Alt}(A_i)$, we have $|M| = l!(k!)^{l-1} k!/2$ and so $n^2 = (lk)!/[l! \cdot (k!^l - 2)^k + 6]$. This is a contradiction.
(l + 1)/2 × ⋯ × 2l/2. From this, l ≤ 3, contrary to the fact that n = kl ≥ 18. Thus k ≥ 3.

As k ≥ 3, z\(n\). 1 for any z \( \in z^{G} \cap M \). Hence |z\(G\) ∩ M| = l\((\frac{n}{2})\) × 3 + (\(\frac{n}{2}\))\((\frac{n}{2})\) and so n = |\(\mathcal{F}(z)\) ∩ \(\mathcal{P}\)| = [4(d - 4)](3\((\frac{n}{2})\) + (\(\frac{n}{2}\))(\(\frac{n}{2}\)))/(l!/2)(k!)\(^{l}\). Here (\(\frac{n}{2}\)) = 0 when k = 3. Hence n = n\^{2}/|\(\mathcal{F}(z)\) ∩ \(\mathcal{P}\)| = [lk(lk - 1)(lk - 2) (lk - 3)]/(lk(k - 1)(k - 2)(k - 3) + l(l - 1)(k(k - 1))\(^{2}\)] = [(lk - 1)(lk - 2)(lk - 3)]/[ (k - 1)(lk\(^{2}\) - (l + 4)k + 6)]\(^{2}\).

Assume k = 3. Then n\(^{2}\) = (3l)!/1!6\(^{l}\) and n = [8 × (3l - 4)!9(\(\frac{n}{2}\))]!/1!6\(^{l}\) = (3l)!/1!6\(^{l}\) × 4/(3l - 1)(3l - 2) and so so{(3l - 1)(3l - 2)/4}\(^{2}\) = {(3l)!/1!6\(^{l}\)}/n\(^{2}\) = (3l)!/1!6\(^{l}\) = (l + 1)/6 × (l + 2)/6 × ⋯ 2/6 × (2l + 1) (2l + 2) ⋯ (3l) > \{(3l - 1)(3l - 2)/4\}\(^{2}\), a contradiction. Thus k ≥ 4.

Assume l = 2. Then n = [(2k - 1)(2k - 2)(2k - 3)]/[ (k - 1)(2k\(^{2}\) - 6k + 6)] < 2\(^{3}\)[k\(^{2}\) - 4k + 2] = k\(^{2}\) - 3k + 3 < 4, a contradiction. Assume l = 3. Then n = [(3k - 1)(3k - 2)(3k - 3)]/[ (k - 1)(3k\(^{2}\) - 7k + 6)] = [3(9k\(^{2}\) - 9k + 2)]/[ (3k\(^{2}\) - 7k + 6] < 16. By Lemma 3.1(iii), \(\sqrt{n} \in \mathbb{N}\) and so n ≤ 9. If n = 9, then k = \(\frac{n}{3}\), a contradiction. Hence n ≤ 8. But it is known that every affine plane of order at most 8 is desarguesian.

Assume l = 4. Then, n = [(4k - 1)(4k - 2)(4k - 3)]/[ (k - 1)(4k\(^{2}\) - 8k + 6)]]. Since (k - 1, 4k - 3) = 1, k - 1 |(4k - 1)(4k - 2) = (k - 1) (16k + 4) + 6. Therefore k - 1 = 3 or 6 as k ≥ 4 and n = (5 · 7 · 13)/19 or (3\(^{3}\)5\(^{2}\) · 13)/(2 · 73), respectively. This is a contradiction.

**Lemma 5.4.** The case (ii) does not occur.

**Proof.** Suppose false. By Lemma 5.3, \{ [(d - 1)(d - 2)(d - 3)]/[ (k - 1) (lk\(^{2}\) - (l + 4)k + 6)]\}\(^{2}\) = n\(^{2}\) = (lk\(^{2}\) - (l + 4)k + 6)]/l(k!)\(^{l}\) > 1/l! × [k! k\((2k)\)k ⋯ ((l - 1)k\(^{k}\))/(k!)\(^{l}\) = (l - 1)!/l(k!)\(^{l}\) - 1/(k!)\(^{l}\). Since k\(^{k}\) ≥ k! and (l - 1)! ≥ \(\frac{44}{125}\) \(l\)^{k\(^{k}\)} - 1(k - 1)\(^{2}\). \(lk\(^{2}\) - (l + 4)k + 6) > \(\frac{44}{125}\) \(l\)^{k\(^{k}\)} - 1(k - 1)\(^{2}\). \(lk\(^{2}\) - (l + 4)k + 6) > \(\frac{44}{125}\) \(l\)^{k\(^{k}\)} - 1(k - 1)\(^{2}\). \(lk\(^{2}\) - (l + 4)k + 6) > \(\frac{44}{125}\) \(l\)^{k\(^{k}\)} - 1(k - 1)\(^{2}\), a contradiction.

By Lemmas 5.2 and 5.4, we have

**Lemma 5.5.** G is not isomorphic to any alternating group of degree at least 5.

6. Sporadic Simple Groups

In this section we consider the case that G is isomorphic to one of the 26 sporadic simple groups (cf. [6, Chapter 2]). Let \( S \) be a Sylow 2-subgroup of G and z an involution of Z(S). Let \( p_{0} \) be a maximal prime dividing |G|. Set \( \pi_{0} = \mathcal{F}(z) \), \( \Delta = \pi_{0} \cap l_{\infty} \), and \( I = l_{\infty} - \Delta \). By Lemma 3.1(iii),
\(|G : M| = m^2 = m^d\) for some odd integer \(m \geq 1\). Hence, it suffices to consider the following cases. (cf. [6, Table 2.4])

\[
\begin{align*}
(i) \quad G &= L_3, \ m | 3 \cdot 5, \ |S| = 2^8, \ p_0 = 67. \\
(ii) \quad G &= M_4, \ m | 3^2 5, \ |S| = 2^{21}, \ p_0 = 23. \\
(iii) \quad G &= M(22), \ m | 3^2, \ |S| = 2^{17}, \ p_0 = 13. \\
(iv) \quad G &= M(23), \ m | 3^3, \ |S| = 2^{18}, \ p_0 = 23. \\
(v) \quad G &= M(24)', \ m | 3^4, \ |S| = 2^{21}, \ p_0 = 29. \\
(vi) \quad G &= F_2, \ m | 3^3 5, \ |S| = 2^{41}, \ p_0 = 47. \\
(vii) \quad G &= F_3, \ m | 3^2, \ |S| = 2^{15}, \ p_0 = 31. \\
(viii) \quad G &= F_5, \ m | 3 \cdot 5, \ |S| = 2^{14}, \ p_0 = 19. \\
n(ix) \quad G &= F_1, \ m | 3^2 5^2, \ |S| = 2^{46}, \ p_0 = 71.
\end{align*}
\]

**Lemma 6.1.** (i) If \(p\) is a prime and \(p \mid |G|\), then \(p \leq m^2\).

(ii) \(|S| \mid (m + 1)! \cdot (m - 1)\).

(iii) If \(\sqrt{m} \not\equiv 1\) \(\pmod{n}\), then \(|S| \mid 2(m - 1)^3\).

**Proof.** Since \(G\) acts faithfully on \(l_\infty\), (i) holds and (iii) follows immediately from Lemma 2.2. Set \(T = S_\Delta\). Since \(T\) contains no nontrivial perspectivities, \(T\) is semiregular on \(I\). Hence \(|T| \mid |I| = m(m - 1)\) and so \(|T| \mid m - 1\). Clearly \(S/T \simeq S^d \leq \text{Sym}(d)\). Thus \(|S| = |T| \\cdot |S/T| \mid (m - 1) \cdot (m + 1)!\). Thus (ii) holds.

By using Lemma 6.1 together with (i)–(ix) as above, we have

**Lemma 6.2.** One of the following holds.

\((G, m) = (M(24)', 3^4), (F_1, 3^4), (F_1, 3^2 5^2), \) or \((F_1, 3^4 5^2)\).

**Lemma 6.3.** Let \(T( \neq 1)\) be a 2-subgroup of \(S\) and set \(\Psi = \mathcal{F}(T) \cap l_\infty\). If \(|\Psi| > \sqrt{m} + 1\), then \(|T| \mid m - 1\).

**Proof.** Let \(t\) be an involution of \(Z(T)\) and set \(z = \mathcal{F}(t), A = \pi_1 \cap l_\infty\) and \(l_\infty = A_1\). Then \(|A_1| = m + 1\) and \(A_1 \supset \Psi\). Let \(u\) be an arbitrary involution of \(T\). Then, as \(u^{z_1}\) is a collineation of \(z_1\) such that \(\mathcal{F}(u^{z_1}) \supset \Psi, \mathcal{F}(u) \cap A_1 = A_1\). Therefore \(T\) is semiregular on \(I\). Thus \(|T| \mid m^2 - m = m(m - 1)\) and the lemma holds.

**Lemma 6.4.** \((G, m) \neq (M(24)', 3^4), (F_1, 3^4)\).

**Proof.** Suppose false. As \(3^4 + 1 \equiv 2 + 16 \pmod{32}\), there exists an \(S\)-invariant subset \(A\) of \(A\) such that \(|A| = 16\). Set \(T = S_\Delta\). Then \(|T| \geq 2^6\)
since \(|S| \geq 2^{21}\) and \(S/T \cong S^4 \leq \text{Sym}(A) \cong \text{Sym}(16)\). By Lemma 6.3, 
\(|T| = m - 1 = 3^4 - 1 = 2^45\), a contradiction.

**Lemma 6.5.** Let \(T \neq \{1\}\) be a 2-subgroup of \(S\) such that \(\mathcal{F}(T)\) is a subplane of \(\pi\). If \(\sqrt{\text{m} \in \mathbb{N}}\) and \(\sqrt{\text{m} \notin \mathbb{N}}\), then \(|T| \bigl(\sqrt{\text{m} - 1}(m - 1)\bigr)\).

**Proof.** Let \(t\) be an involution of \(Z(T)\) and set \(\pi_1 = \mathcal{F}(t)\) and \(K = T_{\pi_1}\). Then \(\pi_1\) is a subplane of \(\pi\) of order \(m\). Since \(K\) is semiregular on \(l_\infty - \pi_1 \cap l_\infty\), \(|K| = m^2 - m = m(m - 1)\) and so \(|K| = m - 1\). Let \(u \in T - K\) such that \(u^2 \in K\). Then \(u^{\pi_1}\) is a collineation of \(\pi_1\) of order 2 and \(\mathcal{F}(\langle u \rangle K)\) is a subplane \(\pi_2\) of \(\pi_1\) of order \(\sqrt{m}\). Since \(\sqrt{\text{m} \notin \mathbb{N}}\) and \(\mathcal{F}(\langle u \rangle K) \supset \mathcal{F}(T)\), \(\mathcal{F}(T)\) is a subplane of \(\pi_1\) of order \(\sqrt{m}\). Therefore \(\mathcal{F}(\langle u \rangle K) = \mathcal{F}(T)\). From this \(T_{\pi_1} \simeq T/K\) is semiregular on \(\pi_1 \cap l_\infty - \pi_2 \cap l_\infty\) and so \(|T/K| = m - \sqrt{m} = \sqrt{m}(\sqrt{m - 1})\). Thus \(|T| = |K| \cdot |T/K| (m - 1)(\sqrt{m - 1})\).

**Lemma 6.6.** \((G, m) \neq (F_2, 3^25^2), (F_1, 3^45^2)\).

**Proof.** Since \(|l_\infty| = 2 \pmod{4}\), there exists a subgroup \(W\) of \(S\) such that \(|S : W| \leq 2\) and \(|\mathcal{F}(W) \cap l_\infty| \geq 2\). Let \(P \in \mathcal{F}(W) \cap \mathcal{P}\) and \(Q_1; Q_2 \in \mathcal{F}(W) \cap l_\infty\). Let \(w\) be an involution of \(Z(W)\). Set \(\pi_1 = \mathcal{F}(w) \supset \{P, Q_1, Q_2\}\). Assume \(m = 3^25^2\) or \(3^45^2\). Then \(2^{6}\bigl(|(\pi_1 \cap PQ_i) - \{P, Q_i\}\bigr)|\) for \(i \in \{1, 2\}\). Hence we can choose \(P_i \in \pi_1 \cap PQ_i - \{P, Q_i\}\) so that \(|W : W_{P_i}| \leq 2^5\) for each \(i \in \{1, 2\}\). Set \(T = W_{P_1, P_2}\). Then \(|W : T| \leq 2^{10}\). Furthermore \(T\) satisfies the hypothesis of Lemma 6.5. From this \(|T| \bigl(\sqrt{m - 1}(m - 1)\bigr)\) and so \(|T| \leq 2^6\). Thus \(|S| = |S : W| \cdot |W : T| \cdot |T| \leq 2^6 \cdot 2^{10} \cdot 2^6 = 2^{17}\), a contradiction.

We have proved the following.

**Lemma 6.7.** \(G\) is not isomorphic to any sporadic simple group.

**Remark.** In this section we only used the assumption that \(G\) is transitive on \(\mathcal{P}\) and has no involutory perspectivities. Hence we have practically shown the following fact.

**Proposition 6.8.** If an affine plane of odd order admits a point transitive collineation group isomorphic to one of the 26 sporadic simple groups, then the group contains an involutory perspectivity.

By Lemmas 4.6, 4.8, 5.5, 6.7, we have a contradiction. This implies that the group \(F\) contains a nontrivial perspectivity. Thus we have obtained the following.

**Theorem 2.** A point primitive collineation group of a finite affine plane contains a nontrivial perspectivity.
Proof of Theorem 1. By Theorem 2 above, together with [8, Corollary 15.1.1], we have Theorem 1.

REFERENCES