Noiselets

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Noiselets are functions which are noise-like in the sense that they are totally uncompressible by orthogonal wavelet packet methods. We describe a library of such functions and demonstrate a few of their noise-like properties.

0. INTRODUCTION

As the reader undoubtedly knows, various effective algorithms exist for using wavelets and wavelet packets to process data, for example, for compression or noise removal. In these algorithms, analysis of data is achieved because one is able to find rapid decay in the distribution of values of the data, when it is transformed into wavelet or wavelet packet bases.

In practice one finds that the few large values in the transformed data describe the interesting part of the data, and the vast majority of values, which are small, represent a noise term. See, for example, [6].

The performance of these algorithms is impressive and might lull one into the belief that analysis of any "interesting" structure can be carried out via wavelet packet analysis. Of course this cannot be so, and this paper gives constructions of large families of functions which give worst case behavior for orthogonal wavelet packet compression schemes.

Noiselets are functions which give worst case behavior for the aforementioned type of orthogonal wavelet packet analysis. In particular, this paper gives explicit examples of (complex-valued) noiselets for which all Haar–Walsh wavelet packet coefficients have exactly the same absolute value. So, in some sense, noiselets are "noise-like," and in particular, noiselets are totally uncompressible by orthogonal wavelet packet methods.

Although noiselets are noise-like in the sense of being spread in time and frequency, there are patterns lurking in them. Certain families of noiselets arise as bases for the spaces of the Haar multiresolution analysis. These bases are computationally good in the same way that wavelet packets are; they come with fast algorithms for forward and inverse transforms and there are trees of bases with the structure needed to support the best-basis algorithm. These good properties of noiselets are no coincidence. Noiselets are

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constructed via a multiscale iteration in exactly the same way as wavelet packets, but with a twist. So in some sense noiselets have the structure of wavelet packets. Because of this fast computational structure, the possibility exists that noiselets will be valuable tools for certain applications, rather than simply representing counterexamples.

Another source of pattern within noiselets is that one finds within their construction certain classical fractal generating mechanisms. In fact, a whole class of noiselets are nothing but the distributional derivatives of the classical paper folding curves (see [4] for an introduction to paper folding). Hence noiselets provide a counterexample to the philosophical view of analysis with which this note began. Indeed, one sees that certain interesting multiscale mechanisms can produce well-organized data which are nonetheless invisible to our standard analysis tools.

This paper provides constructions of families of noiselets which are shown to give bases for the spaces of the Haar multiresolution analysis and to have totally flat Haar wavelet packet coefficients. The Fourier transforms of the noiselets are computed and are seen to also be reasonably flat.

Forthcoming articles by the authors will give generalizations and show that noiselets cannot be "denoised" using local time-frequency methods.

The authors should point out that after this work had been completed, the article by Benke [1] appeared, where certain related but more general constructions were given. However, in the latter article, no connection with wavelet packet analysis was made, and nothing about the constructions in [1] implies the existence of Haar–Walsh totally flat systems of bases, which is the main point of the present article.

1. PRELIMINARIES

Throughout this paper we will need to talk about the binary expansions of nonnegative integers, and the following functions will be used. Define the *binary length of* n by

$$\ell(n) = \lfloor \log_2 n \rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer which is not greater than x. Note that $\ell(0)$ is undefined, and by convention any sum of the form $\sum_{j=0}^{\ell(0)}$ will be taken to be the empty sum and equal to zero, and any product of the form $\prod_{j=0}^{\ell(0)}$ will be taken to be the empty product and equal to one.

We define $\nu_j(n) \in \{0, 1\}$ to be the jth digit in the binary expansion of n, so that

$$n = \sum_{j=0}^{\ell(n)} \nu_j(n) 2^j.$$

We define

$$\epsilon_j(n) = (-1)^{\nu_j(n)}.$$

The Haar multiresolution analysis on $[0, 1]$ is defined by:

$$V_n = \left\{ f \in L^2([0,1]) \mid f \text{ is constant on all intervals of the form } (k2^{-n}, (k+1)2^{-n}) \right\}.$$

The Walsh functions are defined by:

$$\begin{aligned} W_0(x) &= \chi_{[0,1)}(x) \\ W_{2n}(x) &= W_n(2x) + W_n(2x-1) \\ W_{2n+1}(x) &= W_n(2x) - W_n(2x-1). \end{aligned} \tag{1}$$

Let $r_0(x)$ and $r_1(x)$ denote the first two Rademacher functions, extended periodically to all of $\mathbb{R}$. That is, $r_0(x) = 1$, and

$$r_1(x) = \begin{cases} 1, & \text{if } x \in [k, k+\frac{1}{2}) \text{ for some integer } k, \\ -1, & \text{otherwise.} \end{cases}$$

We recall two standard lemmas about these objects.

LEMMA 1. *The functions $W_0, \ldots, W_{2^N-1}$ are an orthonormal basis for V_N.*

LEMMA 2. *For each $n \geq 0$,*

$$W_n(x) = \prod_{j=0}^{\ell(n)} r_{\nu_j(n)}(2^j x),$$

restricted to $[0, 1]$.

Functions of the form $W_n(2^q x - k)$ are called Haar–Walsh wavelet packets. In the Haar–Walsh context, these are the functions from which one chooses subsets to produce bases corresponding to various partitionings of the phase plane. See, for example, [6, 5].

2. NOISELETS

In this section we will construct a family of bases subordinate to the Haar multiresolution analysis. The sequence of bases will be seen to limit to a distributional resolution of the identity. Each of the constructed functions will have all of its Haar–Walsh coefficients be of modulus 1, up to the finest possible scale. The limiting distributions will have well-defined Haar–Walsh coefficients, all of them of modulus 1. The functions will be supported on $[0, 1]$ where they will have constant absolute value. For the extension to $\mathbb{R}$, see Section 6.

Consider the family of functions defined recursively by:

$$\begin{aligned} f_1(x) &= \chi_{[0,1)}(x) \\ f_{2n}(x) &= (1-i)\,f_n(2x) + (1+i)\,f_n(2x-1) \\ f_{2n+1}(x) &= (1+i)\,f_n(2x) + (1-i)\,f_n(2x-1). \end{aligned} \tag{2}$$

Note the similarity with Eq. (1), and beware of the fact that here the iteration starts with f_1, while in Eq. (1), it starts with W_0.

LEMMA 3. *The set $\{f_j \mid j = 2^N, \ldots, 2^{N+1}-1\}$ is an orthogonal basis for V_N.*

Proof. By counting it is enough to show that if $2^N \leq j < k < 2^{N+1}$, then $\langle f_j, f_k\rangle = 0$. When $N = 0$ there is nothing to prove, so suppose the theorem is true for $N - 1$. Now

$$\langle f_j, f_k\rangle = \frac{1}{2}(1-\epsilon_0(j)i)(1+\epsilon_0(k)i)\langle f_{\lfloor j/2\rfloor}, f_{\lfloor k/2\rfloor}\rangle$$
$$+\frac{1}{2}(1+\epsilon_0(j)i)(1-\epsilon_0(k)i)\langle f_{\lfloor j/2\rfloor}, f_{\lfloor k/2\rfloor}\rangle.$$

If $\lfloor j/2\rfloor \neq \lfloor k/2\rfloor$ then the two terms on the right above are zero, by induction. Otherwise, since we have assumed that $j<k$, we have that j is even and $k=j+1$. Let $l=j/2$. Then

$$\langle f_j, f_k\rangle = -i\langle f_l, f_l\rangle + i\langle f_l, f_l\rangle = 0. \quad \blacksquare$$

LEMMA 4. *For each* $n\geq 1$,

$$\int_0^1 f_n(x)\,dx = 1.$$

Proof. This follows immediately from Eqs. (2) by induction. ■

Let $\tilde{r}_0(x) = 1 - ir_1(x)$, and $\tilde{r}_1(x) = 1 + ir_1(x)$.

LEMMA 5. *For each* $n\geq 1$,

$$f_n(x) = \prod_{j=0}^{\ell(n)-1} \tilde{r}_{\nu_j(n)}(2^j x),$$

restricted to $[0,1]$.

Proof. Indeed, when $n=1$ we have an empty product, which equals (by definition) 1. The result follows by induction and the observation that Eqs. (2) may be rewritten:

$$f_{2n}(x) = \tilde{r}_0(x) f_n(2x \mod 1)$$
$$f_{2n+1}(x) = \tilde{r}_1(x) f_n(2x \mod 1). \quad \blacksquare$$

COROLLARY 6. *For each* $n\geq 1$, *and for all* m *such that* $\ell(m)<\ell(n)$,

$$f_n(x)W_m(x) = i^k f_{n'}(x),$$

for all $x\in[0,1]$, *where* $k = -\sum_{j=0}^{\ell(m)} \nu_j(m)\epsilon_j(n)$ *(i.e., the number of places in the binary expansion where* m *is* 1 *and* n *is* 1, *minus the number of places where* m *is* 1 *and* n *is* 0*), and* n' *is defined by* $\epsilon_j(n') = \epsilon_j(n)\epsilon_j(m)$, *for* $j=0,\ldots,\ell(n)$.

Proof. Simply combine Lemmas 2 and 5, and note that

$$r_0(2^j x)\tilde{r}_0(2^j(x)) = \tilde{r}_0(2^j x)$$
$$r_0(2^j x)\tilde{r}_1(2^j(x)) = \tilde{r}_1(2^j x)$$
$$r_1(2^j x)\tilde{r}_0(2^j(x)) = -i\tilde{r}_1(2^j x)$$

and

$$r_1(2^j x)\tilde{r}_1(2^j(x)) = i\tilde{r}_0(2^j x). \quad \blacksquare$$

Combining this with Lemma 4 we get:

COROLLARY 7. *For each $n \geq 1$ and all m such that $\ell(m) < \ell(n)$, we have that*

$$\int_0^1 f_n(x) W_m(x)\, dx = i^k,$$

where $k = -\sum_{j=0}^{\ell(m)} \nu_j(m)\epsilon_j(n)$.

LEMMA 8. *For each $n \geq 1$ and all m such that $\ell(m) \geq \ell(n)$, we have that*

$$\int_0^1 f_n(x) W_m(x)\, dx = 0.$$

Proof. On any dyadic interval of size $2^{-\ell(m)}$, the function W_m has integral zero, and when $\ell(m) \geq \ell(n)$, the function f_n is constant on these intervals. ■

LEMMA 9. *Given $n, m \in \mathbb{Z}^+$, and $l \geq 0$,*

$$\int f_m(x) W_k(x)\, dx = \int f_n(x) W_k(x)\, dx$$

for all $k < 2^l$ if and only if $\nu_j(m) = \nu_j(n)$ for all $0 \leq j < l$.

Proof. Indeed, by Lemmas 7 and 8, the projection of f_n into V_l is f_n' where $\nu_j(n') = \nu_j(n)$ for $0 \leq j < l$, $\nu_l(n') = 1$, and $\nu_j(n') = 0$ otherwise. Hence, both hypotheses translate into the statement that f_n and f_m have the same projection on V_l. ■

Lemma 7 showed that, up to the finest reasonable scale, the functions f_n are flat in the Walsh basis. The next lemma shows that the f_n's are actually flat up to the finest reasonable scale in all Haar–Walsh wavelet packet bases.

LEMMA 10. *Let $q, m, n \in \mathbb{Z}^+$. If $q + \ell(m) < \ell(n)$, and $0 \leq k < 2^q$, then*

$$\int_0^1 f_n(x) 2^{q/2} W_m(2^q x - k)\, dx$$

is an eighth root of unity. Otherwise (if $q + \ell(m) \geq \ell(n)$), it is zero.

Proof. The function $W_m(2^q x - k)$ is supported on the interval $[k2^{-q}, (k+1)2^{-q}]$. The functions $\tilde{r}_0(2^p x)$ and $\tilde{r}_1(2^p x)$ are constant on that interval when $p < q$. Hence,

$$\begin{aligned}\int_0^1 f_n(x) 2^{q/2} W_m(2^q x - k)\, dx &= \int_0^1 \prod_{j=0}^{\ell(n)-1} \tilde{r}_{\nu_j(n)}(2^j x) 2^{q/2} W_m(2^q x - k)\, dx \\ &= \prod_{j=0}^{q-1} (1 \pm i) \int_0^1 \prod_{j=q}^{\ell(n)-1} \tilde{r}_{\nu_j(n)}(2^j x) 2^{q/2} W_m(2^q x - k)\, dx \\ &= e^{i\pi n'/4} \int_0^1 \prod_{j=0}^{l-1-q} \tilde{r}_{\nu_{j+q}(n)}(2^j x) W_m(x)\, dx.\end{aligned}$$

By the previous lemmas, this last quantity is either $e^{i\pi n''/4}$ (when $\ell(n) - q > \ell(m)$) or 0 (when $\ell(n) - q \leq \ell(m)$). ■

Now let $\{\epsilon_i\}_{i=0}^{\infty}$ be any sequence with values in $\{-1, 1\}$. Let $\nu_j = (1 - \epsilon_j)/2$, and $n_k = 2^k + \sum_{j=0}^{k-1} \nu_j 2^j$. By Lemma 9, if $k \geq l$ then f_{n_k} and f_{n_l} act as the same distribution on V_l; hence we expect the limiting distribution to exist. The next result shows that this is so and shows to what extent the spectrum of a noiselet is spread out or flat.

PROPOSITION 11. *The distributional limit of f_{n_i} exists and is the $|\xi|^{1/2}$-tempered distribution $f_{\vec{\epsilon}}$ whose Fourier transform is*

$$\hat{f}_{\vec{\epsilon}}(\xi) = e^{-i\xi/2} \prod_{j=2}^{\infty} \left(\cos(\xi/2^j) + \epsilon_{j-2} \sin(\xi/2^j)\right).$$

Proof. This is essentially the standard argument (see, for example, [3]). One first notes that there is some real number $C > 0$ such that

$$\left| \prod_{j=2}^{N} \left(\cos(\xi/2^j) + \epsilon_{j-2} \sin(\xi/2^j)\right) \right| \leq C\sqrt{1 + |\xi|}, \tag{*}$$

where C does not depend on N, and such that the product converges as $N \to \infty$, uniformly on compact sets.

Now, defining $K(x) = \chi_{[0,1]}$, and $K_N(x) = 2^N K(2^N x)$, one has

$$\begin{aligned} f_{n_N}(x) = 2^{-N}&((1 - \epsilon_0 i)\delta_0 + (1 + \epsilon_0 i)\delta_{2^{-1}}) \\ &\times ((1 - \epsilon_1 i)\delta_0 + (1 + \epsilon_1 i)\delta_{2^{-2}}) \\ &\vdots \\ &\times ((1 - \epsilon_{N-1} i)\delta_0 + (1 + \epsilon_{N-1} i)\delta_{2^{-N-1}}) \\ &\times K_N(x). \end{aligned} \tag{3}$$

But

$$((1 \pm i)\delta_0 + (1 \mp i)\delta_1)^{\xi} = (1 \pm i) + (1 \mp i)e^{-i\xi} = 2e^{-i\xi/2}(\cos(\xi/2) \mp \sin(\xi/2)).$$

Hence

$$\hat{f}_{n_N}(\xi) = \hat{K}_N(\xi) \prod_{j=2}^{N+1} e^{-i\xi/2^j} (\cos(\xi/2^j) + \epsilon_{j-2} \sin(\xi/2^j)).$$

But $|\hat{K}_N(\xi)| \leq 1$ and $\hat{K}_N(\xi) \to 1$, so, by virtue of the uniform bound (*), $\hat{f}_{n_N}$ converges to the well-defined limit stated in the proposition. ∎

Next, we observe that the distributions $f_{\vec{\epsilon}}$ are totally flat in the Haar–Walsh phase plane. That is:

LEMMA 12. *For any $m, q \geq 0$, and $0 \leq k < 2^q$ one has*

$$\left| \int f_{\vec{\epsilon}}(x) 2^{q/2} W_m(x)\, dx \right| = 1.$$

Proof. One only needs to make sense of the integral. Since $f_{\vec{\epsilon}}$ is a tempered distribution we can pair it with any Schwartz function. Although the Haar wavelet packets are not smooth, we can nevertheless pair with them as well. Indeed, $f_{\vec{\epsilon}}$ is a limit of functions f_{n_i}, and f_{n_j} agrees with f_{n_k} in the sense of distributions on all functions in $V_{\min(j,k)}$. In particular, $f_{\vec{\epsilon}}$ agrees with f_{n_j} on V_j. Hence this lemma follows from Lemma 10. ■

The noiselets introduced in this section are complex valued functions of a real variable. As such, they can be graphed as curves in the complex plane. However, any give noiselet assumes only four values, and hence the graph simply looks like an "X" in a box (i.e., a complete graph on four vertices). The intricate structure of various noiselets is made clear by plotting their indefinite integrals. Figure 1 shows graphs of the functions

$$F_n(t) = \int_0^t f_n(x)\,dx$$

as curves in the complex plane, for the n indicated, and arranged in a basis tree with descendants according to Lemma 9.

3. THE DRAGON NOISELETS

The constructions in the preceding section give rise to a basis of distributions which have a totally flat Haar–Walsh spectrum. Since the functions f_n have constant absolute value on the interval $[0, 1]$, these distributions can be thought of as being totally spread out in time and scale (see Section 6 for an extension from $[0, 1]$ to $\mathbb{R}$).

One would ultimately like the distributions to be totally spread in all reasonable notions of phase plane, and in particular in time and frequency.

In a sense which can be made precise, the constructions in the previous section are related to a complexification of the automatic sequence known as the Thue–Morse sequence. If one carries out a similar complexification of another automatic sequence, the Rudin–Shapiro sequence, then one gets a family which does not arise from an infinite convolution product and one has the head-start of basing the construction on a sequence which was designed to produce Fourier-spread sequences.

It turns out that this notion gives rise to the distributional basis whose elements are simply the distributional derivatives of the classical Dragon curves.

Consider the family of functions defined recursively by:

$$\begin{aligned} g_1(x) &= \chi_{[0,1)}(x) \\ g_{2n}(x) &= (1-i)g_n(2x) + (1+i)g_n(2-2x) \\ g_{2n+1}(x) &= (1+i)g_n(2x) + (1-i)g_n(2-2x). \end{aligned} \tag{4}$$

The proofs of the next two lemmas are word-for-word the same as for the f_n's.

LEMMA 13. *The set* $\{g_j \mid j = 2^N, \ldots, 2^{N+1}-1\}$ *is an orthogonal basis for* V_N.

LEMMA 14. *For each* $n \geq 1$,

$$\int_0^1 g_n(x)\,dx = 1.$$

Now the functions g_n are not easily written as a recursive product because the second terms in Eqs. (4) are flipped. But it is easy to compute the Haar–Walsh coefficients of the

g_n's directly from Eqs. (1) and (4). In fact, we could have taken this approach with the f_n's as well.

LEMMA 15. *Let $n \in \mathbb{Z}^+$, $m \geq 0$. If $\ell(m) < \ell(n)$, then*

$$\int_0^1 g_n(x) W_m(x)\, dx = i^{k'},$$

where

$$k' = -\sum_{j=0}^{\ell(m)} \epsilon_j(n) \cdot \frac{1 - \epsilon_j(m)\epsilon_{j+1}(m)}{2}.$$

Otherwise (if $\ell(m) \geq \ell(n)$), it is zero. Note, k' is the number of j such that in the binary expansion, n is 1 at the jth place and m is different at the jth and $j+1$st place, minus the number of j for which n is 0, and m has such a difference.

Proof. The theorem is true for $n = 1$, so we may proceed by induction.

Now, suppose that n is even. Let $l = n/2$, let $m' = \lfloor m/2 \rfloor$. Then

$$\begin{aligned}\langle g_n, W_m\rangle &= 1/2\langle(1-i)g_l, W_{m'}\rangle \pm 1/2\langle(1+i)g_l, W_{m'}\rangle \\ &= i^{l'}\langle g_l, W_{m'}\rangle\end{aligned}$$

for $l' = 0$ or 3. In the first line, the "plus or minus" comes from the fact that the second g_l is flipped, so if m' is odd the sign changes once, by Eqs. (1). Also, if m is odd the sign changes once, by Eqs. (1). So $l' = 0$ or 3 depending on whether m and m' have the same or different parity.

When n is odd, simply exchange the $(1+i)$ and $(1-i)$ above, and get that $l' = 0$ or 1, depending on whether m and m' have the same or different parity, respectively.

In any case,

$$\langle g_n, W_m\rangle = \prod_j i^{l'_j}.$$

Hence the result follows by induction. ■

In the same way as in the previous section, we conclude:

COROLLARY 16. *Given $n, m \in \mathbb{Z}^+$, and $l \geq 0$,*

$$\int g_m(x) W_k(x)\, dx = \int g_n(x) W_k(x)\, dx$$

for all $k < 2^l$ if and only if $\nu_j(m) = \nu_j(n)$ for all $0 \leq j < l$.

LEMMA 17. *Let $n \in \mathbb{Z}^+$, $q, m \geq 0$, and $0 \leq k < 2^q$. If $q + \ell(m) < \ell(n)$, then*

$$\int_0^1 g_n(x) 2^{q/2} W_m(2^q x - k)\, dx$$

is an eighth root of unity. Otherwise (if $q + \ell(m) \geq \ell(n)$), it is zero.

Proof. The case where the integral is zero is the same as before; the rescaled W_m has integral zero on intervals over which g_n is constant.

In the other case, notice that the function $W_m(2^q x - k)$ is supported on the interval $[k2^{-q}, (k+1)2^{-q}]$. It is easy to see by induction that on this interval the function g_n is equal to some g'_n, possibly time-reversed, times q factors of the form $1 \pm i$. Hence it is $2^{q/2}$ times an eighth root of unity times g'_n possibly flipped. A Walsh function flipped either stays the same or is multiplied by -1 (i.e., it is either even or odd about $1/2$). Hence, by a change of variables, it is enough to prove this in the case $q = 0$. But this is just Lemma 15. ■

Next, as in the previous section, we will see that g_{n_i} converges as a tempered distribution. Since the g_n's do not arise from convolutions this will be slightly more involved. So, let $\{\epsilon_i\}_{i=0}^{\infty}$ be any sequence with values in $\{-1, 1\}$. Let $\nu_j = (-1)^{\epsilon_j}$ and $n_k = 2^k + \sum_{j=0}^{k-1} \nu_j 2^j$.

PROPOSITION 18. *The distributional limit of g_{n_i} exists and is a tempered distribution $g_{\vec{\epsilon}}$. If we define the distribution $h_{\vec{\epsilon}}(x) = g_{\vec{\epsilon}}(1-x)$ then the Fourier transforms are given by*

$$\begin{pmatrix} \hat{g}_{\vec{\epsilon}}(\xi) \\ \hat{h}_{\vec{\epsilon}}(\xi) \end{pmatrix} = \prod_{j=0}^{\infty} \begin{pmatrix} \frac{1-\epsilon_j i}{2} & \frac{1+\epsilon_j i}{2} e^{-\pi i \xi/2^j} \\ \frac{1+\epsilon_j i}{2} & \frac{1-\epsilon_j i}{2} e^{-\pi i \xi/2^j} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with later multiplications on the right.

Note. The infinite product of unitary matrices above does not converge as a matrix. However, it converges to a 4-cycle of matrices, each of which take the vector

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

to the same place, so the infinite product of matrices applied to the vector does converge (uniformly on compact sets).

Proof. From Eqs. (4) we know that

$$g_{n_j}(x) = (1-\epsilon_0 i) g_{\lfloor n_j/2 \rfloor}(2x) + (1+\epsilon_0 i) g_{\lfloor n_j/2 \rfloor}(2-2x). \tag{5}$$

Let $h_{n_j}(x) = g_{n_j}(1-x)$. Then, from (5) we have

$$\begin{pmatrix} g_{n_j}(x) \\ h_{n_j}(x) \end{pmatrix} = \begin{pmatrix} (1-\epsilon_0 i) & (1+\epsilon_0 i) \\ (1+\epsilon_0 i) & (1-\epsilon_0 i) \end{pmatrix} \cdot \begin{pmatrix} g_{\lfloor n_j/2 \rfloor}(2x) \\ h_{\lfloor n_j/2 \rfloor}(2x-1) \end{pmatrix}.$$

Taking Fourier transforms, one finds that

$$\begin{pmatrix} \hat{g}_{n_j}(\xi) \\ \hat{h}_{n_j}(\xi) \end{pmatrix} = \begin{pmatrix} \frac{1-\epsilon_0 i}{2} & \frac{1+\epsilon_0 i}{2} e^{-\pi i \xi} \\ \frac{1+\epsilon_0 i}{2} & \frac{1-\epsilon_0 i}{2} e^{-\pi i \xi} \end{pmatrix} \cdot \begin{pmatrix} \hat{g}_{\lfloor n_j/2 \rfloor}(\xi/2) \\ \hat{h}_{\lfloor n_j/2 \rfloor}(\xi/2) \end{pmatrix}.$$

It follows that

$$\begin{aligned} \begin{pmatrix} \hat{g}_{n_N}(\xi) \\ \hat{h}_{n_N}(\xi) \end{pmatrix} &= \prod_{j=0}^{N-1} \begin{pmatrix} \frac{1-\epsilon_j i}{2} & \frac{1+\epsilon_j i}{2} e^{-\pi i \xi/2^j} \\ \frac{1+\epsilon_j i}{2} & \frac{1-\epsilon_j i}{2} e^{-\pi i \xi/2^j} \end{pmatrix} \cdot \begin{pmatrix} \hat{K}(\xi/2^N) \\ \hat{K}(\xi/2^N) \end{pmatrix}, \\ &= \hat{K}(\xi/2^N) \prod_{j=0}^{N-1} \begin{pmatrix} \frac{1-\epsilon_j i}{2} & \frac{1+\epsilon_j i}{2} e^{-\pi i \xi/2^j} \\ \frac{1+\epsilon_j i}{2} & \frac{1-\epsilon_j i}{2} e^{-\pi i \xi/2^j} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Now, let

$$U = \prod_{j=0}^{N-1} \begin{pmatrix} \frac{1-\epsilon_j i}{2} & \frac{1+\epsilon_j i}{2} e^{-\pi i \xi/2^j} \\ \frac{1+\epsilon_j i}{2} & \frac{1-\epsilon_j i}{2} e^{-\pi i \xi/2^j} \end{pmatrix}.$$

Then U is unitary, and for ξ in any fixed compact set,

$$\begin{aligned}
&\left\| \begin{pmatrix} \hat{g}_{n_{N+1}}(\xi) \\ \hat{h}_{n_{N+1}}(\xi) \end{pmatrix} - \begin{pmatrix} \hat{g}_{n_N}(\xi) \\ \hat{h}_{n_N}(\xi) \end{pmatrix} \right\|_2 \\
&\quad = \left\| \left[\hat{K}(\xi/2^{N+1}) U \begin{pmatrix} \frac{1-\epsilon_N i}{2} & \frac{1+\epsilon_N i}{2} e^{-\pi i \xi/2^N} \\ \frac{1+\epsilon_N i}{2} & \frac{1-\epsilon_N i}{2} e^{-\pi i \xi/2^N} \end{pmatrix} - \hat{K}(\xi/2^N) U \right] \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_2 \\
&\quad \le \left\| \left(\hat{K}(\xi/2^{N+1}) - \hat{K}(\xi/2^N) \right) U \begin{pmatrix} \frac{1-\epsilon_N i}{2} & \frac{1+\epsilon_N i}{2} e^{-\pi i \xi/2^N} \\ \frac{1+\epsilon_N i}{2} & \frac{1-\epsilon_N i}{2} e^{-\pi i \xi/2^N} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_2 \\
&\qquad + \left\| \hat{K}(\xi/2^N) U \left[\begin{pmatrix} \frac{1-\epsilon_N i}{2} & \frac{1+\epsilon_N i}{2} e^{-\pi i \xi/2^N} \\ \frac{1+\epsilon_N i}{2} & \frac{1-\epsilon_N i}{2} e^{-\pi i \xi/2^N} \end{pmatrix} - I \right] \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_2 \\
&\quad = \sqrt{2} \left| \hat{K}(\xi/2^{N+1}) - \hat{K}(\xi/2^N) \right| \\
&\qquad + \left| \hat{K}(\xi/2^N) \right| \left\| \left[\begin{pmatrix} \frac{1-\epsilon_N i}{2} & \frac{1+\epsilon_N i}{2} e^{-\pi i \xi/2^N} \\ \frac{1+\epsilon_N i}{2} & \frac{1-\epsilon_N i}{2} e^{-\pi i \xi/2^N} \end{pmatrix} - I \right] \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_2 \\
&\quad \le \frac{C}{2^N},
\end{aligned}$$

where the last inequality follows from the fact that its left-hand side is equal to 0 at $\xi = 0$ and is differentiable, with bounded derivative on any compact set. Hence it is Lipschitz in its argument $\xi/2^N$, and as ξ is supposed to be in some compact set, the inequality follows.

Hence this sequence of products is uniformly Cauchy on compact sets, and it converges to:

$$\prod_{j=0}^{\infty} \begin{pmatrix} \frac{1-\epsilon_j i}{2} & \frac{1+\epsilon_j i}{2} e^{-\pi i \xi/2^j} \\ \frac{1+\epsilon_j i}{2} & \frac{1-\epsilon_j i}{2} e^{-\pi i \xi/2^j} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad \blacksquare$$

Since, as we will soon see, the above construction is related to the classical Rudin–Shapiro sequence, the distribution $g_{\vec{\epsilon}}$ has semiflat Fourier spectrum. Specifically, we have:

COROLLARY 19. *One has that* $|\hat{g}_{\vec{\epsilon}}(\xi)| \le \sqrt{2}$. *When* $\vec{\epsilon} = (1, 1, \ldots)$, $\hat{g}_{\vec{\epsilon}}(\xi)$ *does not decay at infinity.*

Proof. Since each of the matrices in the expansion for $\hat{g}_{\vec{\epsilon}}$ are unitary, we have that

$$\|(g_{\vec{\epsilon}}(\xi), h_{\vec{\epsilon}}(\xi))\|_2 = \|(1,1)\|_2 = \sqrt{2}.$$

Hence the bound on $|\hat{g}_{\vec{\epsilon}}(\xi)|$.

From the infinite product expansion, one sees that for any $m, n \in \mathbb{Z}^+$,

$$\hat{g}_{\vec{\epsilon}}(2^{4n+1} m) = \hat{g}_{\vec{\epsilon}}(2m),$$

since we have that

$$\begin{pmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Also, by explicit computation $\hat{g}_{\vec{\epsilon}}(2) \neq 0$. Hence, $\hat{g}_{\vec{\epsilon}}$ does not decay at infinity. ■

Note that a similar argument for other $\vec{\epsilon}$ shows that if $\vec{\epsilon}$ is a periodic sequence, then $\hat{g}_{\vec{\epsilon}}$ does not decay at infinity.

The nonvanishing of $\hat{g}_{\vec{\epsilon}}$ on the above mentioned lacunary sequence is but the simplest example. Much more can be said. A more detailed analysis will appear in another paper.

Figure 2 shows some dragon noiselets aranged in a basis tree as in Fig. 1.

3.1. *Relation with paper folding and dragon curves*

The function g_n is piecewise constant on intervals of size $2^{-\ell(n)}$ and hence can be identified with a sequence of length $2^{\ell(n)}$ of eighth roots of unity.

Consider taking unit steps in the complex plane. We can interpret the sequences of eighth roots of unity as lists of instructions to take each of these steps in one of eight possible directions. The resulting curves are the classical dragon curves which arise from folding a piece of paper in half, repeatedly, in either of the two possible ways (left over right or right over left) and then unfolding each crease to 90° (see, for example, [4]). The presence of eighth roots of unity, instead of fourth roots, simply rotates by 45° from one stage to the next, so that extending a binary sequence will correspond to a refinement of the curve, up to rescaling. Hence the g_n's, up to a 45° rotation, come from the system:

$$\begin{aligned} \tilde{g}_{2n}(x) &= \tilde{g}_n(2x) + i\,\tilde{g}_n(2-2x) \\ \tilde{g}_{2n+1}(x) &= \tilde{g}_n(2x) - i\,\tilde{g}_n(2-2x). \end{aligned} \tag{4$'$}$$

When one unfolds a folded piece of paper out to 90° creases, one gets a sequence of turns followed by that sequence flipped and rotated by ±90°. Hence the paper-folding dragon curves are exactly the indefinite integrals of the dragon noiselets (since the process of interpreting a piecewise constant function as instructions to take steps of a given size in the indicated direction is the process of integration).

3.2. *Relation with the Rudin–Shapiro sequence*

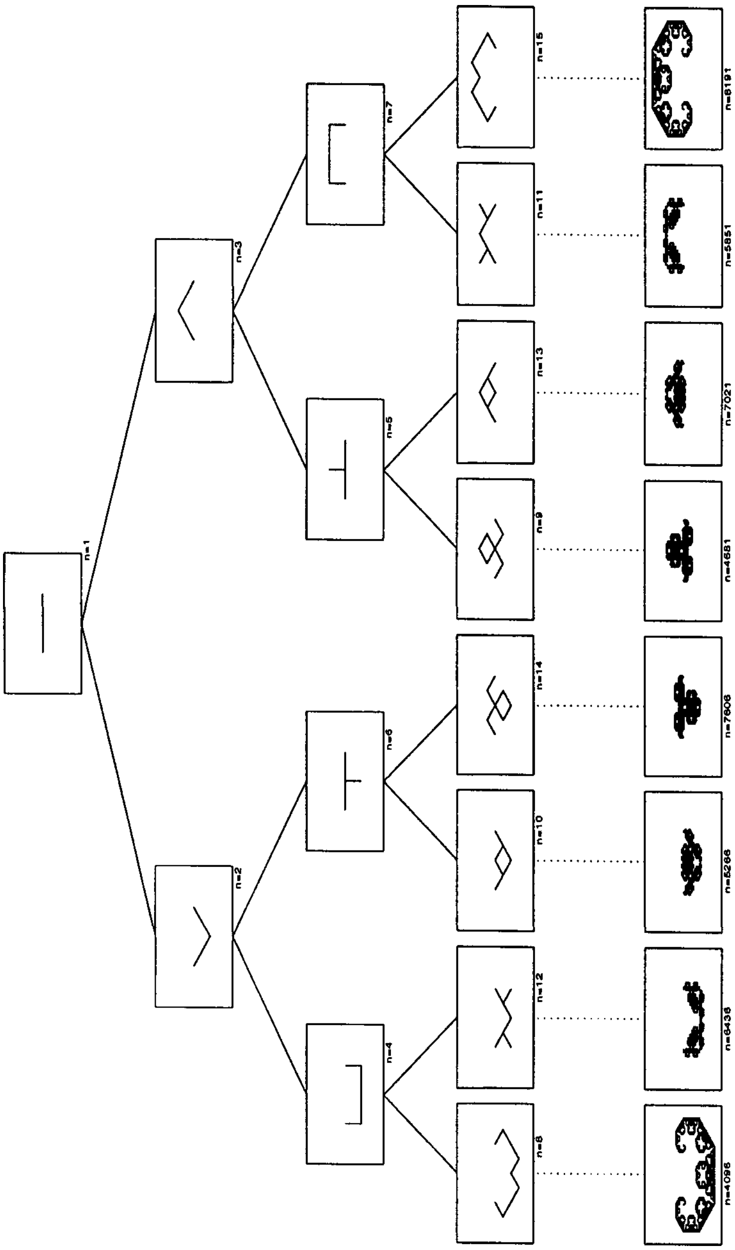
Again thinking of the $\tilde{g}_n$'s as sequences of fourth roots of unity arising from Eqs. (4′), one sees that there is an alternate construction which parallels the classical Rudin–Shapiro construction.

If one defines the sequences $P_0 = (1, i)$ and $Q_0 = (1, -i)$ and for $n \geq 1$,

$$\begin{aligned} P_{2n} &= P_n \wedge Q_n \\ Q_{2n} &= P_n \wedge -Q_n \\ P_{2n+1} &= Q_n \wedge P_n \\ Q_{2n+1} &= Q_n \wedge -P_n, \end{aligned} \tag{4$''$}$$

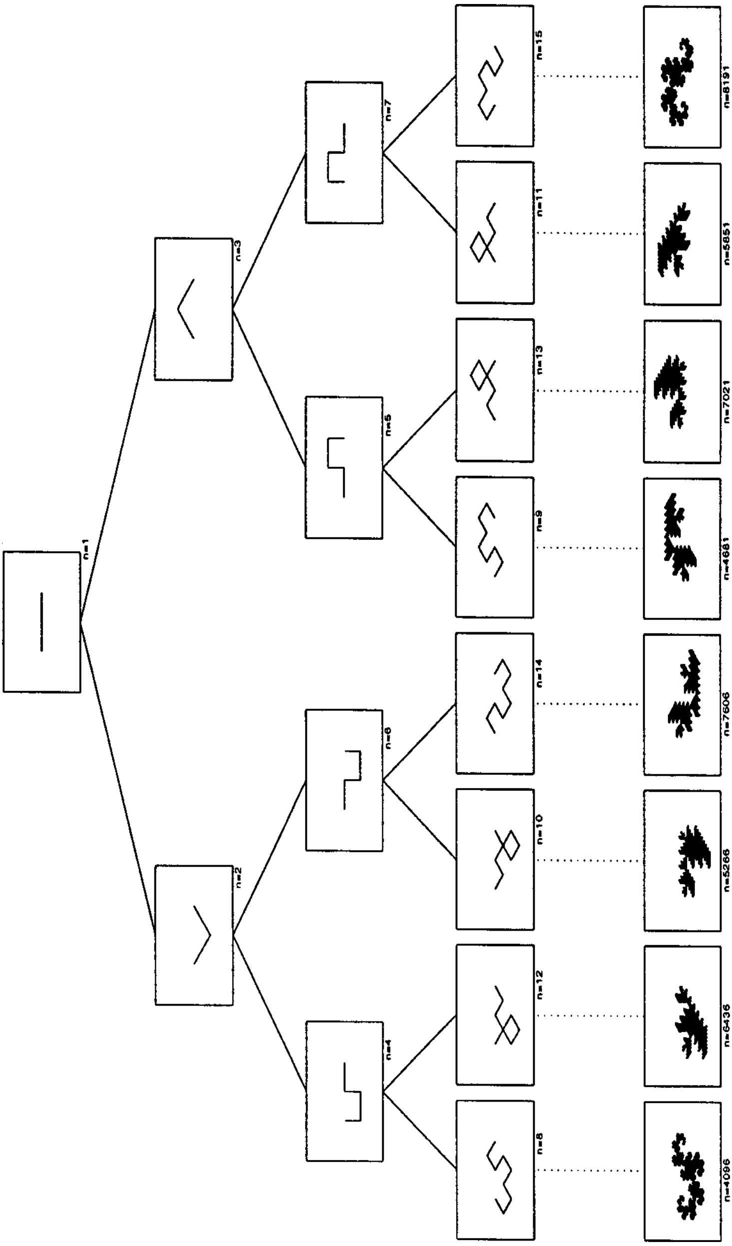
where $\wedge$ denotes string composition and $-$ denotes the obvious operation of pointwise negation, then one gets the same sequences as those arising from the sampled values of the $\tilde{g}_n$'s in (4′).

But (4″) mirrors the classical Rudin–Shapiro construction, with simply a different initial condition. The fact that (4″) is a unitary, basis-producing generalization of the Rudin–Shapiro construction, which produces a basis of semiflat functions, was made by Byrnes in [2], although the complex initial conditions were not made there.



n=1
n=2
n=3
n=4
n=5
n=6
n=7
n=8
n=9
n=10
n=11
n=12
n=13
n=14
n=15
n=4096
n=6436
n=5266
n=7606
n=4681
n=7021
n=5851
n=8191

FIGURE 1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n=4096

FIGURE 2

Note that the noiselets in Section 2 are related to another classical automatic sequence: the Thue–Morse sequence. Automatic sequences are fixed points for string rewriting rules which have the form of substitutions. All of our automatic sequences were also generated by string compositions, so their automatic-sequence nature was not directly evident in the definitions (2) and (4). However, all of our sequences do come from repeated applications of certain substitutions (see the end of the next section). In order to get the noiselets in Section 2 from the Thue–Morse construction (generalized to the Walsh basis), instead of complexifying the initial conditions, we complexify the substitutions.

4. A LARGE NUMBER OF COMBINATIONS

It is worth mentioning that the two different constructions in the two previous sections can be mixed. At each scale we fix a choice of whether to apply the construction in Eqs. (2) or to apply the construction in Eqs. (4). For this fixed sequence of choices, the family produced has all of the basis and Haar–Walsh properties of the preceding examples. However, the semiflat Fourier spectrum property goes away as soon as we mix in any definite amount of Eqs. (2), as these lead to growth of the Fourier transform like some power of ξ (at least along certain sequences that accumulate at infinity).

Specifically, we let $\vec{\beta} = (\beta_1, \beta_2, \ldots)$ and define

$$f_{\vec{\beta},1} = \chi_{[0,1)}$$

and

$$f_{\vec{\beta},n} = (1 - \epsilon_0(n)i) f_{\vec{\beta},\lfloor m/2 \rfloor}(2x) + (1 + \epsilon_0(n)i) f_{\vec{\beta},\lfloor m/2 \rfloor}\big((-1)^{\beta_{\ell(n)}} 2x - 1 + 3\beta_{\ell(n)}\big).$$

Then one sees that when $\beta = (0, 0, 0, \ldots)$, then $f_{\vec{\beta},n} = f_n$, and when $\beta = (1, 1, 1, \ldots)$, we have that $f_{\vec{\beta},n} = g_n$. More generally, the nth digit of β determines whether we apply Eqs. (2) or (4) to compute $f_{\vec{\beta},m}$ from $f_{\vec{\beta},\lfloor m/2 \rfloor}$ when $\ell(m) = n$.

LEMMA 20. *For fixed $\vec{\beta}$, the set $\{f_{\vec{\beta},j} \mid j = 2^N, \ldots, 2^{N+1} - 1\}$ is an orthogonal basis for V_N.*

LEMMA 21. *For each $\vec{\beta}$ and each $n \geq 1$,*

$$\int_0^1 f_{\vec{\beta},n}(x)\, dx = 1.$$

LEMMA 22. *Let $n \in \mathbb{Z}^+$, $m \geq 0$. If $\ell(m) < \ell(n)$, then*

$$\int_0^1 f_{\vec{\beta},n}(x) W_m(x)\, dx = i^{k'},$$

where

$$k' = -\sum_{j=0}^{\ell(m)} \epsilon_j(n) \cdot \frac{1 - \epsilon_j(m)(\epsilon_{j+1}(m))^{\beta_j}}{2}.$$

Otherwise (if $\ell(m) \geq \ell(n)$), it is zero.

COROLLARY 23. *For fixed $\vec{\beta}$, given $n, m \in \mathbb{Z}^+$, and $l \geq 0$,*

$$\int f_{\vec{\beta},m}(x) W_k(x)\,dx = \int f_{\vec{\beta},n}(x) W_k(x)\,dx$$

for all $k < 2^l$ if and only if $\nu_j(m) = \nu_j(n)$ for all $0 \leq j \leq l$.

LEMMA 24. *Let $n \in \mathbb{Z}^+$, $q, m \geq 0$, and $0 \leq k < 2^q$. If $q + \ell(m) < \ell(n)$, then*

$$\int_0^1 f_{\vec{\beta},n}(x) 2^{q/2} W_m(2^q x - k)\,dx$$

is an eighth root of unity. Otherwise (if $q + \ell(m) \geq \ell(n)$), it is zero.

As usual, let $\{\epsilon_i\}_{i=0}^{\infty}$ be any sequence with values in $\{-1, 1\}$. Let $\nu_j = (-1)^{\epsilon_j}$ and $n_i = \sum_{j=0}^{i} \nu_j 2^j$.

PROPOSITION 25. *The distributional limit of $f_{\vec{\beta},n_i}$ exists and is a tempered distribution $f_{\vec{\beta},\vec{\epsilon}}$. If we define the distribution $g_{\vec{\beta},\vec{\epsilon}}(x) = f_{\vec{\beta},\vec{\epsilon}}(1-x)$ then the Fourier transforms are given by*

$$\begin{pmatrix} \hat{f}_{\vec{\beta},\vec{\epsilon}}(\xi) \\ \hat{g}_{\vec{\beta},\vec{\epsilon}}(\xi) \end{pmatrix} = \prod_{j=0}^{\infty} A_{\vec{\beta},\vec{\epsilon},j}(\xi/2^j) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with later multiplications on the right, where

$$A_{\vec{\beta},\vec{\epsilon},j}(\xi) = \begin{pmatrix} \frac{1-\epsilon_j i}{2} & \frac{1+\epsilon_j i}{2} e^{-\pi i \xi} \\ \frac{1+\epsilon_j i}{2} & \frac{1-\epsilon_j i}{2} e^{-\pi i \xi} \end{pmatrix},$$

when $\beta_j = 1$, and

$$A_{\vec{\beta},\vec{\epsilon},j}(\xi) = \begin{pmatrix} \cos(\xi/4) + \epsilon_j \sin(\xi/4) & 0 \\ 0 & \cos(\xi/4) - \epsilon_j \sin(\xi/4) \end{pmatrix},$$

when $\beta_j = 0$.

Figure 3 shows a basis tree of one family of mixed noiselets (with β alternating between 1 and 0).

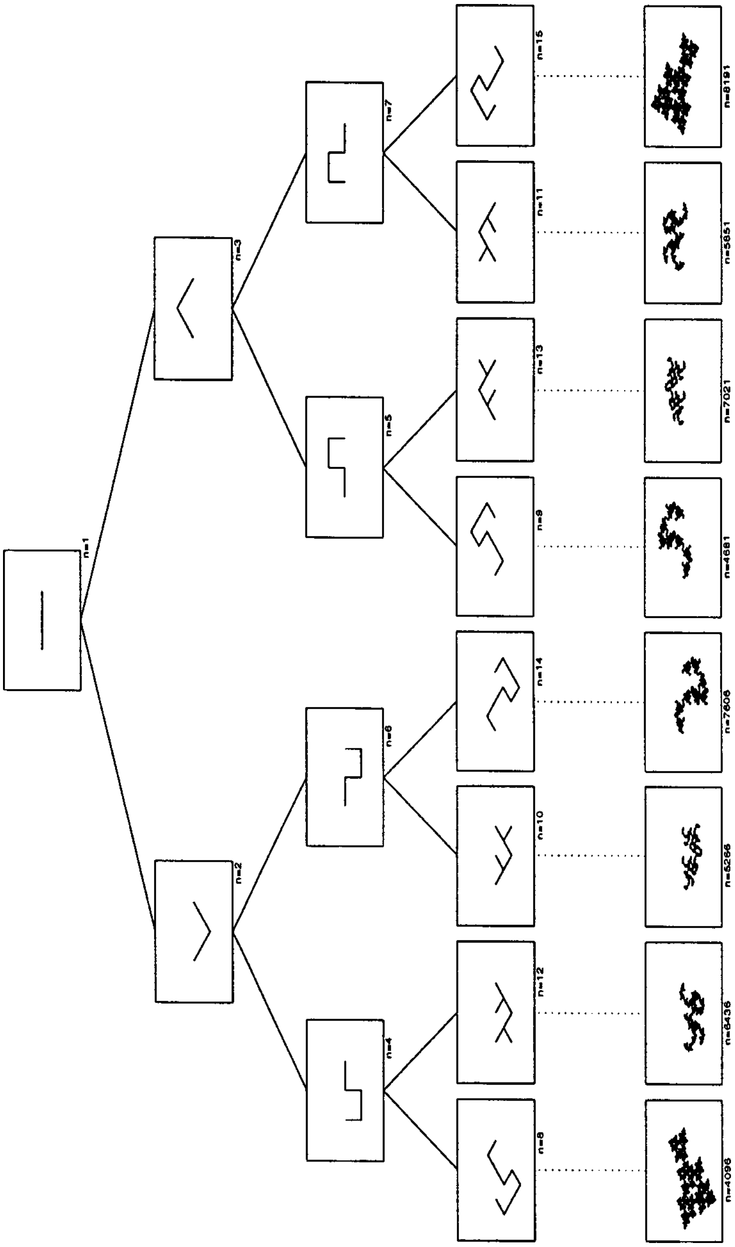
As pointed out, the preceding constructions can each be described in a few different ways. When one combines the constructions using (2) and (4) (analogous to mixing the string composition rules in the automatic sequence definitions), the proofs go over essentially unmodified.

It is possible to combine the constructions in another way, analogous to mixing the string rewriting rules. The constructions are most easily described in terms of the sequences of sampled values of the functions. We define the substitutions:

$$\begin{aligned} s_0&: \quad 1 \mapsto 1+i,\ -1 \mapsto -1-i,\ i \mapsto i-1,\ -i \mapsto -i+1, \qquad \text{and} \\ s_1&: \quad 1 \mapsto 1-i,\ -1 \mapsto -1+i,\ i \mapsto i+1,\ -i \mapsto -i-1, \end{aligned} \tag{S01}$$

and

$$\begin{aligned} s_2&: \quad 1 \mapsto 1+i,\ -1 \mapsto -1-i,\ i \mapsto -1+i,\ -i \mapsto -1-i, \qquad \text{and} \\ s_3&: \quad 1 \mapsto 1-i,\ -1 \mapsto -1+i,\ i \mapsto 1+i,\quad -i \mapsto -1-i, \end{aligned} \tag{S23}$$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n=4096
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n=5266
n=5851
n=6436
n=7021
n=7606
n=8191

FIGURE 3

where a substitution acts on a sequence of length n of fourth roots of unity to produce a sequence of length $2n$ in the obvious way.

Up to multiplication by powers of $1 \pm i$, the functions f_n arise from repeated applications of s_0 and s_1 according to the binary expansion of n. The functions g_n arise from repeatedly applying s_2 and s_3 according to the binary expansion of n. However, if we apply either (S01) or (S23) according to the digits of β, and the binary expansion of n, we get something other than $f_{\vec{\beta},n}$. The resulting functions, when rescaled by appropriate multiples of $1 \pm i$ can be shown to have all of the Haar–Walsh, basis and distributional limit properties of the $f_{\vec{\beta},n}$.

In this form, it is easier to analyze the resulting primitives (indefinite integrals) directly to determine that the mixed constructions converge. Indeed, one can show that the sequences of primitives have a martingale property with exponentially shrinking displacement from one scale to the next. In fact the side lengths shrink like 2^{-n} while the displacements shrink like $2^{-n/2}$, so that all of the resulting curves are Hölder-$1/2$. The convergence is uniform, so that such a sequence of noiselets converges to the distributional derivative of the limit of its primitives. Hence each of these limiting primitives gives a sort of deterministic Brownian motion, of which our deterministic white noise is the derivative.

5. COMMENTS

Since the noiselets are built from equations like (2) and (4), one has a fast algorithm for noiselet packets and best noiselet packet bases, as in the wavelet packet case (see [6]). One needs to observe that $f_{\vec{\beta},n}(1-x) = f_{\vec{\beta},n'}(x)$ for some n' such that $\ell(n') = \ell(n)$. This is proved easily by induction. Hence in the recursive discrete algorithm, in situations where one is at a stage where it is necessary to apply Eqs. (4), it is possible to proceed.

All of our functions and distributions are supported on the interval [0, 1]. It is possible to lift this restriction by simply repeating the construction out to infinity, hence producing a distribution which has uniform absolute value and which has uniformly large Haar–Walsh wavelet packet coefficients. The only change needed, in order to get convergence, is the elimination of the eighth roots of unity. In other words, we take sequences such as

$$f_{\vec{\beta},\vec{\epsilon},1}(x) = f_{\vec{\beta},\vec{\epsilon}}(x)$$

$$f_{\vec{\beta},\vec{\epsilon},2n}(x) = f_{\vec{\beta},\vec{\epsilon},n}(x) + i f_{\vec{\beta},\vec{\epsilon},n}(x - 2^{\ell(n)}) \qquad \text{and}$$

$$f_{\vec{\beta},\vec{\epsilon},2n+1}(x) = f_{\vec{\beta},\vec{\epsilon},n}(x) - i f_{\vec{\beta},\vec{\epsilon},n}(x - 2^{\ell(n)}).$$

Then $f_{\vec{\beta},\vec{\epsilon},n}(x)$ is supported on $[0, 2^{\ell(n)}]$, and along sequences of increasing n_i as in the previous sections, there is obviously convergence to distributions which are supported on $[0,\infty)$, and have Haar–Walsh coefficients all of modulus 1.

Similarly, we can take

$$g_{\vec{\beta},\vec{\epsilon},1}(x) = f_{\vec{\beta},\vec{\epsilon}}(x)$$

$$g_{\vec{\beta},\vec{\epsilon},2n}(x) = g_{\vec{\beta},\vec{\epsilon},n}(x) + i g_{\vec{\beta},\vec{\epsilon},n}(2^{\ell(n)+1} - x) \qquad \text{and}$$

$$g_{\vec{\beta},\vec{\epsilon},2n+1}(x) = g_{\vec{\beta},\vec{\epsilon},n}(x) - i g_{\vec{\beta},\vec{\epsilon},n}(x - 2^{\ell(n)+1} - x)$$

as in the dragon constructions, and the subsequences converge to distributions on $[0,\infty)$, which are Haar–Walsh totally flat.

Finally, the above two constructions can be mixed, as in the previous section.

The proofs of all of the above facts are essentially the same as in previous sections.

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