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# Helly's selection theorem and the principle of local reflexivity of ordered type

by Yau-Chuen Wong and Chi-Keung Ng

*Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong*

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ABSTRACT

Let $(E, E_+, \|\cdot\|)$ be an ordered normed space with a positive cone $E_+$, let $0 \leq \psi \in E''$, let $N$ be a finite-dimensional subspace of $E'$ and $\varepsilon > 0$. In terms of the notions of half-full injections and half-decomposable surjections, sufficient conditions for $N$ to ensure the existence of $x \in E_+$ with

$$\|x\| \leq \|\psi\| + \varepsilon \quad \text{and } \psi = K_E\, x \text{ on } N$$

have been found (Theorems 3.5 and 3.6). As an application of Helly's selection theorem of ordered type, the principle of local reflexivity of ordered type is obtained (Theorem 4.7).

## 1. INTRODUCTION

The classical Helly's selection theorem (see [17] or [14, p. 63]) says that for a normed space $E$, if $\psi \in E''$ and $N$ is a finite-dimensional subspace of $E'$, then for any $\varepsilon > 0$, there exists an $x \in E$ with $\|x\| \leq \|\psi\| + \varepsilon$ such that

$$(*) \qquad \psi = K_E\, x \quad \text{on } N.$$

Now if $(E, E_+)$ is an ordered normed space with a positive cone $E_+$, and $\psi \in E''$ is positive, it is natural to ask whether there exists an $x \in E_+$ with $\|x\| \leq \|\psi\| + \varepsilon$ such that $(*)$ holds. The answers are, in general, negative. One of the purposes of this paper is devoted to give some sufficient conditions for the finite-dimensional subspace $N$ of $E'$ to ensure that the answer is affirmative. To this end, we introduce the notions of half-full injection modulus and half-decomposable surjection modulus which are useful for normal cones and conormal cones (open decomposition in the terminology of Jameson [6]) respectively (Propositions 2.9 and

2.8), and then establish their duality result (Theorem 2.4). In terms of these two notions, sufficient conditions for $N$ to answer the above question have been found (Theorems 3.5 and 3.6). On the other hand, the 'principle of local reflexivity', due to Lindenstrauss and Rosenthal [5], can be derived from the equation $\mathcal{L}(E,F)'' = \mathcal{L}(E,F'')$ (where $\dim E < \infty$), by using Helly's selection theorem (see [2]). As an application of Helly's selection theorem of ordered type, the principle of local reflexivity of ordered type is obtained (Theorem 4.6). Also a necessity and sufficient condition for a subspace $M$ of an ordered normed space $(E, E_+, \|\cdot\|)$ such that each positive continuous linear functional on $M$ can be extended positively and continuously on the whole space is also found (Theorem 2.5) by means of the notion of half-full injection modulus.

## 2. HALF-FULL INJECTIONS AND HALF-DECOMPOSABLE SURJECTIONS

We shall assume throughout this paper that the scalar field for vector spaces is the real field $\mathbb{R}$, and that all topological vector spaces (abbreviated by TVS) will be Hausdorff.

By an *ordered convex space* (abbreviated by ordered LCS), denoted by $(X, X_+, \mathcal{P})$, is meant a locally convex space $(X, \mathcal{P})$ equipped with a cone $X_+$; and a normed (resp. Banach) space with a cone is called an *ordered normed space* (resp. *ordered B-space*), abbreviated by ordered NS (resp. ordered BS) and denoted by $(E, E_+, \|\cdot\|)$ or $(E, E_+)$. In general, we shall not assume that cones are closed or proper.

Let $(X, X_+, \mathcal{P})$ and $(Y, Y_+, \mathcal{T})$ be ordered LCS. Throughout this paper, $\mathcal{L}(X, Y)$ will denote the vector space of all continuous linear maps (called operators) from $X$ into $Y$, and

$$\mathcal{L}_+(X, Y) = \{T \in \mathcal{L}(X, Y) : T(X_+) \subset Y_+\}$$

(elements in $\mathcal{L}_+(X, Y)$ are called *positive operators*); in particular, we write the topological dual of $X$ and the dual cone of $X_+$ by:

$$X' = \mathcal{L}(X, \mathbb{R}) \quad \text{and} \quad X'_+ = \mathcal{L}_+(X, \mathbb{R}).$$

If $(E, E_+, \|\cdot\|)$ and $(F, F_+, \|\cdot\|)$ are ordered normed spaces. We denote by $U_E$ (resp. $U_{E'}$) the closed unit ball in $E$ (resp. in $E'$); when $E$ and $F$ are isometric or metrically isomorphic (resp. isometric and order preserving, called *order isometric*), we write $E \equiv F$ (resp. $E \overset{(0)}{\equiv} F$).

Let $(E, E_+, \|\cdot\|)$ and $(F, F_+, \|\cdot\|)$ be ordered normed spaces and $T \in \mathcal{L}_+(E, F)$. For a given $\tau > 0$, we say that $T$ is a (or an):

(a) *$\tau$-half-full injection* (resp. *almost $\tau$-half-full injection*) if

$$\tau T^{-1}(U_F - F_+) \subset U_E - E_+ \quad (\text{resp. } \tau T^{-1}(U_F - F_+) \subset \overline{U_E - E_+});$$

(b) *$\tau$-half-decomposable surjection* (resp. *almost $\tau$-half-decomposable surjection*) if

$$\tau(U_F \cap F_+) \subset T(U_E \cap E_+) \quad \text{resp. } \tau(U_F \cap F_+) \subset \overline{T(U_E \cap E_+)}).$$

A $T \in \mathcal{L}_+(E, F)$ is called a *half-full injection* (resp. *almost half-full injection, half-decomposable surjection, almost half-decomposable surjection*) if it is a $\lambda$-half-full injection (resp. almost $\lambda$-half-full injection, $\lambda$-half-decomposable surjection, almost $\lambda$-half-decomposable surjection) for some $\lambda > 0$.

Let $T \in \mathcal{L}_+(E, F)$. It is easily seen that:

(a) $T$ is an almost $\tau$-half-full injection if and only if $T$ is a $(\tau/1 + \varepsilon)$-half-full injection for any $\varepsilon > 0$;

(b) $T' \in \mathcal{L}_+(F', E')$ is an almost $\tau$-half-full injection if and only if $T'$ is also a $\tau$-half-full injection (since $U_{F'}$ is weakly compact).

Denote by $p_E$ (resp. $q_F$) the guage of $U_E - E_+$ (resp. $U_F - F_+$). It is easily seen that

$$\begin{aligned} p_E(x) &= \inf\{\|u\| : u \geq x\} \\ &= \inf\{\|x + v\| : v \in E_+\} \ (\leq \|x\|) \quad \text{(for any } x \in E), \end{aligned}$$

hence a $T \in \mathcal{L}_+(E, F)$ is an almost $\tau$-half-full injection if and only if

$$\tau p_E(x) \leq q_F(Tx) \quad \text{(for any } x \in E).$$

This observation enables us to induce the following notions analogues to that of injection modulus and surjection modulus, introduced by Pietsch in [8, (B.3)].

**Definition.** Let $(E, E_+, \|\cdot\|)$ and $(F, F_+, \|\cdot\|)$ be ordered normed spaces. For any $T \in \mathcal{L}_+(E, F)$, we write:

$$\begin{aligned} j_{hf}(T) &= \sup\{\tau \geq 0 : \tau T^{-1}(U_F - F_+) \subset U_E - E_+\}; \\ j_{hf}^{(a)}(T) &= \sup\{\tau \geq 0 : \tau T^{-1}(U_F - F_+) \subset \overline{U_E - E_+}\}; \\ q_{hf}(T) &= \sup\{\tau \geq 0 : \tau(U_F \cap F_+) \subset T(U_E \cap E_+)\}; \\ q_{hf}^{(a)}(T) &= \sup\{\tau \geq 0 : \tau(U_F \cap F_+) \subset \overline{T(U_E \cap E_+)}\}. \end{aligned}$$

$j_{hf}(T)$ (resp. $j_{hf}^{(a)}(T)$) is called the *half-full injection modulus* (resp. *almost half-full injection modulus*) of $T$; while $q_{hf}(T)$ (resp. $g_{hf}^{(a)}(T)$) is called the *half-decomposable surjection modulus* (resp. *almost half-decomposable surjection modulus*) of $T$.

It is clear that $T \in \mathcal{L}_+(E, F)$ is a half-full injection (resp. almost half-full injection) if and only if $j_{hf}(T) > 0$ (resp. $j_{hf}^{(a)}(T) > 0$). Similar conclusions hold for half-decomposable surjections or almost half-decomposable surjections. It is also clear that

$$j_{hf}(T) = j_{hf}^{(a)}(T) \quad \text{and} \quad q_{hf}(T) \leq q_{hf}^{(a)}(T).$$

In order to study the duality relationship between half-full injections and half-decomposable surjections, we require the following result:

**Lemma 2.1.** *Let $(X, X_+, \mathcal{P})$ be an ordered LCS with the topological dual $X'$ and dual cone $X'_+$, let $0 \in V \subset X$ and $0 \in D \subset X'$. Then*:

(a) $(V - X_+)^0 = V^0 \cap (-X_+^0) = V^0 \cap X'_+$ *and* $(V + X_+)^0 = V^0 \cap X_+^0 = V^0 \cap (-X'_+)$.

(b) *If, in addition, $V$ is a convex $\tau(X, X')$-neighbourhood of* 0, *then*:

$$(V \cap X_+)^0 = V^0 + X_+^0 = V^0 - X'_+.$$

(c) $(D - X'_+)^0 = D^0 \cap (-(X'_+)^0) = D^0 \cap X_+^{00} = D^0 \cap \overline{X_+}$.

**Proof.** The proofs of (a) and (b) can be found in [12, (1.1.5)]; while (c) follows directly from the definition of polars. □

As a consequence of Lemma 2.1 and the bipolar theorem, we get the following:

**Corollary 2.2.** *Let $(X, X_+, \mathcal{P})$ be an ordered LCS. For any convex $\tau(X, X')$-neighbourhood $V$ of* 0, *one has*:

$$\overline{V \cap X_+} = \overline{V} \cap \overline{X_+}.$$

**Proof.** As $0 \in V \cap X_+$ and $X_+$ is convex, it follows from the bipolar Theorem, 2.1(b) and (c) that

$$\begin{aligned}\overline{V \cap X_+} &= (V \cap X_+)^{00} = (V^0 + X_+^0)^0 = (V^0 - X'_+)^0 \\ &= V^{00} \cap \overline{X_+} = \overline{V} \cap \overline{X_+}. \qquad \square\end{aligned}$$

As an immediate consequence of Corollary 2.2, we obtain:

**Corollary 2.3.** *Let $(E, E_+)$ and $(F, F_+)$ be ordered normed spaces, let $T \in \mathcal{L}_+(E, F)$ and $\tau > 0$. Then $T : (E, E_+) \to (F, F_+)$ is an almost $\tau$-half-decomposable surjection if and only if $T : (E, E_+) \to (F, \overline{F_+})$ is an almost $\tau$-half-decomposable surjection.*

We now describe the duality relationship between almost half-full injections (resp. almost half-decomposable surjections) and half-decomposable surjections (resp. half-full injections).

**Theorem 2.4.** *Let $(E.E_+)$ and $(F, F_+)$ be ordered normed spaces and $T \in \mathcal{L}_+(E, F)$. Then*

$$j_{hf}^{(a)}(T) = q_{hf}(T') \quad \text{and} \quad q_{hf}^{(a)}(T) = j_{hf}(T').$$

*In particular, $T$ is an almost $\tau$-half-full injection (resp. almost $\tau$-half-decomposable surjection) if and only if* (its dual operator) *$T'$ is a $\tau$-half-decomposable surjection (resp. $\tau$-half-full injection) for some $\tau > 0$.*

**Proof.** The equality $j_{hf}^{(a)}(T) = q_{hf}(T')$ follows from

$$(1) \qquad \tau T^{-1}(U_F - F_+) \subset \overline{U_E - E_+} \quad \Leftrightarrow \quad \tau(U_{E'} \cap E'_+) \subset T'(U_{F'} \cap F'_+).$$

Thus, we are going to verify (1).

In fact, let $\tau > 0$ be such that $\tau T^{-1}(U_F - F_+) \subset \overline{U_E - E_+}$. As $U_F - F_+$ is convex and contains $U_F$, it follows that

$$(2) \qquad \tau(\overline{U_E - E_+})^0 \subset (T^{-1}(U_F - F_+))^0 = T'((U_F - F_+)^0).$$

By Lemma 2.1(a), we obtain

$$(3) \qquad (\overline{U_E - E_+})^0 = U_E^0 \cap (-E_+^0) = U_{E'} \cap E'_+$$

and

$$(4) \qquad T'((U_F - F_+)^0) = T'(U_F^0 \cap (-F_+^0)) = T'(U_{F'} \cap F'_+);$$

it then follows from (2) that

$$\tau(U_{E'} \cap E'_+) \subset T'(U_{F'} \cap F'_+).$$

Conversely, let $\tau > 0$ be such that $\tau(U_{E'} \cap E'_+) \subset T'(U_{F'} \cap F'_+)$. As

$$\tau(U_{E'} \cap E'_+) = \tau(U_E - E_+)^0 \subset T'((U_F - F_+)^0) = (T^{-1}(U_F - F_+))^0$$

(by Lemma 2.1(a) or (3) and (4)), it follows that

$$\tau T^{-1}(U_F - F_+) \subset \tau(T^{-1}(U_F - F_+))^{00} \subset (U_E - E_+)^{00} = \overline{U_E - E_+}.$$

Therefore (1) holds.

The equality $q_{hf}^{(a)}(T) = j_{hf}(T')$ follows from

$$(5) \qquad \tau(U_F \cap F_+) \subset \overline{T(U_E \cap E_+)} \quad \Leftrightarrow \quad \tau(T')^{-1}(U_{E'} - E'_+) \subset U_{F'} - F'_+.$$

Therefore, we are going to verify (5).

In fact, we first notice from Lemma 2.1(b) that

$$(6) \qquad (\overline{T(U_E \cap E_+)})^0 = (T')^{-1}((U_E \cap E_+)^0) = (T')^{-1}(U_{E'} - E'_+)$$

and

$$(7) \qquad (U_F \cap F_+)^0 = U_{F'} - F'_+.$$

Formula (5) now follows from (6) and (7) and the polar calculations. □

It is amusing to compare the preceding result with classical well-known results [8, (B.3.8), p. 27].

The preceding result enables us to give a criteria for vector subspaces of ordered normed spaces such that positive continuous linear functionals can be extended positively to the whole spaces.

**Theorem 2.5.** *Let $(E, E_+, \|\cdot\|)$ be an ordered normed space, let $M$ be a vector subspace of $E$ and $\tau > 0$. Then the canonical injection $J_M : M \to E$ is an almost $\tau$-half-full injection if and only if for any $g \in M'_+$ there exists an $f \in E'_+$ such that*

$$(1) \qquad g = fJ_M \quad \text{and} \quad \|f\| \le \frac{1}{\tau}\|g\|_M.$$

*In particular, if $J_M$ is an almost half-full injection, then any positive continuous linear functional on $M$ has a positive, continuous extension.*

**Proof.** The result follows from the following equivalent statements:

$$\begin{aligned} J_M : M \to E \quad & \text{is an almost } \tau\text{-half-full injection} \\ \Leftrightarrow \quad & J'_M : E' \to (M', M'_+) \quad \text{is a } \tau\text{-half-decomposable surjection} \\ \Leftrightarrow \quad & \tau(U_{M'} \cap M'_+) \subset J'_M(U_{E'} \cap E'_+) \\ \Leftrightarrow \quad & \text{(1) holds.} \qquad \square \end{aligned}$$

**Remarks.** (i) It is clear that if $f \in E'_+$ is an extension of $g \in M'_+$ then $\|g\|_M \leq \|f\|$. Thus $J_M : M \to E$ is an almost 1-half-full injection if and only if each positive continuous linear functional on $M$ has a norm preserving, positive extension on $E$.

(ii) The preceding result is still true for an arbitrary ordered normed space $M$ that is not a subspace of $E$.

(iii) It is not hard to verify the dual result for quotient maps; more general, one can show that if $T \in \mathcal{L}_+(E, F)$ and $\tau > 0$, then $T$ is a $\tau$-half-decomposable surjection if and only if for any $S_0 \in \mathcal{L}_+(\mathbb{R}, F)$ there exists an $S \in \mathcal{L}_+(\mathbb{R}, E)$ such that

$$TS = S_0 \quad \text{and} \quad \|S\| \leq \frac{1}{\tau} \|S_0\|.$$

Because of the preceding result, we call a vector subspace $M$ of an ordered normed space $(E, E_+)$ to be a (resp. an *almost*) *$\tau$-half-full subspace* of $E$ if $J_M$ is a (resp. an almost) $\tau$-half-full injection. Dually a closed subspace $N$ of $(E, E_+, \|\cdot\|)$ is called a (resp. an *almost*) *$\tau$-half co-subspace* of $E$ if the quotient map $Q_N : E \to E/N$ is an (almost) $\tau$-half-decomposable surjection. A subspace (resp. closed subspace) $M$ of $(E, E_+)$ is called a *half-full subspace* (resp. (*almost*) *half co-subspace*) if $J_M : M \to E$ is a $\tau$-half-full injection (resp. $Q_M : E \to E/M$ is an (almost) $\tau$-half-decomposable surjection) for some $\tau > 0$.

As another application of Theorem 2.4, we are going to verify that $l_\infty(\Lambda)$ (resp. $l_1(\Lambda)$) has some sort of positive extension property (resp. positive lifting property) as follows:

**Proposition 2.6.** *Let $(E, E_+)$, $(F, F_+)$ and $(G, G_+)$ be ordered B-spaces, let $\Lambda$ be a non-empty index set and $\tau > 0$.*

(a) *Let $J \in \mathcal{L}_+(G, E)$ be an almost $\tau$-half-full injection. Then for any $T_0 \in \mathcal{L}_+(G, l_\infty(\Lambda))$, there exists a positive $\tilde{T} \in \mathcal{L}_+(E, l_\infty(\Lambda))$ such that*

(1) $$\tilde{T}J = T_0 \quad \textit{and} \quad \|\tilde{T}\| \leq \frac{1}{\tau} \|T_0\|.$$

(b) *Dually let $Q \in \mathcal{L}_+(F, G)$ be a $\tau$-half-decomposable surjection. Then for any $S_0 \in \mathcal{L}_+(l_1(\Lambda), G)$, there exists a positive $\tilde{S} \in \mathcal{L}_+(l_1(\Lambda), F)$ such that*

(2) $$Q\tilde{S} = S_0 \quad \textit{and} \quad \|\tilde{S}\| \leq \frac{1}{\tau} \|S_0\|.$$

**Proof.** (a) For any $i \in \Lambda$, let $P_i : l_\infty(\Lambda) \to \mathbb{R}$ be the $i$-th projection and $f_i = P_i T_0$. Then $f_i \in G'_+$ is such that

$$(3)\qquad \begin{cases} |f_i(x)| = |P_i T_0(x)| \le \|T_0\, x\|_\infty \le \|T_0\|\ \|x\| \quad \text{and} \\ T_0\, x = [P_i(T_0\, x),\ i \in \Lambda] \quad (x \in G), \end{cases}$$

hence

$$(4)\qquad \sup_{i\in\Lambda} \|f_i\| \le \|T_0\|.$$

As $J \in \mathcal{L}_+(G, E)$ is an almost $\tau$-half-full injection, it follows from Theorem 2.4 that $J' \in \mathcal{L}_+(E', G')$ is a $\tau$-half-decomposable surjection, i.e.,

$$\tau(U_{G'} \cap G'_+) \subset J'(U_{E'} \cap E'_+).$$

For any $i \in \Lambda$, there exists a $g_i \in U_{E'} \cap E'_+$ such that

$$(5)\qquad \frac{\tau}{\|T_0\|} f_i = J' g_i,$$

hence the map $\tilde{T} : E \to l_\infty(\Lambda)$, defined by

$$\tilde{T}(z) = \left[\frac{\|T_0\|}{\tau} g_i(z),\ i \in \Lambda\right] \quad (\text{for any } z \in E),$$

is a positive operator such that

$$\|\tilde{T}\| = \sup_{i\in\Lambda} \frac{\|T_0\|}{\tau} \|g_i\| \le \frac{1}{\tau} \|T_0\|$$

and

$$\tilde{T}(Jx) = \left[\frac{\|T_0\|}{\tau} g_i(Jx),\ i \in \Lambda\right] = [f_i(x),\ i \in \Lambda] = T_0\, x \quad (\text{for any } x \in G)$$

(by (3) and (5)).

(b) As $S_0 : l_1(\Lambda) \to G$ is an operator, it follows from a well-known result (see [7, p. 280] or [14, (3.j)(i)′]) that there exists a bounded family $[y_i,\ i \in \Lambda]$ in $G$ such that

$$(6)\qquad S_0([\zeta_i,\ i \in \Lambda]) = \sum_\Lambda \zeta_i y_i \quad (\text{for any } [\zeta_i,\ i \in \Lambda] \in l_1(\Lambda))$$

and

$$(7)\qquad \sup_{i\in\Lambda} \|y_i\| = \|S_0\|.$$

Since $S_0$ is positive, it is easily seen from (6) that $y_i \in G_+$ $(i \in \Lambda)$. On the other hand, since $\tau(U_G \cap G_+) \subset Q(U_F \cap F_+)$, it follows from (7) that there exists a $z_i \in U_F \cap F_+$ such that

$$(8)\qquad \frac{\tau}{\|S_0\|} y_i = Q(z_i) \quad (\text{for all } i \in \Lambda),$$

and hence that $[(\|S_0\|/\tau) z_i,\ i \in \Lambda]$ is a bounded family in $F_+$. Now the map $\tilde{S} : l_1(\Lambda) \to F$, defined by

$$\tilde{S}([\zeta_i,\ i \in \Lambda]) = \sum_{\Lambda} \zeta_i \frac{\|S_0\|}{\tau} z_i \quad (\text{for any } [\zeta_i,\ i \in \Lambda] \in l_1(\Lambda)),$$

satisfies all requirements. $\square$

In order to give some examples of half-full injections and half-decomposable surjections, let us recall the following terminology: Let $(E, E_+, \|\cdot\|)$ be an ordered normed space. A vector subspace $M$ of $E$ is a *positive complemented subspace* of $E$ if there exists a positive projector $P \in \mathcal{L}_+(E, E)$ such that $M = P(E)$.

**Proposition 2.7.** *Let E be an ordered normed space.*

(a) *Suppose that M is a positive complemented subspace of E. Then the canonical injection $J_M : M \to E$ is a half-full injection, and the positive projector $P : E \to M$ is a 1-half-decomposable surjection.*

(b) *The canonical embedding $K_E : (E, E_+) \to E'', E''_+)$ is an almost 1-half-full injection.*

**Proof.** (a) We have

$$\begin{aligned} J_M^{-1}(U_E - E_+) &= (U_E - E_+) \cap M \subset P(U_E - E_+) \\ &\subset P(U_E) - M_+ \subset \|P\|\, U_M - M_+ = \|P\|\,(U_M - M_+), \end{aligned}$$

hence $J_M$ is a $(1/\|P\|)$-half-full injection.

As $U_M \cap M_+ \subset U_E \cap E_+$ and $P^2 = P$, we conclude that

$$U_M \cap M_+ = P(U_M \cap M_+) \subset P(U_E \cap E_+),$$

and hence that $P$ is a 1-half-decomposable surjection.

(b) In view of Theorem 2.4, it is required to show that $K_E' : E''' \to E'$ is a 1-half-decomposable surjection, or, equivalently

(1) $\qquad U_{E'} \cap E'_+ \subset K_E'(U_{E'''} \cap E'''_+).$

In fact, let $f \in U_{E'} \cap E'_+$ and let us define

$$g = K_{E'}(f)$$

where $K_{E'} : E' \to E'''$ is the canonical embedding. Then $g \in U_{E'''} \cap E'''_+$ is such that $f = K_E'(g)$ [since $I_{E'} = (K_E)' K_{E'}$]. $\square$

**Examples.** (a) Let $E$ be a normed vector lattice. Any sublattice of $E$ is a 1-half-full subspace, and any closed lattice ideal in $E$ is a $1/(1+\varepsilon)$-half co-subspace (for any $\varepsilon > 0$).

(b) Any ordered and topological isomorphism between ordered normed spaces must be a half-full injection as well as a half-decomposed surjection.

(c) Metric injections are, in general, not half-full injections, and dually open maps are, in general, not almost half-decomposable surjections.

**Proof.** Parts (a) and (b) are obvious. To prove (c), consider the ordered B-space $E = l_\infty$ and define

$$(*) \qquad M = \{[\mu_n] \in l_\infty : \mu_{2n} = -n\mu_{2n-1} \text{ (for all } n = 1, 2, \ldots)\}.$$

Then $M_+ = (l_\infty)_+ \cap M = \{0\}$ [since $\mu_{2n-1} \geq 0$ and $-n\mu_{2n-1} \geq 0$ will imply that $\mu_{2n-1} = 0$]. Moreover, we claim that $(U_E - E_+) \cap M$ is not a subset of $nU_M$ (for all $n \geq 1$), thus the canonical embedding $J_M : M \to E$ is a metric injection which is not a half-full injection.

In fact, let us take $\zeta = [\zeta_n] \in U_E$ with $\zeta_k = 1$ (for all $k \geq 1$). For a fixed $n \geq 1$, let us define $v^{(n)} = [v_k^{(n)}] \in E_+$ by

$$v_k^{(n)} = \begin{cases} 0, & \text{if } k \text{ is odd and } k \leq 2n+1; \\ (k/2)+1, & \text{if } k \text{ is even and } k \leq 2n+2; \\ 1, & \text{if } k \geq 2n+3. \end{cases}$$

Then the $k$-th coordinate of $\zeta - v^{(n)}$, denoted by $\alpha_k^{(n)}$, is

$$\alpha_k^{(n)} = \begin{cases} 1, & \text{if } k \text{ is odd and } k \leq 2n+1; \\ -(k/2), & \text{if } k \text{ is even and } k \leq 2n+2; \\ 0, & \text{if } k \geq 2n+3. \end{cases}$$

hence $\zeta - v^{(n)} \in M$ and $\|\zeta - v^{(n)}\|_\infty = n+1 > n$, this shows that $\zeta - v^{(n)} \notin nU_E$.

Dually, let $G = l_1$. Then $G' \overset{(0)}{\equiv} l_\infty = E$. Observe that the subspace $M$, defined by $(*)$, is $\sigma(E,G)$-closed [since, if $\zeta^{(\lambda)} = [\zeta_n^{(\lambda)}]$, $\zeta = [\zeta_n]$ are in $E$ such that $\zeta = \sigma(E,G) - \lim_\lambda \zeta^{(\lambda)}$, then $\lim_\lambda \zeta_n^{(\lambda)} = \zeta_n$ (for all $n \geq 1$)]. Let $N = M^\top$ $(\subset G)$ and $Q_N : G \to G/N$ the quotient map. Then $Q_N' = J_M$ and $Q_N$ is not an almost half-decomposable surjection (by Theorem 2.4). $\square$

Because of Example (b), it is natural to ask that under what conditions on $(E, E_+)$ or $(F, F_+)$, half-full injections (resp. half-decomposable surjections) must be injections (resp. open).

**Proposition 2.8.** *Let* $(E, E_+)$ *and* $(F, F_+)$ *be ordered normed spaces, and let* $T \in \mathcal{L}_+(E,F)$ *be a* $\tau$*-half-decomposable surjection for some* $\tau > 0$. *Then*

(a) $T(E_+) = F_+$.

(b) *If* $F = F_+ - F_+$, *then* $T$ *is onto.*

(c) *If* $F_+$ *is a conormal cone* (i.e., $F_+$ is $\lambda$-generating [6, p. 112] or [16]), *then* $T$ *is an order topological surjection* (i.e., $T$ is open and $T(E_+) = F_+$).

**Proof.** (a) For any $0 \neq v \in F_+$, $\tau(v/\|v\|) \in \tau(U_F \cap F_+)$, it then follows from $\tau(U_F \cap F_+) \subset T(U_E \cap E_+)$ that there exists an $x \in U_E \cap E_+$ such that $\tau(v/\|v\|) = Tx$, thus $v = T((\|v\|/\tau)x) \in T(E_+)$.

(b) Follows from (a).

(c) As $F_+$ is conormal, there exists a $\lambda > 0$ such that $\lambda U_F \subset D(U_F)$ (where $D(U_F) = \Gamma(U_F \cap F_+)$, (the disked hull)), it then follows that

$$\begin{aligned} \lambda\tau U_F &\subset \tau D(U_F) \subset \tau(U_F \cap F_+ - U_F \cap F_+) \subset T(U_E \cap E_+) - T(U_E \cap E_+) \\ &= T(U_E \cap E_+ - U_E \cap E_+) \subset 2T(U_E), \end{aligned}$$

and hence that $T$ is open. Consequently, $T$ is an ordered topological surjection by part (a). $\square$

**Remark.** By a similar argument given in the proof of part (c), one can show that if $T \in \mathcal{L}_+(E,F)$ is an *almost* $\tau$-half-decomposable surjection and if $F_+$ is conormal then $T$ is almost open. (Hence $T$ is open provided that $E$ is complete).

As another application of Theorem 2.4, we are able to verify the dual result of Proposition 2.8 as follows:

**Proposition 2.9.** *Let $(E, E_+)$ and $(F, F_+)$ be ordered normed spaces, and let $T \in \mathcal{L}_+(E,F)$ be a $\tau$-half-full injection for some $\tau > 0$.*

(a) *If $E_+$ is closed then $T$ is order preserving, i.e., $T(E_+) = F_+ \cap T(E)$.*

(b) *If $E_+$ is proper and closed then $T$ is one-one.*

(c) *If $E_+$ is a closed normal cone* (i.e., $E_+$ is $\lambda$-normal [6, p. 112]), *then $T$ is an order topological injection* (i.e., order preserving and $\mu T^{-1}(U_F) \subset U_E$ (for some $\mu > 0$)).

**Proof.** (a) Let $x \in E$ be such that $Tx \in F_+$. It is required to show by the closedness of $E_+$ that $f(x) \geq 0$ (for all $f \in E'_+$).

In fact, it is clear that $T$ is an almost $\tau$-half-full injection, it then follows from Theorem 2.4 that $T' \in \mathcal{L}_+(F', E')$ is a $\tau$-half-decomposable surjection, i.e., $\tau(U_{E'} \cap E'_+) \subset T'(U_{F'} \cap F'_+)$. Now for any $0 \neq f \in E'_+$, there exists a $g \in U_{F'} \cap F'_+$ such that $T'g = \tau(f/\|f\|)$; it then follows from $T \in \mathcal{L}_+(E,F)$ and $g \in F'_+$ that

$$f(x) = \frac{\|f\|}{\tau}(T'g)\,x = \frac{\|f\|}{\tau}\,g(Tx) \geq 0,$$

which obtains our requirement.

(b) Suppose that $Tx = 0$. Then $\pm Tx \geq 0$, hence $x \in E_+$ and $-x \in E_+$ (by part (a)), thus $x = 0$.

(c) We first show that $T$ is a topological injection or, equivalently, there exists a $\mu > 0$ such that $\mu T^{-1}(U_F) \subset U_E$.

In fact, as $E_+$ is a normal cone, there exists a $\lambda > 0$ such that $F(U_E) \subset \lambda U_E$ (where $F(U_E) = (U_E + E_+) \cap (U_E - E_+)$). On the other hand, the symmetry and $\tau T^{-1}(U_F - F_+) \subset U_E - E_+$ imply that $\tau T^{-1}(U_F + F_+) \subset U_E + E_+$. It then follows that

$$(1)\qquad \left\{\begin{aligned} T^{-1}(U_F) &\subset T^{-1}((U_F - F_+) \cap (U_F + F_+)) \\ &= (T^{-1}(U_F - F_+)) \cap (T^{-1}(U_F + F_+)) \\ &\subset \frac{1}{\tau}(U_E - E_+) \cap (U_E + E_+) \subset \frac{\lambda}{\tau}\,U_E, \end{aligned}\right.$$

which proves our assertion by taking $\mu = \tau/\lambda$.

The closedness of $E_+$ ensures that $T(E_+) = F_+ \cap T(E)$ (by part (a)), thus $T$ is an ordered topological injection. $\square$

**Remark.** By a similar argument given in the proof of part (c), one can show that if $T \in \mathcal{L}_+(E, F)$ is an *almost* $\tau$-half-full injection and if $E_+$ is a normal cone then $T$ is still a topological injection. [Observe that $U_E - E_+$ and $U_E + E_+$ contain $U_E$, hence $\overline{(U_E - E_+)} \cap \overline{(U_E + E_+)} = \overline{F(U_E)}$ (see [12, (1.1.4)])].

**Proposition 2.10.** *Let* $(G, G_+, \|\cdot\|)$ *be an ordered B-space, let*

$$U_{G_+} = U_G \cap G_+ \quad \textit{and} \quad U_{G'_+} = U_{G'} \cap G'_+.$$

(a) *If* $G_+$ *is* 1*-generating* (i.e., $U_G = \Gamma(U_G \cap G_+) = D(U_G)$), *then the map* $Q_G : l_1(U_{G_+}) \to (G, G_+, \|\cdot\|)$, *defined by*

$$Q_G([\zeta_x : x \in U_{G_+}]) = \sum_{x \in U_{G_+}} \zeta_x \cdot x \quad (\textit{for any } [\zeta_x : x \in U_{G_+}] \in l_1(U_{G_+})),$$

*is an order metric surjection* (i.e., order preserving and metric surjection), *and also an* almost 1*-half-decomposable surjection.*

(b) *If* $G_+$ *is* 1*-normal* (i.e., $U_G = (U_G - G_+) \cap (U_G + G_+) = F(U_G)$) *and closed, then the map* $J_G : (G, G_+, \|\cdot\|) \to l_\infty(U_{G'_+})$, *defined by*

$$J_G(x) = [\langle x, u' \rangle : u' \in U_{G'_+}] \quad (\textit{for any } x \in G),$$

*is an order metric injection* (i.e., order preserving, isometry into), *and also an almost* 1*-half-full injection. Moreover, if, in addition,* $K_G$ *is a* 1*-half-full injection* (*and surely* $G$ *is a normed vector lattice*), *then* $J_G$ *is a* 1*-half-full injection.*

**Proof.** (a) As $G_+$ is 1-generating, it is not hard to show that $U_G \subseteq Q_G(U_{l_1(U_{G_+})})$; consequently $Q_G$ is a metric surjection (since $Q_G(U_{l_1(U_{G_+})}) \subset U_G$ is always true).

By a routine argument, it is easily seen that $Q_G$ is order preserving and also 1-half-decomposable surjection.

(b) As $G_+$ is 1-normal, it follows that

$$\|x\| = \sup\{|\langle x, u' \rangle| : u' \in U_{G'_+}\} = \|J_G(x)\|_\infty \quad (\text{for any } x \in G)$$

(see [12, (1.2.1)(a)]). The closedness of $G_+$ implies that $J_G$ is order preserving.

To prove that $J_G$ is also an almost 1-half-full injection, we first notice from Grosberg-Krein's theorem (see [16, (5.15)]) that $G'_+$ is 1-generating, hence part (a) of this result shows that the mapping

$$Q_{G'} : l_1(U_{G'_+}) \to (G', G'_+, \|\cdot\|) : [\zeta_{u'} : u' \in U_{G'_+}] \to \sum_{u' \in U_{G'_+}} \zeta_{u'} \cdot u'$$

is a 1-half-decomposable surjection, hence its dual mapping $(Q_{G'})' : (G'', G''_+, \|\cdot\|) \to (l_1(U_{G'_+}))'$ is a 1-half-full injection (by Theorem 2.4), thus $J_G = (Q_{G'})' K_G$ is an almost 1-half-full injection (by Proposition 2.7(b)).

Finally, if $K_G$ is a 1-half-full injection, then so does $J_G$. □

**Remarks.** (i) If $G_+$ is $\tau$-normal and closed, then $J_G$ is one-one, order preserving with

$$\tau^{-1} \leq j(J_G) \quad \text{and} \quad \|J_G\| \leq 1,$$

and also a $(1/\tau)$-half-full injection (where $j(J_G)$ is the injection modulus (see [8, (B.3.1), p. 26]).

(ii) If $G_+$ is $\tau$-generating, then $Q_G$ is onto, order preserving with

$$\tau^{-1} \leq q(Q_G) \quad \text{and} \quad \|Q_G\| \leq 1,$$

and also an $(1/\tau)$-half-decomposable surjection (where $q(Q_G)$ is the surjection modulus of $Q_G$ (see [8, (B.3.4), p. 26])).

## 3. HELLY'S SELECTION THEOREM OF ORDERED TYPE

The classical Helly's selection theorem (see [17] or [14, p. 63]) says that for a normed space $E$, if $\psi \in E''$ and $N$ is a finite-dimensional subspace of $E'$, then for any $\varepsilon > 0$, there exists an $x \in E$ with $\|x\| \leq \|\psi\| + \varepsilon$ such that

$$(*) \qquad \psi = K_E x \quad \text{on } N.$$

This section is devoted to a study of this theorem with the consideration of an ordering in $E$ (see Theorems 3.5 and 3.6). Before doing this, let us first recall some well-known facts about the duality between subspaces and quotients of ordered normed spaces.

Let $(E, E_+)$ be an ordered vector space, let $M$ be a vector subspace of $E$, let $Q_M : E \to E/M$ be the quotient map and

$$Q_M(E_+) = (E/M)_+ = \hat{E}_+$$

(the *quotient cone* of $E$ (by $M$)). It is clear that

$$Q_M(x) \in \hat{E}_+ \quad \text{if and only if } (x + M) \cap E_+ \neq \phi.$$

$Q_M(E_+)$ is proper if and only if $M = (M + E_+) \cap (M - E_+)$ (i.e., order-convex).

**Lemma 3.1.** *Let $(E, E_+, \|\cdot\|)$ be an ordered normed space, let $M$ be a closed vector subspace, let $Q_M : E \to E/M$ be the quotient map, and $M_+^\perp = M^\perp \cap E'_+$. Then $Q'_M : (E/M, \hat{E}_+)' \to (M^\perp, M_+^\perp)$ is an order isometry.*

**Proof.** It is well-known that $Q'_M$ is a metric isomorphism. It is also clear that $Q'_M$ is positive. To prove that $Q'_M$ is order preserving, let $g \in M_+^\perp$ and let $\varphi \in (E/M)'$ be such that $g = Q'_M(\varphi) = \varphi Q_M$. We claim that $\varphi$ is positive.

In fact, let $Q_M(x) \in \hat{E}_+$. Then there exists a $u \in E_+$ such that $Q_M(x) = Q_M(u)$, hence

$$\varphi(Q_M(x)) = \varphi(Q_M(u)) = g(u) \geq 0. \qquad \square$$

Let $(E, E_+, \|\cdot\|)$ be ordered normed space, let $M$ be a vector subspace of $E$, let $J_M : M \to E$ be the canonical injection, and let $\widehat{J'_M} : E'/M^\perp \to M'$ be the injection associated with $J'_M$. It is well-known that $\widehat{J'_M}$ is a metric isomorphism. It is easily seen that

$$\widehat{J'_M}(Q_{M^\perp}(E'_+)) \subset M'_+ \ (= M' \cap E'_+),$$

i.e., $\widehat{J'_M}$ is positive. It is clear that $\widehat{J'_M}$ is order preserving (i.e., $(E'/M^{\perp}, Q_{M^{\perp}}(E'_+)) \overset{(0)}{\equiv} (M', M'_+)$ (order isometric) under $\widehat{J'_M}$) if and only if $M'_+ \subset J'_M(E'_+)$ (and surely $M'_+ = J'_M(E'_+)$ since $J_M$ is positive). Hence we obtain the following:

**Lemma 3.2.** *Let $(E, E_+, \|\cdot\|)$ be an ordered normed space, let $M$ be a vector subspace of $E$, let $J_M : M \to E$ be the canonical injection, and let $\widehat{J'_M} : E'/M^{\perp} \to M'$ be the injection associated with $J'_M$. Then $\widehat{J'_M}$ is an order isometry if and only if*

$$M'_+ \subset J'_M(E'_+).$$

*In particular, if $J_M$ is an almost $\tau$-half-full injection* (and surely $\tau$-half-full injection) *for some $\tau > 0$, then $\widehat{J'_M}$ is order preserving.*

**Lemma 3.3.** *Let $(E, E_+)$ and $(F, F_+)$ be ordered normed spaces, let $Q \in \mathcal{L}_+(E, F)$ be an open* (positive) *operator and $\tau > 0$. Then $Q$ is an almost $\tau$-half-decomposable surjection if and only if for any $v \in F_+$ and $\varepsilon > 0$, there exist $u \in E_+$ and $z \in \varepsilon U_E$ such that*

$$(1) \qquad \|u\| \le \frac{1}{\tau}\|v\| \quad \textit{and} \quad Q(u+z) = v.$$

**Proof.** We first notice from the openness of $Q$ that

$$(2) \qquad \overline{Q(U_E \cap E_+)} \subset Q(U_E \cap E_+) + \varepsilon Q(U_E) \quad (\text{for any } \varepsilon > 0).$$

*Necessity.* For any $0 \neq v \in F_+$, since $\tau(U_F \cap F_+) \subset \overline{Q(U_E \cap E_+)}$, it follows from (2) that there exist $x_0 \in U_E \cap E_+$ and $w \in (\tau\varepsilon/\|v\|)\, U_E$ such that $(\tau/\|v\|)\, v = Q(x_0 + w)$, so that $v = Q((\|v\|/\tau)(x_0 + w))$. Now let

$$u = \frac{\|v\|}{\tau}\, x_0 \quad \text{and} \quad z = \frac{\|v\|}{\tau}\, w.$$

Then $u$ and $z$ have the required property (1).

*Sufficiency.* By (2), it is required to show that

$$(3) \qquad \tau(U_F \cap F_+) \subset Q(U_E \cap E_+) + \varepsilon Q(U_E) \quad (\text{for any } \varepsilon > 0).$$

Indeed, let $0 \neq v \in \tau(U_F \cap F_+)$ and $\varepsilon > 0$. There exist $u \in E_+$ and $z \in \varepsilon U_E$ such that (1) holds, hence $u \in U_E \cap E_+$ (since $\|u\| \le (1/\tau)\,\|v\| \le 1$), and thus

$$v = Q(u+z) \in Q(U_E \cap E_+) + \varepsilon Q(U_E). \qquad \square$$

**Lemma 3.4.** *Let $(E, E_+)$ be an ordered normed space, let $M$ be a closed subspace of $E$, let $Q_M : E \to E/M$ be the quotient map and $\tau > 0$. Then $Q_M$ is a $\tau/(1+\varepsilon)$-half-decomposable surjection* (for any $\varepsilon > 0$) *if and only if for any $\hat{x} = Q_M(x) \in Q_M(E_+)$, one has*

$$(1) \qquad \inf\{\|x + z\| : z \in M \text{ and } x + z \in E_+\} \le \frac{1}{\tau}\|\hat{x}\|.$$

**Proof.** *Necessity.* For any $\hat{0} \neq \hat{x} \in Q_M(E_+)$, since $(\tau/1+\varepsilon)(U_{E/M} \cap \hat{E}_+) \subset Q_M(U_E \cap E_+)$, there exists a $u \in U_E \cap E_+$ such that

$$\frac{\tau}{(1+\varepsilon)\,\|\hat{x}\|}\, Q_M(x) = Q_M(u);$$

it then follows that there exists a $z \in M$ such that $x - ((1+\varepsilon)/\tau)\,\|\hat{x}\|\, u = -z$, and hence that

$$x + z = \frac{(1+\varepsilon)\,\|\hat{x}\|}{\tau}\, u \in E_+ \quad \text{and} \quad \|x+z\| \leq \frac{1+\varepsilon}{\tau}\, \|\hat{x}\|,$$

thus (1) holds.

*Sufficiency.* For any $\hat{0} \neq \hat{x} \in \tau(U_{E/M} \cap \hat{E}_+)$ and $\varepsilon > 0$, there exists a $z \in M$ such that

(2) $\qquad x + z \in E_+ \quad \text{and} \quad \|x+z\| < \dfrac{1}{\tau}\, \|\hat{x}\| + \varepsilon \leq 1 + \varepsilon.$

It then follows that $(1/1+\varepsilon)\,\hat{x} \in (\tau/1+\varepsilon)(U_{E/M} \cap \hat{E}_+)$ and

$$\frac{1}{1+\varepsilon}\,\hat{x} = \frac{1}{1+\varepsilon}\, Q_M(x) = \frac{1}{1+\varepsilon}\, Q_M(x+z) \in Q_M(U_E \cap E_+)$$

(by (2)); thus $(\tau/1+\varepsilon)(U_{E/M} \cap \hat{E}_+) \subset Q_M(U_E \cap E_+)$. $\square$

**Remark.** For any $\hat{x} = Q_M(x) \in Q_M(E_+)$, we always have

$$\|\hat{x}\| \leq \inf\{\|x+z\| : z \in M \text{ and } x + z \in E_+\},$$

thus (1) becomes equality when $\tau = 1$.

By means of the previous lemmas, we are able to verify the following interesting result which can be regarded as *Helly's selection theorem of ordered type.*

**Theorem 3.5.** *Let* $(E, E_+, \|\cdot\|)$ *be an ordered normed space, let* $N$ *be a finite-dimensional subspace of* $E'$, *let* $\psi \in E''_+$ *and* $\tau > 0$. *Suppose that the annihilator* $N^\top$ *of* $N$ (in $E$) *is a* $\tau/(1+\delta)$*-half co-subspace of* $E$ (i.e., $Q_{N^\top} : E \to E/N^\top$ is a $\tau/(1+\delta)$-half-decomposable surjection) *for any* $\delta > 0$, *and that the quotient cone* $Q_{N^\top}(E_+)$ *is closed in* $E/N^\top$. *Then for any* $\varepsilon > 0$, *there exists a* $u \in E_+$ *with* $\|u\| < (1/\tau)\,\|\psi\| + \varepsilon$ *such that*

(1) $\qquad f(u) = \psi(f) \quad (\text{for all } f \in N).$

**Proof.** Let $M = N^\top$. Then $M$ is closed in $E$ such that $M^\perp = (N^\top)^\perp = N$ (since $\dim N < \infty$). Let $M^{\perp\perp} = N^\perp$ be the annihilator of $N$ taken in $E''$. Then Lemma 3.1 shows that

(2) $\qquad (E/M, Q_M(E_+))' \overset{(0)}{\equiv} (N, N \cap E'_+) \quad$ (order isometric)

and

(3) $\qquad N' \equiv E''/M^{\perp\perp} \quad$ (isometric).

As $\dim N < \infty$, it follows from (2) that $\dim E/M < \infty$, and hence from (2) and (3) that

(4) $\qquad E/M \equiv E''/M^{\perp\perp}$ under $K_{E/M}$.

Thus it is easily seen that the following diagram commutes:

$$\begin{tikzcd}(E, E_+) \arrow[r, "K_E"] \arrow[d, "Q_M"'] & (E'', E''_+) \arrow[d, "Q_{M^{\perp\perp}}"] \\ (E/M, \overline{Q_M(E_+)}) \arrow[r, "K_{E/M}"] & (E''/M^{\perp\perp}, Q_{M^{\perp\perp}}(E''_+))\end{tikzcd} \tag{5}$$

where $K_{E/M}$ is an order isometry (see (4)). while $K_E$ is a positive and metric injection (for definition, see [8, (B.3.2)]). As $\psi \in E''_+$, it follows that there is an $x \in E$ such that

(6) $\qquad Q_M(x) \in \overline{Q_M(E_+)}$ and $K_{E/M}(Q_M(x)) = Q_{M^{\perp\perp}}(\psi)$.

Suppose now that $M = N^\top$ is a $\tau/(1+\varepsilon)$-half co-subspace of $E$ such that $Q_M(E_+)$ is closed in $E/M$. For this $Q_M(x) \in \overline{Q_M(E_+)} = Q_M(E_+)$, Lemma 3.4 ensures that there exists a $z \in M$ such that

$$x + z \in E_+ \quad \text{and} \quad \|x+z\| < \frac{1}{\tau}\,\|Q_M(x)\| + \varepsilon.$$

Now let $u = x + z$. Then $u \in E_+$ is such that

$$\begin{aligned}\|u\| = \|x+z\| &< \frac{1}{\tau}\,\|Q_M(x)\| + \varepsilon \\ &= \frac{1}{\tau}\,\|K_{E/M}(Q_M(x))\| + \varepsilon \le \frac{1}{\tau}\,\|\psi\| + \varepsilon\end{aligned}$$

(by (6)) and $Q_M(u) = Q_M(x+z) = Q_M(x)$ (since $z \in M$), hence

$$Q_{M^{\perp\perp}}(\psi) = K_{E/M}(Q_M(x)) = K_{E/M}(Q_M(u)) = Q_{M^{\perp\perp}}\,K_E(u),$$

in other words,

$$f(u) = \langle f, K_E\,u\rangle = \langle f, \psi\rangle \quad (\text{for all } f \in N). \qquad \square$$

**Remark.** It is interesting to apply Theorem 3.5 to the case when $E$ is a B-lattice and $N$ is a finite-dimensional sublattice of $E'$ such that the canonical injection is almost interval preserving (i.e., for any $u \in N_+$, $[0, u]$ (w.r.t. $(N, N_+)$) is dense in $[0, u]$ (w.r.t. $(E, E_+)$)). In this case, $N^\perp$ is an ideal of $E''$ (by [11, (II.2.20), p. 74]) and so is $M$, thus, $Q_M(E_+)$ is closed and $Q_M$ is a surjective Reisz homomorphism (which implies that it is a 1-half-decomposable surjection).

Using a result of V.A. Geiler and I.I. Chuchaev [3], one can easily find another version of Helly's selection theorem of ordered type with slightly weaker conclusion provided by a simpler assumption. Here, we would like to thank the referee for suggesting us to compare the original results of Theorem 3.5 to the results in [1] and [3].

**Theorem 3.6.** *Let $(E, E_+)$ be an ordered normed space and $N$ be a finite dimensional subspace of $E'$. Then for any $\varepsilon > 0$ and $\psi \in E''_+$, there exist $w \in E_+$ and $z \in \varepsilon U_E$ with $\|w\| \leq \|\psi\|$ such that*

$$\langle w + z, f \rangle = \langle f, \psi \rangle \quad (\text{for all } f \in N).$$

**Proof.** Let $C = \|\psi\| \, U_E \cap E_+$ and $H$ be the 1-dimensional subspace of $E''$ spanned by $\psi$. Then $C$ is a convex subset of $E$ containing 0 and the bipolar, $C^{00}$, of $C$ equal to $\|\psi\| \, U_{E''} \cap E''_+$. By Theorem 1.2 (more precisely, parts (i) and (ii) of Theorem 1.2) of [3], for any $\eta > 0$, there exist $u \in (1 + \eta) \, C$ and $v \in (\varepsilon/2) \, U_E$ such that

$$\langle u + v, f \rangle = \langle f, \psi \rangle \quad (\text{for all } f \in F);$$

in particular, for this $\eta = \varepsilon/(2 \|\psi\|)$, one can take $w = u/(1 + \eta)$ and $z = v + u - w$, then the result follows. □

## 4. THE PRINCIPLE OF LOCAL REFLEXIVITY OF ORDERED TYPE

The famous principle of local reflexivity, (found by Lindenstrauss, J. and H.P. Rotsenthal [5] and strengthened by Johnson, W.B., H.P. Rosenthal and M. Zipping, [4]) says that if $E$, $F$ and $D$ are Banach spaces such that $E$ is finite dimensional and $D$ is a finite dimensional subspace of $F'$, then for any $T \in \mathcal{L}(E, F'')$ and any $\varepsilon > 0$, there exists an $S \in \mathcal{L}(E, F)$ with $\|S\| \leq \|T\| + \varepsilon$ such that

$$\langle y', Tx \rangle = \langle y', K_F \, Sx \rangle \quad (\text{for all } x \in E \text{ and } y' \in D).$$

Moreover,

$$K_F \, Sx = Tx \quad (\text{for all } x \in E \text{ with } Tx \in K_F(F)).$$

It is well-known that this principle has many important applications (see [8, (28.1) p. 383]) and that the simple proof is given by Dean, D.W. [2] who observes that local reflexivity theorem (for Banach spaces) can be derived from one-dimensional ones. In this section, we will borrow Dean's idea to deduce this famous theorem for ordered type (Theorem 4.7). Our proof of Theorem 4.7 rests on the following lemmas which examine the order structure of $\mathcal{F}(E, F)$ and its dual and bidual, and are also interesting themselves (in particular, Lemmas 4.4 and 4.6).

Let $(E, E_+, \|\cdot\|)$ and $(F, F_+, \|\cdot\|)$ be ordered B-spaces and let $\mathcal{F}(E, F)$ be the vector space of all all finite operators from $(E, \|\cdot\|)$ into $(F, \|\cdot\|)$. We always identify $\mathcal{F}(E, F)$ with the algebraic tensor product $E' \otimes F$. Denote by $C_\pi(E' \otimes F)$ the projective cone in $E' \otimes F$, i.e.,

$$C_\pi(E' \otimes F) = \operatorname{co}(E'_+ \otimes F_+) = \left\{ \sum_{i=1}^{n} u'_i \otimes v_i : u'_i \in E'_+ \text{ and } v_i \in F_+ \right\};$$

and by $C_i(E' \otimes F)$ the biprojective cone in $E' \otimes F$, i.e.,

$$C_i(E' \otimes F) = \left\{ \sum_{i=1}^{n} x_i' \otimes y_i : \sum_{i=1}^{n} \langle x_i', \psi'' \rangle \langle y_i, v' \rangle \geq 0 \ (\psi'' \in E_+'', v' \in F_+') \right\}.$$

It is easily seen that $C_\pi(E' \otimes F) \subset C_i(E' \otimes F)$ and that

$$\text{(4.A)} \quad \begin{cases} C_i(E' \otimes F) \\ \quad = \left\{ \sum_{i=1}^{n} x_i' \otimes y_i : \sum_{i=1}^{n} \langle u, x_i' \rangle \langle y_i, v' \rangle \geq 0 \ (u \in E_+, v' \in F_+') \right\}. \end{cases}$$

(since $E_+''$ is the $\sigma(E'', E')$-closure of $K_E\, E_+$). If $F_+$ is closed, then

$$\text{(4.B)} \qquad C_i(E' \otimes F) = \mathcal{F}_+(E, F).$$

Moreover, if $E$ is a finite-dimensional Banach lattice, then we have the following:

**Lemma 4.1.** *Let $E$ be an $n$-dimensional B-lattice. For any ordered normed space $(G, G_+)$, one has $\mathcal{F}_+(E, G) = C_\pi(E' \otimes G)$ and $\mathcal{F}_+(G, E) = C_\pi(G' \otimes E)$.*

**Proof.** It is well-known (see [9, Ex. (V.21(a)), p. 254]) that $E$ is a norm and lattice isomorphic to $\mathbb{R}^n$, ordered in the usual way and normed so that $\|e_i\| = 1$, where $e_i = [\delta_{ij}]_{1 \leq j \leq n}$ $(i = 1, \ldots, n)$, under $\psi$. Then $u_i = \psi^{-1}(e_i) \in E_+$ $(i = 1, \ldots, n)$ forms a basis in $E$ such that

$$\text{(1)} \qquad \sum_{i=1}^{n} \alpha_i u_i \in E_+ \quad \text{if and only if } \alpha_i \geq 0 \ (i = 1, \ldots, n);$$

moreover, (1) shows that any dual basis of $\{u_1, \ldots, u_n\}$ must be positive, hence we assume that $\{f_1, \ldots, f_n\}$ is a positive dual basis of $\{u_1, \ldots, u_n\}$ (i.e., $f_i \in E_+'$).

The result now is clear since any $T \in \mathcal{F}(E, G)$ can be represented as

$$T = \sum_{i=1}^{n} f_i \otimes (Tu_i)$$

and any $S \in \mathcal{F}(G, E)$ can be represented as

$$S = \sum_{i=1}^{n} (S'f_i) \otimes u_i. \qquad \square$$

Denote by $\|\cdot\|_{(\pi)}$ the finite nuclear norm (or $\pi$-norm) on $\mathcal{F}(E, F)$ (for definition, see [8, (6.8.1)]). It is well-known that

$$\text{(3)} \qquad (\mathcal{F}(E, F), C_\pi(E' \otimes F), \|\cdot\|_{(\pi)})' \overset{(0)}{\equiv} (\mathcal{L}(F, E''), \mathcal{L}_+(F, E''), \|\cdot\|)$$

(order isometric), under the map $T \to \langle \cdot, T \rangle_{\mathrm{tr}}$ (where $T \in \mathcal{L}(F, E'')$), defined by

$$(4) \qquad \begin{cases} \langle R, T \rangle_{\mathrm{tr}} = \mathrm{trace}(TR) = \sum_{i=1}^{n} \langle x_i', Ty_i \rangle \\ \qquad\qquad \left(\text{for any } R = \sum_{i=1}^{n} x_i' \otimes y_i \in \mathcal{F}(E, F)\right), \end{cases}$$

so that $\langle \mathcal{F}(E, F), \mathcal{L}(F, E'') \rangle$ is a dual pair and

$$(5) \qquad \mathcal{L}_{+}(F, E'') = -(C_{\pi}(E' \otimes F))^{0} = (C_{\pi}(E' \otimes F))' \quad \text{(the dual cone)}.$$

Denote by $\mathcal{J}(E, F)$ the vector space of all integral operators from $E$ into $F$ (for definition, see [8, (6.4.1)]), and by $\mathbb{I}(\cdot)$ the integral norm on $\mathcal{J}(E, F)$. It is well-known (see [10, (IV.5.9)] or [13, (4.1.3)]) that

$$(6) \qquad (\mathcal{F}(E, F), \|\cdot\|)' \equiv (\mathcal{J}(F, E''), \mathbb{I}(\cdot)) \quad \text{(isometric)}$$

under the trace duality defined by (4).

Suppose now that $E$ is finite-dimensional (denoted by $\dim E < \infty$). Then

$$\mathbb{I}(\cdot) = \|\cdot\|_{(\pi)} \quad \text{on } \mathcal{F}(E, F)$$

(see [8, (6.8.3)]). It then follows from (6) and (3) that

$$(7) \qquad \begin{cases} (\mathcal{L}(E, F), \|\cdot\|)'' \equiv (\mathcal{F}(E, F), \|\cdot\|)'' \equiv (\mathcal{J}(F, E), \mathbb{I}(\cdot))' \\ \qquad\qquad \equiv (\mathcal{F}(F, E), \|\cdot\|_{(\pi)})' \equiv (\mathcal{L}(E, F''), \|\cdot\|) \end{cases}$$

under the trace duality defined similarly by (4).

The dual cone and bidual cone of $\mathcal{L}_{+}(E, F)$ can be calculated as follows:

**Lemma 4.2.** *Let $E$ be a finite dimensional B-lattice and $(F, F_{+}, \|\cdot\|)$ be an ordered B-space with closed cone $F_{+}$. Then*

$$(8) \qquad (\mathcal{L}(E, F), \mathcal{L}_{+}(E, F), \|\cdot\|)' \overset{(0)}{\equiv} (\mathcal{F}(F, E), \overline{C_{\pi}(F' \otimes E)}, \|\cdot\|_{(\pi)})$$

*(order isometric) under the map $S \to \langle \cdot, S \rangle_{\mathrm{tr}}$ (where $S \in \mathcal{F}(F, E)$), defined by*

$$(9) \qquad \begin{cases} \langle R, TS \rangle_{\mathrm{tr}} = \mathrm{trace}(SR) = \sum_{i=1}^{n} \langle Sy_i, x_i' \rangle \\ \qquad\qquad \left(\textit{for any } R = \sum_{i=1}^{n} x_i' \otimes y_i \in \mathcal{F}(E, F)\right), \end{cases}$$

*(where the 'bar' is the $\sigma(\mathcal{F}(F, E), \mathcal{F}(E, F))$-closure), and*

$$(10) \qquad (\mathcal{L}(E, F), \mathcal{L}_{+}(E, F), \|\cdot\|)'' \overset{(0)}{\equiv} (\mathcal{L}(E, F''), \mathcal{L}_{+}(E, F''), \|\cdot\|).$$

*In other words,*

$$(11) \qquad \overline{C_{\pi}(F' \otimes E)} = -(\mathcal{F}_{+}(E, F))^{0} = \mathcal{F}_{+}(E, F)' \quad (\textit{the dual cone})$$

*and*

$$(12) \qquad \mathcal{L}_{+}(E, F'') = (\mathcal{L}_{+}(E, F))'' \quad (\textit{the bidual cone}).$$

**Proof.** It has been observed (see (6)) that $\mathcal{F}(F,E) = \mathcal{J}(F,E'')$ can be identified with the Banach dual of $(\mathcal{F}(E,F), \|\cdot\|)$ (since $\dim E < \infty$); in other words, $\langle E' \otimes F, F' \otimes E\rangle$ is a dual pair (under the trace duality (9)). It then follows that $C_i(E' \otimes F) = -(C_\pi(F' \otimes E))^0$ (with respect to $\langle E' \otimes F, F' \otimes E\rangle$), and hence from the bipolar theorem that

$$\overline{C_\pi(F' \otimes E)} = -(C_i(E' \otimes F))^0 = -(\mathcal{F}_+(E,F))^0,$$

thus (11) holds, consequently (8) is true.

Moreover, by Lemma 4.1 and the fact that $\mathcal{F}_+(F,E) = (C_\pi(E' \otimes F))^0$ is $\sigma(\mathcal{F}(F,E), \mathcal{F}(E,F))$-closed, we have

$$C_\pi(F' \otimes E) = \overline{C_\pi(F' \otimes E)} = \overline{\mathcal{F}_+(F,E)} = \mathcal{F}_+(F,E).$$

Finally, we get from (3) and the above that

$$(13) \qquad (\mathcal{F}(F,E), \overline{C_\pi(F' \otimes E)}, \|\cdot\|_{(\pi)})' \overset{(0)}{\equiv} (\mathcal{L}(E,F''), \mathcal{L}_+(E,F''), \|\cdot\|),$$

so that $\langle \mathcal{F}(F,E), \mathcal{L}(E,F'')\rangle$ is a dual pair and

$$(14) \qquad \mathcal{L}_+(E,F'') = -(\overline{C_\pi(F' \otimes E)})^0 = -(C_\pi(F' \otimes E))^0 \quad \text{(the dual cone)}$$

We then conclude from (11), (13) and (14) that (10) and (12) hold. □

**Remark.** It is easily seen from this proof that $C_\pi(F' \otimes E) = \overline{C_\pi(F' \otimes E)}$.

Let $E$ be an $n$-dimensional B-lattice, let $l_1^n(\mathbb{R}) = (\mathbb{R}^n, \|\cdot\|_{l_1})$, and let us write

$$d(E) = \inf\{\|\psi\|\,\|\psi^{-1}\| : \psi : E \to l_1^n(\mathbb{R}) \text{ is topological and lattice isomorphic}\}.$$

**Lemma 4.3.** *Let $(F, F_+, \|\cdot\|)$ be an ordered normed space, let $N$ be a closed vector subspace of $F$ and let $Q_N : F \to F/N$ be the quotient map. If $Q_N$ is an $\tau$-half-decomposable surjection* (for some $\tau > 0$), *then for any $n$-dimensional B-lattice $E$, the map $Q_{(l)} : \mathcal{L}(E,F) \to \mathcal{L}(E,F/N)$, defined by*

$$Q_{(l)}(T) = Q_N T \quad (\textit{for all } T \in \mathcal{L}(E,F)),$$

*is an $\tau/(d(E)+\varepsilon))$-half-decomposable surjection for any $\varepsilon > 0$.*

**Proof.** It is required to show that

$$(1) \qquad \frac{\tau}{d(E)+\varepsilon}\,(U_{\mathcal{L}(E,F/N)} \cap \mathcal{L}_+(E,F/N)) \subset Q_{(l)}(U_{\mathcal{L}(E,F)} \cap \mathcal{L}_+(E,F)).$$

For any $\varepsilon > 0$, there exists a topological and lattice isomorphism $\psi : E \to l_1^n(\mathbb{R})$ such that

$$\|\psi\|\,\|\psi^{-1}\| < d(E) + \varepsilon.$$

Now let $S_0 \in \tau/(d(E)+\varepsilon)\,(U_{\mathcal{L}(E,F/N)} \cap \mathcal{L}_+(E,F/N))$. Then

$$S_0\,\psi^{-1} \in \mathcal{L}_+(l_1^n(\mathbb{R}), F/N),$$

hence there exists (see Proposition 2.6(b)) an $\tilde{S} \in \mathcal{L}_+(l_1^n(\mathbb{R}), \mathbb{F})$, such that

$$S_0\psi^{-1} = Q_N\,\tilde{S} \quad \text{and} \quad \|\tilde{S}\| \le \frac{1}{\tau}\,\|S_0\,\psi^{-1}\|.$$

Let $S = \tilde{S}\psi$. Then $S \in \mathcal{L}_+(E,F)$ is such that

(2) $\qquad Q_N\,S = Q_N\,\tilde{S}\psi = S_0\,\psi^{-1}\psi = S_0$

and

$$\|S\| \le \|\tilde{S}\|\,\|\psi\| \le \frac{1}{\tau}\,\|S_0\|\,\|\psi^{-1}\|\,\|\psi\| \le \frac{1}{\tau}\,\|S_0\|(d(E)+\varepsilon)$$
$$\le \frac{1}{\tau}\,(d(E)+\varepsilon)\,\frac{\tau}{d(E)+\varepsilon} = 1,$$

thus (1) holds by (2). $\square$

**Lemma 4.4.** *Let $E$ be an $n$-dimensional B-lattice, let $(F,F_+)$ be an ordered B-space with closed cone $F_+$, and let $D$ be a finite-dimensional subspace of $F'$. If $D$ is a $\tau$-half-full subspace of $F'$.* (i.e., $J_D : D \to F'$ is a $\tau$-half-full injection) *for some $\tau > 0$, then the subspace $D \otimes E$, defined by*

(1) $$D \otimes E = \left\{ \sum_{i=1}^{n} d_i' \otimes x_i : d_i' \in D \text{ and } x_i \in E \right\},$$

*is a finite-dimensional $\tau/(d(E)+\varepsilon)$-half-full subspace of $(\mathcal{F}(F,E), \overline{C_\pi(F' \otimes E)}, \|\cdot\|_{(\pi)})$, where the* 'bar' *is the $\sigma(\mathcal{F}(F,E),\mathcal{F}(E,F))$-closure of the projective cone $C_\pi(F',E)$* (see Lemma 4.2).

**Proof.** It is clear that $\dim(D \otimes E) < \infty$. Let $p$ be the restriction of $\|\cdot\|_{(\pi)}$ on $D \otimes E$, let

$$C = \overline{C_\pi(F' \otimes E)} \cap (D \otimes E) \quad \text{(the relative cone)},$$

and let $J : (D \otimes E, C, p) \to (\mathcal{F}(F,E), \overline{C_\pi(F' \otimes E)}, \|\cdot\|_{(\pi)})$ be the canonical injection. In order to verify the result, it has to show (by Theorem 2.4 and the fact that $j_{hf}(T) = j_{hf}^{(a)}(T)$) that the dual operator

(2) $\qquad J' : (\mathcal{F}(F,E), \overline{C_\pi(F' \otimes E)}, \|\cdot\|_{(\pi)})' \to (D \otimes E, C, p)'$

is a $\tau/(d(E)+\varepsilon)$-half-decomposable surjection. To this end, we first calculate these two ordered Banach dual spaces. It is well-known that

(3) $\qquad (\mathcal{F}(F,E), C_\pi(F' \otimes E), \|\cdot\|_{(\pi)})' \overset{(0)}{\equiv} (\mathcal{L}(E,F''), \mathcal{L}_+(E,F''), \|\cdot\|)$

(order isometric) under the trace duality

(4) $\qquad \langle L, T\rangle_{\text{tr}} = \text{trace}(TL) \quad \left(T \in \mathcal{L}(E,F'') \text{ and } L = \sum_{j=1}^{m} y_j' \otimes x_j \in \mathcal{F}(F,E)\right).$

To calculate the ordered Banach dual $(D \otimes E, C, p)'$, we first observe the following facts. If $N = D^{\top}$ (the annihilator of $D$ taken in $F$), then $D \equiv (F/N)'$ (isometric), hence we identify the tensor product $D \otimes E$ with $\mathcal{F}(F/N, E)$. Let $q_{(\pi)}$ be the finite nuclear norm on $(F/N)' \otimes E$, i.e.,

$$q_{(\pi)}(z) = \inf\left\{ \sum_{i=1}^{n} \|d_i'\|\,\|x_i\| : z = \sum_{i=1}^{n} d_i' \otimes x_i,\ d_i' \in D \text{ and } x_i \in E \right\},$$

and let $C_\pi((F/N)' \otimes E)$ be the projective cone in $(F/N)' \otimes E$. Then

$$(5)\qquad \begin{cases} (\mathcal{F}(F/N, E), C_\pi((F/N)' \otimes E), q_{(\pi)})' \\ \quad \overset{(0)}{\equiv} (\mathcal{L}(E, F''/N^{\perp\perp}), \mathcal{L}_+(E, F''/N^{\perp\perp}), \|\cdot\|) \end{cases}$$

(on account of $(F/N)'' \equiv (D, \|\cdot\|)' \equiv F''/N^{\perp\perp}$), hence

$$(6)\qquad (D \otimes E, p)' \equiv (\mathcal{F}(F/N, E), q_{(\pi)})' \equiv (\mathcal{L}(E, F''/N^{\perp\perp}), \|\cdot\|)$$

(since $p$ and $q_{(\pi)}$ are equivalent on $D \otimes E$ with $\dim(D \otimes E) < \infty$). On the other hand, it is clear that $C_\pi((F/N)' \otimes E) \subset C \subset \mathcal{F}_+(F/N, E)$, it then follows from (6) that

$$\begin{aligned} C_\pi(E' \otimes F/N) &\subseteq -(\mathcal{F}_+(F/N, E))^0 \subset -C^0 \\ &\subset -(C_\pi((F/N)' \otimes E))^0 = \mathcal{F}_+(E, F''/N^{\perp\perp}), \end{aligned}$$

and hence from Lemma 4.1 and $F/N$ is finite dimensional that

$$(7)\qquad C_\pi(E' \otimes F/N) = -C^0 = \mathcal{L}_+(E, F''/N^{\perp\perp})$$

(i.e., $\mathcal{L}_+(E, F''/N^{\perp\perp})$ is the dual cone of $C$). Combining (5), (6) and (7), we obtain

$$(8)\qquad \begin{cases} (D \otimes E, C, p)' \overset{(0)}{\equiv} (\mathcal{F}(F/N, E), C_\pi((F/N)' \otimes E), q_{(\pi)})' \\ \qquad\qquad\qquad \overset{(0)}{\equiv} (\mathcal{L}(E, F''/N^{\perp\perp}), \mathcal{L}_+(E, F''/N^{\perp\perp}), \|\cdot\|) \end{cases}$$

under the trace duality

$$(9)\qquad \langle R, S\rangle_{\mathrm{tr}} = \operatorname{trace}(SR) \quad \left( S \in \mathcal{L}(E, F''/N^{\perp\perp}),\ R = \sum_{i=1}^{n} d_i' \otimes x_i \in D \otimes E \right).$$

According to (3) and (8), now we have to show that

$$(10)\qquad \begin{cases} J' : (\mathcal{L}(E, F''), \mathcal{L}_+(E, F''), \|\cdot\|) \\ \qquad \to (\mathcal{L}(E, F''/N^{\perp\perp}), \mathcal{L}_+(E, F''/N^{\perp\perp})\|\cdot\|) \end{cases}$$

is a $\tau/(d(E) + \varepsilon)$-half-decomposable surjection.

In fact, as $J_D : (D, D \cap F_+') \to (F', F_+')$ is a $\tau$-half-full injection, $J_D' : (F'', F_+'') \to (D, D \cap F_+')'$ is a $\tau$-half-decomposable surjection (by Theorem 2.4) and $(D, D \cap F_+')' \overset{(0)}{\equiv} (F''/N^{\perp\perp}, Q_{N^{\perp\perp}}(F_+''))$ (by Lemma 3.2), thus $J_D'$ can be identified with the quotient map $Q_{N^{\perp\perp}} : F'' \to F''/N^{\perp\perp}$, which is a $\tau$-half-decomposable surjection. By Lemma 4.3, the map $Q_{(l)} : \mathcal{L}(E, F'') \to \mathcal{L}(E, F''/N^{\perp\perp})$, defined by

$$(11)\qquad Q_{(l)}(T) = Q_{N^{\perp\perp}} \circ T \quad (\text{for all } T \in \mathcal{L}(E, F'')),$$

is a $\tau/(d(E)+\varepsilon)$-half-decomposable surjection for any $\varepsilon > 0$.

By Theorem 2.4, we complete the proof by showing that

$$(12) \qquad J' = Q_{(l)}$$

(on account of $J_D : (F/N)' \to F'$). Indeed, let $T \in \mathcal{L}(E, F'')$. Then $J'(T) \in \mathcal{L}(E, F''/_{N^{\perp\perp}}) \equiv (\mathcal{F}(F/N, E), q_{(\pi)})'$ (by (8)), hence we have for any $R = \sum_{i=1}^{n} d_i' \otimes x_i \in \mathcal{F}(F/N, E)$, that (see (9))

$$(13) \qquad \begin{cases} \langle R, J'(T)\rangle_{\mathrm{tr}} = \langle JR, T\rangle_{\mathrm{tr}} = \sum\limits_{i=1}^{n} \langle d_i', Tx_i\rangle = \sum\limits_{i=1}^{n} \langle J_D d_i', Tx_i\rangle \\ \qquad = \sum\limits_{i=1}^{n} \langle d_i', J_D' Tx_i\rangle = \sum\limits_{i=1}^{n} \langle d_i', Q_{N^{\perp\perp}} Tx_i\rangle \end{cases}$$

(on account of $Q_{N^{\perp\perp}} = J_D'$). On the other hand, for any $T \in \mathcal{L}(E, F'')$, we have $Q_{(l)}(T) = Q_{N^{\perp\perp}} T \in \mathcal{L}(E, F''/_{N^{\perp\perp}}) = (\mathcal{F}(F/N, E), q_{(\pi)})'$ (by (8)), hence we obtain for any $R = \sum_{i=1}^{n} d_i' \otimes x_i \in \mathcal{F}(F/N, E)$ that

$$Q_{N^{\perp\perp}} TR = \sum_{i=1}^{n} (Q_{N^{\perp\perp}} Tx_i) \otimes d_i',$$

so that

$$(14) \qquad \langle R, Q_{N^{\perp\perp}} T\rangle_{\mathrm{tr}} = \sum_{i=1}^{n} \langle d_i', Q_{N^{\perp\perp}} Tx_i\rangle.$$

Combining (13) and (14), we obtain the requirement. □

To verify the dual result of Lemma 4.4, we need the following:

**Lemma 4.5.** *Let $E$, $G$ and $N$ be ordered Banach spaces such that $G' \overset{(0)}{\equiv} N$* (metric and order isomorphic). *If there is an order and topological surjection $Q : E \to G$ and an order and metric injection $J : N \to E'$ such that $Q' = J$, then*

$$G \overset{(0)}{\equiv} E/[J(N)]^{\top} \quad (\textit{metric and order isomorphic}).$$

**Proof.** Let $Q_K$ be the canonical quotient map from $E$ to $E/\mathrm{Ker}\, Q$, and let $\Psi$ be the induced map from $E/\mathrm{Ker}\, Q$ to $G$ such that $\Psi \circ Q_K = Q$. Since $Q$ is an order and topological surjection, $G$ is order and topologically isomorphic to $E/\mathrm{Ker}\, Q$ under $\Psi$. Since $\mathrm{Ker}\, Q = [Q'(N)]^0 = [J(N)]^{\top}$, it has to show that $\Psi$ is isometric.

In fact, as $J$ is a metric isomorphism and $N$ is norm complete, it follows that

$$(E/\mathrm{Ker}\, Q)' \overset{(0)}{\equiv} \overline{J(N)} \overset{(0)}{\equiv} J(N) \overset{(0)}{\equiv} N \quad \text{(metric and order isomorphic).}$$

For any $\hat{x} \in E/\mathrm{Ker}\, Q$, there exists a $x \in E$ such that $Q_K(x) = \hat{x}$, hence

$$\begin{aligned} \|\Psi(\hat{x})\| &= \sup\{|f(\Psi(\hat{x}))| : f \in U_N\} = \sup\{|f(Q(x))| : f \in U_N\} \\ &= \sup\{|[Jf](x)| : f \in U_N\} = \sup\{|[Jf](\hat{x})| : f \in U_N\} \end{aligned}$$

(the last equality following from the isometric isomorphism between $J(N)$ and $(E/\mathrm{Ker}\, Q)'$). Hence, $\|\Psi(\hat{x})\| = \|\hat{x}\|$. □

**Remark.** The above proof actually gives the following fact: if $Q$ is an order and topological surjection from $E$ to $G$ such that $G'$ is order and metric isomorphic to $(E/\mathrm{Ker}\, Q)'$, then $G \overset{(0)}{\equiv} E/\mathrm{Ker}\, Q$ (metric and order isomorphic) under the induced map $\hat{Q}$ from $E/\mathrm{Ker}\, Q$ to $G$.

**Lemma 4.6.** *Let $E$ be an $n$-dimensional B-lattice, let $(F, F_+)$ be an ordered B-space with closed cone $F_+$, let $D$ be a finite-dimensional subspace of $F'$ and let*

$$J : D \otimes E \to (F' \otimes E, \overline{C_\pi(F' \otimes E)}, \|\cdot\|)$$

*be the canonical injection. If $N = D^\top$ (in $F$) is a $\tau$-half co-subspace of $F$* (i.e., $Q_N : F \to F/N$ is a $\tau$-half-decomposable surjection) *for some $\tau > 0$, then the annihilator $(J(D \otimes E))^\top$ in $\mathcal{L}(E, F)$* (i.e., with respect to $\langle E' \otimes F, F' \otimes E\rangle$) *is a $\tau/((d(E) + \varepsilon))$-half co-subspace of $(\mathcal{L}(E, F), \mathcal{L}_+(E, F))$* (for any $\varepsilon > 0$).

**Proof.** By Lemma 4.3, the map $Q_{(l)} : \mathcal{L}(E, F) \to \mathcal{L}(E, F/N)$, defined by

$$Q_{(l)}(T) = Q_N T \quad (\text{for any } T \in \mathcal{L}(E, F)),$$

is a $\tau/(d(E) + \varepsilon)$-half-decomposable surjection for any $\varepsilon > 0$. A similar argument given in the proof of (12) in Lemma 4.4 shows that $J = Q'_{(l)}$. Now if $q_{(\pi)}$ denotes the finite nuclear norm on $(F/N)' \otimes E$, then Lemmas 4.2 and 4.1 show that

$$\begin{aligned}(\mathcal{F}(E, F/N), \mathcal{F}_+(E, F/N), \|\cdot\|)' &\overset{(0)}{\equiv} (\mathcal{F}(F/N, E), \overline{C_\pi((F/N)' \otimes E)}, q_{(\pi)}) \\ &\overset{(0)}{\equiv} (D \otimes E, C, p)\end{aligned}$$

(where $C = \overline{C_\pi(F' \otimes E)} \cap (D \otimes E)$ and $p$ is the restriction of $\|\cdot\|_{(\pi)}$ (defined on $F' \otimes E$) on $D \otimes E$), so that $Q_{(l)}$ is also a metric surjection (see [8, (B.3.8)]). Thus the injection $\hat{Q}_{(l)}$ associated with $Q_{(l)}$ is an order and metric isomorphism from $(\mathcal{L}(E, F)/\mathrm{Ker}\, Q_{(l)}, \hat{\mathcal{L}}_+(E, F))$ onto $(\mathcal{L}(E, F/N), \mathcal{L}_+(E, F/N))$ by Lemma 4.5, consequently the quotient map $Q_q : (\mathcal{L}(E, F), \mathcal{L}_+(E, F)) \to (\mathcal{L}(E, F)/\mathrm{Ker}\, Q_{(l)}, \hat{\mathcal{L}}_+(E, F))$ is a $\tau/((d(E) + \varepsilon))$-half-decomposable surjection. The result then follows from

$$(J(D \otimes E))^\top = (Q'_{(l)}(D \otimes E))^0 = Q_{(l)}^{-1}((D \otimes E)^0) = Q_{(l)}^{-1}(\{0\})$$

(on account of $\langle E' \otimes F/N, (F/N)' \otimes E\rangle$). $\square$

Now we are able to verify a very interesting result, regarded as the principle of local reflexivity of ordered type, as follows:

**Theorem 4.7.** *Let $E$ be an $n$-dimensional B-lattice, let $(F, F_+)$ be an ordered B-space with closed cone $F_+$, let $D$ be a finite-dimensional subspace of $F'$ and $T \in \mathcal{L}_+(E, F'')$.*

(a) (The 1st version) *Suppose that $D$ is a $\tau$-half-full subspace* (i.e., $J_D : D \to F'$ is a $\tau$-half-full injection) *for some $\tau > 0$. Then for any $\varepsilon > 0$, there exist $S \in \mathcal{L}_+(E, F)$ and $R \in \mathcal{L}(E, F)$ with*

$$(1) \qquad \|S\| \leq \frac{d(E) + \varepsilon}{\tau} \|T\| \quad \textit{and} \quad \|R\| \leq \varepsilon$$

*such that*

$$Tx = (K_F(S+R))\,x \quad \textit{on } D \quad (\textit{for all } x \in E),$$

*i.e.*,

(2) $$\langle d', Tx\rangle = \langle (S+R)\,x, d'\rangle \quad (\textit{for all } d' \in D \textit{ and } x \in E).$$

(b) (The 2nd version) *Suppose that the annihilator $N = D^{\top}$* (in $F$) *is a $\tau$-half co-subspace of $F$ for some $\tau > 0$, and that the quotient cone $Q_N(F_+) = \hat{F}_+$ is closed in $F/N$. Then for any $\varepsilon > 0$, there exists an $S \in \mathcal{L}_+(E,F)$ with $\|S\| < (d(E)/\tau)\,\|T\| + \varepsilon$ such that*

(3) $$Tx = (K_F\, S)\,x \quad \textit{on } D \quad (\textit{for all } x \in E),$$

*i.e.*,

(4) $$\langle d', Tx\rangle = \langle Sx, d'\rangle \quad (\textit{for all } d' \in D \textit{ and } x \in E).$$

**Proof.** We employ Theorems 3.5 and 3.6 to verify this result. For this, we first notice from Lemma 4.2 (on account of $\dim E < \infty$) that

(5) $$(\mathcal{L}(E,F), \mathcal{L}_+(E,F), \|\cdot\|)' \overset{(0)}{\equiv} (\mathcal{F}(F,E), \overline{C_\pi(F' \otimes E)}, \|\cdot\|_{(\pi)})$$

(order isometric) under the map $R_0 \to \langle\cdot, R_0\rangle_{\mathrm{tr}}$ (where $R_0 \in \mathcal{F}(F,E)$), defined by

(6) $$\langle L, R_0\rangle_{\mathrm{tr}} = \mathrm{trace}(R_0 L) \quad (\text{for all } L \in \mathcal{L}(E,F) = \mathcal{F}(E,F)),$$

and that

(7) $$\begin{cases} (\mathcal{L}(E,F''), \mathcal{L}_+(E,F''), \|\cdot\|) \overset{(0)}{\equiv} (\mathcal{L}(E,F), \mathcal{L}_+(E,F), \|\cdot\|)'' \\ \qquad\qquad\qquad\qquad\qquad \overset{(0)}{\equiv} (\mathcal{L}(F,E), \overline{C_\pi(F' \otimes E)}, \|\cdot\|_{(\pi)})' \end{cases}$$

under the map $T_0 \to \langle\cdot, T_0\rangle_{\mathrm{tr}}$ (where $T_0 \in \mathcal{L}(E,F'')$), defined by

(8) $$\langle R, T_0\rangle_{\mathrm{tr}} = \mathrm{trace}(T_0\, R) \quad (\text{for all } R \in \mathcal{L}(F,E) = \mathcal{F}(F.E)).$$

It then follows from (7) that $T \in (\mathcal{L}_+(E,F))''$ (the bidual cone of $\mathcal{L}_+(E,F)$).

(a) Suppose now that $D$ is a $\tau$-half-full subspace of $F'$ and $\varepsilon > 0$. Then Lemma 4.4 shows that $D \otimes E$ is a finite-dimensional $\tau/(d(E)+\varepsilon)$-half-full subspace of $(\mathcal{F}(F,E), \overline{C_\pi(F' \otimes E)}, \|\cdot\|_{(\pi)})$. By Theorem 3.6 there exist $S \in \mathcal{L}_+(E,F)$ and $R \in \mathcal{L}(E,F)$ with

$$\|S\| \le \frac{d(E)+\varepsilon}{\tau}\,\|T\| \quad \text{and} \quad \|R\| \le \varepsilon$$

such that

$$T = K_{\mathcal{L}(E,F)}(S+R) \quad \text{on } D \otimes E.$$

As $D \otimes E \subset \mathcal{F}(F,E)$, it follows from (8) and (6) that

$$\begin{aligned}\langle d', Tx\rangle &= \langle d' \otimes x, T\rangle_{\mathrm{tr}} = \langle d' \otimes x, K_{\mathcal{L}(E,F)}(S+R)\rangle_{\mathrm{tr}} = \langle S+R, d' \otimes x\rangle_{\mathrm{tr}} \\ &= \langle (S+R)\,x, d'\rangle \quad (\text{for all } d' \in D \text{ and } x \in E).\end{aligned}$$

(b) It is clear that the closedness of $Q_N(F_+)$ in $F/N$ implies that $\mathcal{L}_+(E, F/N)$ is closed, and hence the quotient cone $\hat{\mathcal{L}}_+(E,F)$ is closed in $\mathcal{L}(E,F)/_{(J(D\otimes E))^0}$ (where $J : D \otimes E \to F' \otimes E$ is the canonical injection). By Lemma 4.6 and Theorem 3.5, there exists an $S \in \mathcal{L}_+(E,F)$ such that it satisfies the requirements in the theorem. $\square$

It is interesting to apply Theorem 4.7(a) to the case that when $F$ is a B-lattice and $D$ is a finite-dimensional sublattice of $F'$.

**Remarks.** (i) If $F$ is assumed to be a Banach lattice and $D$ is a closed lattice ideal in $F'$ ($D^{\top}$ is an $1/(1+\varepsilon)$-half-full co-subspace of $F$ (see Example (a) in §2) such that $F/D^{\top}$ is a Banach lattice), then for any $\varepsilon > 0$, there exists an $S \in \mathcal{L}_+(E,F)$ with $\|S\| \leq d(E)\,\|T\| + \varepsilon$ such that (3) hold.

(ii) Using Theorem 3.6 one can improve part (a) as follows: Let $E$ be an $n$-dimensional B-lattice, let $(F, F_+)$ be an ordered B-space and $D$ a finite-dimensional subspace of $F'$. Then for any $T \in \mathcal{L}_+(E, F'')$ and $\varepsilon > 0$, there exist $S \in \mathcal{L}_+(E,F)$ and $R \in \mathcal{L}(E,F)$ with $\|S\| \leq \|T\|$ and $\|R\| \leq \varepsilon$ such that

$$Tx = (K_F(S+R))\,x \quad \text{on } D \quad (\text{for all } x \in E).$$

In fact, by (5) and (7) (in Theorem 4.7) and the fact that $D \otimes E$ is finite-dimensional subspace of $\mathcal{F}(E,F)$, the result follows.

In [1], Behrends gives a version of local reflexivity theorem (for Banach spaces) which is, in a sense, the most general one. It is well known that the classical local reflexivity theorem (also Behrends' version of local reflexivity theorem) is false if we simply replace Banach space by Banach lattices, finite-dimensional subspace by finite-dimensional sublattice and linear isomorphism by lattice isomorphism (resp. as Behrends says [1, p. 110]). Part (a) of Theorem 4.7 is a version of (some sort of) local reflexivity theorem for ordered type which can be deduced from Theorem 1.2 of [3] (see Remark (ii) of Theorem 4.7). While part (b) of Theorem 4.7 is another version of some sort of local reflexivity theorem for ordered type.

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