MATHEMATICS

THE ASYMPTOTIC DISTRIBUTION OF PRIMES

BY

I. S. GÁL

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This paper contains proofs of the prime number theorem, the prime ideal theorem and the prime number theorem of arithmetic progressions. Wiener's Tauberian theorem and a related theorem of Pitt will be adapted to the special situation when the functions occurring in these theorems are determined by the coefficients of Dirichlet series. The results so obtained will be first applied in the case of the Riemann Zeta function and the prime number theorem will be derived in several ways using only the fact that

$$\alpha(s) = (2 - 2^{1-s} - 3^{1-s})\, \zeta(s)$$

is convergent in the half-plane $\mathscr{R}s > 0$ and it does not vanish on the line $\mathscr{R}s = 1$. In the third section of this paper the methods are extended to the Zeta functions of number fields and the prime ideal theorem is proved. Finally in the fourth section we prove the prime number theorem of arithmetic progressions.

It is assumed that the reader knows the theorems of Wiener and Pitt. Both are discussed in Hardy's Divergent series and their group theoretic background can be found in Loomis' Abstract harmonic analysis. We suppose also familiarity with the elementary results and techniques of algebraic and analytic number theory. The second section dealing with the prime number theorem requires considerably less number theoretic background than the ones following it. Thus the first two sections will be accessible to a much larger group of readers than the later ones.

1. *Two principles concerning coefficient sums*

Given a Dirichlet series $\alpha(s) = \sum a_n n^{-s}$ it is customary to denote by A the coefficient sum of α, so that A is a function on the positive half-line $G = (0, +\infty)$ with values

$$A(u) = \sum_{n \leqslant u} a_n \qquad (u > 0).$$

We fix a real number $\sigma > 0$ and define the complex valued a on G by

$$a(u) = u^{-\sigma} A(u).$$

Suppose that σ is inside the half plane of convergence of α i.e. $\sigma > \gamma^{\alpha}$

where γ^α denotes the convergence abscissa of α. Then α is convergent at $\sigma-\varepsilon$ for some small $\delta>0$ and so we have $A(u)=O(u^{\sigma-\varepsilon})$. Therefore

$$\int_0^{+\infty} |a(u)| \frac{du}{u} = \int_0^{+\infty} |A(u)|\, u^{-\sigma} \frac{du}{u}$$

exists. G is a locally compact commutative topological group under the ordinary multiplication and du/u is the Haar measure of G. In view of this we see that if $\sigma>\gamma^\alpha$ and $\sigma>0$ then $a \in L^1(G)$.

Since we shall perform various computations involving a it is convenient to introduce

$$f_m(u) = \begin{cases} 0 & \text{for } 0<u<m \\ u^{-\sigma} & \text{for } m \leqslant u < +\infty \end{cases}$$

where $m>0$. Then a can be written as

(1) $$a = \sum a_m f_m$$

and for any $u>0$ we have

(2) $$a(u) = \sum_{m\leqslant u} a_m f_m(u).$$

It is clear that $f_m \in L^1(G)$ and

(3) $$\hat{f}_m(t) = \frac{m^{-(\sigma+it)}}{\sigma+it} \qquad (-\infty<t<+\infty).$$

Moreover if $m, n>0$ then

$$(f_m \star f_n)(u) = \begin{cases} 0 & \text{for } 0<u<mn \\ u^{-\sigma}(\log u - \log mn) & \text{for } mn \leqslant u. \end{cases}$$

The Fourier transform $\hat{a}$ can be computed from (2) and (3): Since $\sigma>\gamma^\alpha$ we have $A(x)=o(x^\sigma)$ as $x \to +\infty$ and so for integer $x>0$ we get

$$\begin{aligned}\int_0^x u^{-it}\, a(u) \frac{du}{u} &= \sum_{m\leqslant x} a_m \hat{f}_m(t) - \sum_{m\leqslant x} a_m \int_x^\infty u^{-it} f_m(u) \frac{du}{u} \\ &= \sum a_m \hat{f}_m(t) + A(x)(\sigma+it)^{-1} x^{-\sigma-it}.\end{aligned}$$

Therefore

$$\hat{a}(t) = \sum a_m \hat{f}_m(t) = \frac{\alpha(\sigma+it)}{\sigma+it}.$$

Now let α and β be Dirichlet series with coefficients $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ respectively. Given $u>0$ we have $a(u/v)=0$ for $u<v$ and $b(v)=0$ for $0<v<1$. Therefore $a \star b$ exists and

$$(a \star b)(u) = \sum_{mn\leqslant u} a_m b_n (f_m \star f_n)(u).$$

Using the expression obtained for $f_m \star f_n$ we get

$$(a \star b)(u) = u^{-\sigma} \log u \sum_{mn\leqslant u} a_m b_n + u^{-\sigma} \sum_{mn\leqslant u} - a_m b_n \log mn.$$

The sums which occur here have simple interpretations in terms of the product series $\gamma = \alpha\beta$ and its derivative $\gamma' = (\alpha\beta)'$: The first is the coefficient sum $C(u)$ of γ and the second is that of the differentiated series γ'. Therefore

$$(a \star b)(u) = (\log u)\, c(u) + c'(u). \tag{4}$$

This interpretation is very useful in actual computations.

Wiener's Tauberian theorem leads to the following general principle:

Theorem 1. *Let $\sigma > 0$ and let α, β be Dirichlet series such that a is bounded, $\sigma > \gamma^{\beta}$ and β does not vanish on the line $\mathscr{R}s = \sigma$. Then $(a \star b)(u) \to 0$ as $u \to +\infty$ implies $(a \star k)(u) \to 0$ for any Dirichlet series $\varkappa$ with $\sigma > \gamma^{\varkappa}$.*

A second useful principle is obtained from Pitt's theorem:

Theorem 2. *Let $\sigma > 0$ and let α, β be Dirichlet series such that a is bounded and slowly oscillating, $\sigma > \gamma^{\beta}$ and β does not vanish on the line $\mathscr{R}s = \sigma$. Then $(a \star b)(u) \to 0$ as $u \to +\infty$ implies*

$$A(x) = \sum_{n \leqslant x} a_n = o(x^{\sigma}).$$

2. *The prime number theorem*

First we use the principle expressed in Theorem 1 to derive the prime number theorem in the form $\eta(x) \sim x^2/2$ where

$$\eta(x) = \sum_{n \leqslant x} \psi(n) \qquad (x > 0).$$

The elementary equivalence of this proposition to the prime number theorem is well known; it is discussed for instance in Ingham's book on The distribution of prime numbers. In the second half of this section we shall prove directly that $\psi(x) \sim x$ and $M(x) = o(x)$. The proof of these statements will be based on Theorem 2. Both of these propositions are elementary equivalents of the prime number theorem.

We start from Theorem 1 where we let $\sigma = 1$ and $\alpha = \zeta^{-1}$ so that the coefficients of α are $\mu(1), \mu(2), \ldots$ and $a(u) = u^{-1}M(u)$. Then a is bounded, as a matter of fact $|a(u)| \leqslant u$. Next we let

$$\beta(s) = (2 - 2^{1-s} - 3^{1-s})\, \zeta(s). \tag{5}$$

The first factor could be replaced by a variety of others, e.g. instead of the bases 2,3 we could choose any pair of distinct primes $p, q > 1$. The only essential requirement is that the factor should not vanish on the line $\mathscr{R}s = 1$. We have $\gamma^{\beta} = 0 < \sigma = 1$ and using the fact that the Riemann Zeta function does not vanish on the line $\mathscr{R}s = 1$ and has a pole at $s = 1$ we see that $\beta(s) \neq 0$ on $\mathscr{R}s = \sigma = 1$. Since $\gamma = \alpha\beta$ is the finite series

$$\gamma(s) = 2 - \frac{2}{2^s} - \frac{3}{3^s}$$

we see that $C(u)$ and $C'(u)$ are bounded functions and so $c(u) = O(u^{-1})$

and $c'(u)=O(u^{-1})$. Therefore $(a \star b)(u) \to 0$ as $u \to +\infty$ and by Theorem 1 $(a \star k)(u) \to 0$ as $u \to +\infty$ for any $\varkappa$ satisfying $\gamma^{\varkappa} < \sigma$.

Let $\varkappa$ denote the Dirichlet series

$$\varkappa = -\zeta' - \zeta^2 + 2\gamma\zeta \tag{6}$$

where γ denotes the Euler constant. Then the nth coefficient of $\varkappa$ is $\log n - d(n) + 2\gamma$ where as usual $d(n)$ denotes the number of divisors of n. Using the elementary estimates

$$\sum_{n \leqslant x} d(n) = x \log x + (2\gamma - 1)\, x + O(x^{\frac{1}{2}}) \qquad (x \to +\infty)$$

and

$$\sum_{n \leqslant x} \log n = x \log x - x + O(\log x) \qquad (x \to +\infty)$$

we see that the coefficient sum K of $\varkappa$ is

$$K(u) = O(u^{\frac{1}{2}}). \tag{7}$$

From this estimate we obtain $\gamma^{\varkappa} \leqslant \frac{1}{2} < 1 = \sigma$ so that $\varkappa$ satisfies the requirement $\gamma^{\varkappa} < \sigma$. The convolution $a \star k$ can be easily computed from (4): The coefficients of $\alpha\varkappa$ are $c_n = \Lambda(n) - 1$ for $n > 1$ and so $C(u) = \psi(u) - u + O(1)$. Moreover

$$C'(u) = -\sum_{n \leqslant u} c_n \log n = \sum_{n \leqslant u} C(n) \log(1 + 1/n) - C(u) \log([u] + 1).$$

Using $C(u) = O(u)$ and (4) we obtain

$$u\,(a \star k)\,(u) = \sum_{n \leqslant u} \frac{C(n)}{n} + o(u).$$

Hence by $(a \star k)(u) \to 0$ we have

$$\sum_{n \leqslant u} \frac{\psi(n)}{n} = u + o(u) \quad \text{as} \quad u \to +\infty. \tag{8}$$

The prime number theorem is an easy consequence of (8). For if δ denotes the left hand side of (8) then

$$\eta(x) = \sum_{n \leqslant x} \psi(n) = \sum_{n \leqslant x} \{\delta(n) - \delta(n-1)\}\, n =$$

$$= -\sum_{n \leqslant x} \delta(n) + \delta(x)\, O(x) = \frac{x^2}{2} + o(x^2) + \sum_{n \leqslant x} o(n).$$

We have in general

$$\sum_{n \leqslant x} o(n) = o(x^2)$$

and so we proved that $\eta(x) \sim x^2/2$ as $x \to \infty$.

Now we turn to Theorem 2 and use it to give a simple proof of the proposition $M(x) = o(x)$ $(x \to +\infty)$. Indeed this result follows immediately because if we let $\sigma = 1$ then for α we may choose ζ^{-1} and as earlier we may let β be the function given in (5). The fact that $m(u)=M(u)/u$ is bounded and slowly oscillating can be proved in a few lines.

In order to prove $\psi(x)\sim x$ by our second principle we let $\sigma=1$ and

$$\alpha=-\frac{\zeta'}{\zeta}-\zeta+2\gamma$$

where γ stands for the Dirichlet series whose first coefficient is the Euler constant and whose remaining coefficients all vanish. For β we choose again the series given in (5). We show that $(a\star b)(u)\to 0$ as $u\to+\infty$. Since

$$\gamma(s)=\alpha\beta(s)=\left(2-\frac{2}{2^s}-\frac{3}{3^s}\right)\varkappa(s)$$

where $\varkappa$ denotes the series given in (6), by (7) we have $C(u)=O(u^{\frac{1}{2}})$ and $c(u)=O(u^{-\frac{1}{2}})$. Moreover

$$\begin{aligned} C'(u) &= -\sum_{n\leqslant u} c_n\log n=-\sum_{n\leqslant u}\{C(n)-C(n-1)\}\log n= \\ &=\sum_{n\leqslant u} C(n)\log\left(1+\frac{1}{n}\right)+O(u^{\frac{1}{2}}\log u)= \\ &=\sum_{n\leqslant u} O(n^{-\frac{1}{2}})+O(u^{\frac{1}{2}}\log u)=O(u^{\frac{1}{2}}\log u). \end{aligned}$$

Hence by (4) we obtain

$$(a\star b)(u)=c(u)\log u+c'(u)=O(u^{-\frac{1}{2}}\log u)\ \text{ as }\ u\to+\infty.$$

Since $\sigma=1$ we have $u\,a(u)=\psi(u)-u+O(1)$ and so the boundedness of a follows from the elementary estimate $\psi(u)=O(u)$. Hence $\psi(u)-u=o(u)$ will be obtained by proving that a is slowly oscillating. This can be achieved by having the following:

Lemma 1. *Let K be a real or complex valued function on* $(0,+\infty)$ *which is of bounded variation on every finite interval* $(0,x)$ *and is* $O(x^{\frac{1}{2}})$ *as* $x\to+\infty$. *Then* $a_n=O(1)$ *implies that* $1/x\sum_{n\leqslant x} a_n K(x/n)$ *is slowly oscillating.*

Proof. We wish to show that

$$\frac{1}{y}\sum_{n\leqslant y}-\frac{1}{x}\sum_{n\leqslant x}=\frac{1}{y}\sum_{n\leqslant y}a_n\left\{K\left(\frac{y}{n}\right)-K\left(\frac{x}{n}\right)\right\}-\left(\frac{1}{x}-\frac{1}{y}\right)\sum_{n\leqslant x}a_n K\left(\frac{x}{n}\right)- \\ -\frac{1}{y}\sum_{x<n\leqslant y}a_n K\left(\frac{x}{n}\right)$$

approaches 0 as $x\to+\infty$ and $y/x\to 1+0$. The second term on the right hand side is

$$\frac{1}{y}\left(\frac{y}{x}-1\right)\sum_{n\leqslant x}O\left(\frac{x}{n}\right)^{\frac{1}{2}}=O\left(\frac{y}{x}-1\right)=o(1) \tag{9}$$

as $x \to +\infty$ and $y/x \to 1+0$. The third term is

$$\text{(10)} \qquad \frac{1}{y} \sum_{x<n\leqslant y} O\left(\frac{x}{n}\right)^{\frac{1}{2}} = O\left(\frac{x^{\frac{1}{2}}}{y}\right)(y^{\frac{1}{2}} - x^{\frac{1}{2}}) = O\left(\left(\frac{y}{x}\right)^{\frac{1}{2}} - 1\right) = o(1).$$

The first term can be estimated by splitting it up in the form

$$\frac{1}{y} \sum_{n\leqslant y} = \frac{1}{y} \sum_{n\leqslant \delta y} + \frac{1}{y} \sum_{\delta y<n\leqslant y}$$

where $0<\delta<1$. Then we have

$$\text{(11)} \qquad \frac{1}{y} \sum_{n\leqslant \delta y} = \frac{1}{y} \sum_{n\leqslant \delta y} O\left(\frac{y}{n}\right)^{\frac{1}{2}} = O(1)\, \delta^{\frac{1}{2}}$$

where $O(1)$ is bounded by a constant which is independent of x, y and δ. Finally

$$\text{(12)} \qquad \frac{1}{y} \sum_{\delta y<n\leqslant y} = O\left(\frac{1}{y}\right) \operatorname{Var}\left(K;\, 0, \frac{1}{\delta}\right)$$

where "Var" stands for the variation of K on the interval $(0, 1/\delta)$. Now it is clear that the inequality

$$\left| \frac{1}{y} \sum_{n\leqslant y} - \frac{1}{x} \sum_{n\leqslant x} \right| < \varepsilon$$

will hold for all sufficiently large x and $y \geqslant x$ such that y/x is sufficiently close to 1. For by choosing $y/x-1$ small (9) and (10) can be made less than $\varepsilon/4$. Then we choose $\delta\,(0<\delta<1)$ so small that (11) is less than $\varepsilon/4$. Now δ being fixed we can make (12) less than $\varepsilon/4$ by letting x, and ipso facto y, sufficiently large.

If we let $a_n = \mu(n)$ and K be the coefficient sum of $\varkappa$ then

$$\sum_{n\leqslant x} a_n\, K\left(\frac{x}{n}\right)$$

is the coefficient sum of

$$-\frac{\zeta'}{\zeta} - \zeta + 2\gamma.$$

Then a is slowly oscillating and the prime number theorem is proved once more.

We would like to add that the prime number theorem can be derived from the proposition "$M(x) = o(x)$ as $x \to +\infty$" by elementary reasoning. It can be found for instance in Appendix IV of Hardy's Divergent series.

3. *The prime ideal theorem*

Given an algebraic number field Φ we let $H(x)$ denote the number of integral ideals $\mathfrak{a}$ whose norm $N\mathfrak{a}$ is at most x:

$$H(x) = \sum_{N\mathfrak{a}\leqslant x} 1.$$

It is known that

$$H(x) = \mu x + O(x^{1-1/n}) \qquad (x \to +\infty) \tag{13}$$

where μ is a positive constant and n is the degree of Φ over the rationals. This formula is proved for example in Weyl's Algebraic theory of numbers.

Now we derive a few simple consequences of (13), all of which will be needed in the proof of the prime ideal theorem: First of all we have

$$\sum_{N\mathfrak{a} \leqslant x} \log N\mathfrak{a} = \mu x \log x - \mu x + O(x^{1-1/n} \log x). \tag{14}$$

For, the left hand side can be rearranged as

$$\sum_{n \leqslant x} \{H(n) - H(n-1)\} \log n = H(x) \log [x] - \sum_{n \leqslant x} H(n) \log \left(1 + \frac{1}{n}\right)$$

and the first term on the right hand side will give the main term $\mu x \log x$ while the second contributes $-\mu x$.

Next, we prove the existence of a constant β such that

$$\sum_{N\mathfrak{a} \leqslant x} \frac{1}{N\mathfrak{a}} = \mu \log x + \beta + O(x^{-1/n}). \tag{15}$$

Indeed we have for any exponent

$$\sum_{N\mathfrak{a} \leqslant x} (N\mathfrak{a})^{\alpha} = \sum_{m \leqslant x} H(m) \{m^{\alpha} - (m+1)^{\alpha}\} + H(x) ([x]+1)^{\alpha}.$$

We let $\alpha = -1$, substitute from (13) and get

$$\begin{aligned} \sum_{N\mathfrak{a} \leqslant x} (N\mathfrak{a})^{-1} &= \sum_{m \leqslant x} H(m) \left(\frac{1}{m} - \frac{1}{m+1}\right) + \mu + O(x^{-1/n}), \\ &= \int_1^{[x]} H(t)\, t^{-2}\, dt + \mu + O(x^{-1/n}). \end{aligned}$$

Using (13) once more we obtain (15). There is a good reason to write an arbitrary exponent α instead of $\alpha = -1$. For we shall need the estimate

$$\sum_{N\mathfrak{a} \leqslant x} (N\mathfrak{a})^{1/n-1} = O(x^{1/n}) \tag{16}$$

which follows via the mean value theorem.

Formulae (15) and (16) yield

$$\sum_{N\mathfrak{a}\mathfrak{b} \leqslant x} 1 = \mu^2 \log x + \varepsilon x + O(x^{1-1/2n}): \tag{17}$$

(Summation is extended over all pairs of integral ideals $\mathfrak{a}$, $\mathfrak{b}$ such that $N\mathfrak{a}\mathfrak{b} \leqslant x$.) Indeed the left hand side of (17) is the difference

$$2 \sum_{N\mathfrak{a} \leqslant \sqrt{x}} \; \sum_{N\mathfrak{b} \leqslant x/N\mathfrak{a}} 1 - \sum_{N\mathfrak{a}, N\mathfrak{b} \leqslant \sqrt{x}} 1.$$

The second sum being $H(\sqrt{x})^2$, it can be estimated by using (13) and one gets $\mu^2 x + O(x^{1-1/2n})$. The first double sum is

$$\sum_{N\mathfrak{a} \leqslant \sqrt{x}} H(x/N\mathfrak{a}) = \sum_{N\mathfrak{a} \leqslant \sqrt{x}} \left\{ \mu \frac{x}{N\mathfrak{a}} + O\left(\frac{x}{N\mathfrak{a}}\right)^{1-1/n} \right\}.$$

Hence by (15) and (16) we obtain (17).

The prime ideal theorem states that

$$\pi_\Phi(x) = \sum_{N\mathfrak{p} \leqslant x} 1 \sim \frac{x}{\log x} \qquad (x \to \infty).$$

This proposition is an elementary equivalent of

$$\psi_\Phi(x) = \sum_{N\mathfrak{a} \leqslant x} \Lambda(\mathfrak{a}) \sim x \qquad (x \to \infty)$$

where

$$\Lambda(\mathfrak{a}) = \begin{cases} \log N\mathfrak{p} & \text{if } \mathfrak{a} = \mathfrak{p}^m \text{ for some } m = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

The prime ideal theorem can be proved along the same lines as the prime number theorem was derived in the preceding section: One can start with

$$\beta(s) = (2 - N\mathfrak{p}^{1-s} - N\mathfrak{q}^{1-s})\, \zeta_\Phi(s)$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are proper integral ideals of distinct prime norms. Choosing $\alpha(s) = \zeta_\Phi^{-1}(s)$ one heads for one of two alternatives. The first leads to the proposition

$$M_\Phi(x) = \sum_{N\mathfrak{a} \leqslant x} \mu(\mathfrak{a}) = o(x) \qquad (x \to \infty)$$

where $\mu(\mathfrak{a})$ denotes the generalized the Möbius function: $\mu(\mathfrak{a}) = (-1)^r$ if $\mathfrak{a} = \mathfrak{p}_1 \ldots \mathfrak{p}_r$ with distinct $\mathfrak{p}_i$'s and $\mu(\mathfrak{a}) = 0$ otherwise. The second alternative leads immediately to

$$\eta_\Phi(x) = \sum_{n \leqslant x} \psi_\Phi(x) \sim \frac{x^2}{2} \qquad (x \to \infty).$$

In either case one needs the Dirichlet series

$$\varkappa = -\zeta_\Phi' - \frac{1}{\mu}\zeta_\Phi{}^2 + \frac{\mu - \varepsilon}{\mu}\zeta_\Phi.$$

The coefficient sum of $\varkappa$ is

$$K(x) = \sum_{N\mathfrak{a}\mathfrak{b} \leqslant x} 1 - \sum_{N\mathfrak{a} \leqslant x} \log N - \frac{\varepsilon}{\mu} H(x)$$

and so by (13), (14) and (17) one obtains $K(x) = O(x^{1-1/2n} \log x)$.

4. *The prime number theorem of arithmetic progressions*

The object is to prove that

$$\pi_a(x) = \pi(x, m, a) \sim \frac{1}{\varphi(m)} \frac{x}{\log x}$$

as $x \to \infty$ for every prime residue class $\bar{a}$ modulo m. Given any $a > 0$ we let

$$\psi_a(x) = \psi(x, m, a) = \sum_{\substack{n \leqslant x \\ n \equiv a}} \Lambda(n).$$

The prime number theorem of arithmetic progressions is a simple elementary equivalent of

$$\psi_a(x) = \psi(x, m, a) \sim \frac{x}{\varphi(m)} \qquad (x \to \infty). \tag{18}$$

In order to apply Theorem 1 we need two results from elementary number theory. The first is

$$\sum_{n \leqslant x} d\left(\frac{n}{(m, n)}\right) = \frac{\varphi(m)}{m} x \,(\log x + 2\gamma - 1) + O(x^{\frac{1}{2}}). \tag{19}$$

Here $d(n/(m, n))$ is the number of those divisors of n which are prime to the modulus m.

Proof: First of all we have

$$\begin{aligned}\sum_{n \leqslant x} \sum_{\substack{d/n \\ (d, m) = 1}} 1 &= 2 \sum_{\substack{d \leqslant \sqrt{x} \\ (d, m) = 1}} \left[\frac{x}{d} - d\right] + O(\sqrt{x}) \\ &= 2x \sum_{\substack{d \leqslant \sqrt{x} \\ (d, m) = 1}} \frac{1}{d} - 2 \sum_{\substack{d \leqslant \sqrt{x} \\ (d, m) = 1}} d + O(\sqrt{x}).\end{aligned}$$

Thus it is sufficient to show that the condition $(d, m) = 1$ can be dropped by introducing a factor $\varphi(m)/m$. This modification is justified in view of the following:

Lemma 2. *If the number theoretic function $f(n)$ is such that for each fixed a*

$$f(n+a) - f(n) = O(1) \text{ resp. } O\left(\frac{1}{n}\right) \tag{20}$$

where the constant of $O(\cdot)$ may depend on a then

$$\sum_{\substack{n \leqslant x \\ (m, n) = 1}} f(n) - \frac{\varphi(m)}{n} \sum_{n \leqslant x} f(n) = O(x) \text{ resp. } O(\log x). \tag{21}$$

Proof. By (20) we have

$$\sum_{\substack{n \leqslant x \\ n \equiv a}} f(n) - \sum_{\substack{n \leqslant x \\ n \equiv a'}} f(n) = O(x) \text{ resp. } O(\log x)$$

and so it is clear that

(22) $$\sum_{\substack{n\leqslant x\\ n\equiv a}} f(n)-\frac{1}{m}\sum_{n\leqslant x} f(n)=O(x) \text{ resp. } O(\log x).$$

Thus (21) follows immediately from

$$\sum_{\substack{n\leqslant x\\ (m,n)=1}} f(n)=\sum_{\substack{0<a<m\\ (a,m)=1}}\ \sum_{\substack{n\leqslant x\\ n\equiv a}} f(n).$$

If we let $f(n)=\log n$ we obtain the second elementary estimate needed in the proof:

(23) $$\sum_{\substack{n\leqslant x\\ (m,n)=1}} \log n=\frac{\varphi(m)}{m}\, x(\log x-1)+O(\log x).$$

Now comes the proof of (18) in a rapid succession of steps. We shall apply Theorem 1 $\varphi(m)$-times with $\sigma=1$. We fix a residue class character χ and let $\alpha=L^{-1}$. The nth coefficient of α is $\chi(n)\mu(n)$, so $A(u)=O(u)$ and $a\in L^{\infty}(G)$. If $\chi\neq\varepsilon$ we let $\beta=L$ so that the convergence abscissa $\gamma^{\beta}=0<\sigma=1$ and so $b\in L^{1}(G)$. Since $\alpha\beta(s)\equiv 1$ we obtain $(a\star b)(+\infty)=0$. Therefore by Theorem 1 $(a\star k)(+\infty)=0$ for every Dirichlet series $\varkappa$ such that $\gamma^{\varkappa}<\sigma=1$. If we let $\varkappa=-L'$ then by the boundedness of $\sum\chi(n)$ we have

$$K(u)=\sum_{n\leqslant u}\chi(n)\log n=O(\log u)$$

and so the abscissa of $\varkappa$ is $\gamma^{\varkappa}=0<\sigma=1$. Since $\alpha\varkappa=-L'/L$ the nth coefficient of $\gamma=\alpha\varkappa$ is $c_n=\chi(n)\Lambda(n)$ and so $C(u)=O(u)$. Partial summation gives

(24) $$\sum_{n\leqslant u}\frac{\psi_{\chi}(n)}{n}=o(u)\qquad (u\to\infty)$$

where for simplicity

(25) $$\psi_{\chi}(n)=\sum_{k\leqslant n}\chi(k)\,\Lambda(k).$$

If χ is the principal character ε then a few modifications are necessary. First of all the convergence abscissa of L is 1 and so instead of $\beta=L$ we must choose for instance

$$\beta(s)=(2-p^{1-s}-q^{1-s})\,L(s\,|\,\varepsilon)$$

where p and q are prime to the modulus m. Then β is holomorphic in the positive half plane and $\gamma^{\beta}=0$. Since $\alpha\beta$ is a finite Dirichlet series $(a\star b)(+\infty)=0$ and so by Theorem 1 we have $(a\star k)(+\infty)=0$ for every Dirichlet series $\varkappa$ with $\gamma^{\varkappa}<1$. We choose

$$\varkappa=-L'-L\zeta+2\gamma\,\frac{\varphi(m)}{m}\,L$$

where γ denotes the Euler constant. The nth coefficient of $\varkappa$ is

$$c_n = \varepsilon(n) \log n - \sum_{d/n} \varepsilon(d) + 2\gamma \frac{\varphi(m)}{m}.$$

Since $\varepsilon(n)=1$ or 0 according as (m, n) is 1 or not we have

$$C(u) = \sum_{\substack{n \leqslant u \\ (m,n)=1}} \log n - \sum_{n \leqslant u} d\left(\frac{n}{(m, n)}\right) + 2\gamma \frac{\varphi(m)}{m} [u].$$

Hence by (19) and (23) we obtain $C(u)=O(u^{\frac{1}{2}})$. Consequently $\gamma^{\varkappa} \leqslant \frac{1}{2} < \sigma = 1$ and $\varkappa$ is admissible. The rest of the situation is similar to the one which we met in the case $\chi \neq \varepsilon$: Since

$$\gamma = \alpha\varkappa = -\frac{L'}{L} - \zeta + 2\gamma \frac{\varphi(m)}{m}$$

its coefficient sum is

$$C(u) = \sum_{n \leqslant u} \varepsilon(n)\, \Lambda(n) - u + O(1).$$

We have $C(u)$ and so $(a \star k)(+\infty)=0$ gives

$$\sum_{n \leqslant n} \frac{\psi_\varepsilon(n)}{n} = u + o(u) \qquad (u \to +\infty) \tag{26}$$

where

$$\psi_\varepsilon(n) = \sum_{k \leqslant n} \varepsilon(k)\, \Lambda(k). \tag{29}$$

The rest is straight forward: From (24) and (26) we get

$$\sum_{n \leqslant x} \psi_\chi(n) = \begin{cases} o(x^2) & \text{if } \chi \neq \varepsilon \\ \dfrac{x^2}{2} + o(x^2) & \text{if } \chi = \varepsilon. \end{cases} \tag{28}$$

The left hand side can be rearranged as follows:

$$\sum_{n \leqslant x} \sum_{k \leqslant n} \chi(k)\, \Lambda(k) = \sum_{\bar{a} \in G_m} \chi(\bar{a}) \sum_{n \leqslant x} \sum_{\substack{k \leqslant n \\ k \equiv a}} \Lambda(k).$$

Thus introducing the functions ψ_a (28) becomes

$$\sum_{\bar{a} \in G_m} \chi(\bar{a}) \sum_{n \leqslant x} \psi_a(n) = \begin{cases} o(x^2) & \text{if } \chi \neq \varepsilon \\ \dfrac{x^2}{2} + o(x^2) & \text{if } \chi = \varepsilon. \end{cases}$$

As χ varies over $\widehat{G}_m$ we have a system of linear equations which is solved by

$$\sum_{n \leqslant x} \psi_a(n) = \frac{x^2}{2\varphi(m)} + o(x^2). \tag{29}$$

From the last equation we obtain

$$\psi_a(x) \sim \frac{x}{\varphi(m)} \qquad (x \to +\infty) \tag{30}$$

for each prime residue a modulo m. Thus the prime number theorem of arithmetic progressions is proved.

Instead of relying on Theorem 1 we could have used Theorem 2. In Section 2 we gave two proofs of the ordinary prime number theorem which were based on Theorem 2. Both variants can be extended to a proof of the prime number theorem of arithmetic progressions: One choice of the Dirichlet series α leads very rapidly to a result which turns out to be an elementary equivalent of the prime number theorem of arithmetic progressions. The passage from this equivalent to the familiar form $\psi_a(x) \sim x/\varphi(m)$ is via an analogue of Lemma 1 due to Axer. A second choice of the α's gives immediately $\psi_a(x) \sim x/\varphi(m)$ but then it is harder to verify that the a's are slowly oscillating. Namely one needs Lemma 1 with a suitable choice of u. The two methods are closely related to each other and the third variant given above.

In the first case we let $\alpha = L^{-1}$ for every character χ including the principal character ε. For $\chi \neq \varepsilon$ we can let $\beta = L$ but if $\chi = \varepsilon$ we have to choose

$$\beta(s) = (2 - p^{1-s} - q^{1-s})\, L(s \mid \chi) \tag{31}$$

where p and q are prime to m. Of course we could use this second choice of β also when $\chi \neq \varepsilon$. The conclusion is

$$\sum_{n \leqslant x} \chi(n)\, \psi(n) = o(x) \quad \text{as} \quad (x \to +\infty) \tag{32}$$

for every residue class character χ modulo m. In order to derive (30) from (32) first we use the lemma of Axer with $\varkappa = -L'$ if $\chi \neq \varepsilon$. Next for $\chi = \varepsilon$ we let

$$\varkappa = -L' - L\zeta + 2\gamma \frac{\varphi(m)}{m} L$$

where L denotes the L-function corresponding to the principal character ε. These choices will yield

$$\sum_{n \leqslant x} \chi(n)\, \Lambda(n) = \begin{cases} o(x) & \text{if } \chi \neq \varepsilon \\ x + o(x) & \text{if } \chi = \varepsilon. \end{cases} \tag{33}$$

We rearrange terms so that the left hand side becomes

$$\sum_{\bar{a} \in G_m} \chi(\bar{a})\, \psi_a(x)$$

and solve the resulting equation system to obtain (30). We see that the prime number theorem of arithmetic progressions is an elementary consequence of (32).

In the second variant of the proof based on Theorem 2 we let $\alpha = -L'/L$ for each character $\chi \neq \varepsilon$. For the principal character ε we choose

$$\alpha = -\frac{L'}{L} - \zeta + 2\gamma \frac{\varphi(m)}{m}$$

where the last term indicates the Dirichlet series whose only non-vanishing term is the constant $2\gamma\varphi(m)/m$. Upon applying the theorem we obtain (33). As we had already seen this is an elementary equivalent of the prime number theorem of arithmetic progressions. The boundedness of the associated functions a can be seen from the elementary estimate $\psi(x) = O(x)$. We let β denote the Dirichlet series given in (31). If $\chi \neq \varepsilon$ we may simplify the definition by letting $\beta = L$ instead of (31).

We notice that $\alpha\beta = -L'$ for $\chi \neq \varepsilon$ and

$$\alpha\beta = (2 - p^{1-s} - q^{1-s}) \left(-L' - L\zeta + 2\gamma \frac{\varphi(m)}{m} L \right)$$

for $\chi = \varepsilon$. In the first version of the proof we had already seen that the coefficient sum of $-L'$ is $O(\log x)$ and of $-L' - L\zeta + 2\gamma\ \varphi(m)/m\ L$ is $O(x^{\frac{1}{2}})$. This is sufficient information to conclude that $\alpha\beta$ and $(\alpha\beta)'$ have coefficient sums $O(x^{\frac{1}{2}})$ and $O(x^{\frac{1}{2}} \log x)$, respectively. Therefore, $a \star b$ vanishes at $+\infty$. The fact that the a's are slowly oscillating follows immediately from Lemma 1. We let $a_n = \chi(n)$ and $\varkappa = -L'$ or

$$\varkappa = -L' - L\zeta + 2\gamma \frac{\varphi(m)}{m}$$

according as χ is principal or not. The coefficient sum K of $\varkappa$ is of bounded variation on every finite interval $(0, x)$ and satisfies $K(x) = O(x^{\frac{1}{2}})$. Therefore, the a's are slowly oscillating and Theorem 2 can be applied.

University of Minnesota
Minneapolis, Minnesota