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The Probabilistic Method for Problems of Radiative Transfer XIII. Diffusion Matrix

SUEO UENO*

Institute of Astrophysics, University of Kyoto, Kyoto, Japan

Submitted by Richard Bellman

I. INTRODUCTION

In the theory of radiative transfer the equation of transfer formulated from the local viewpoint is expressed in terms of two optical parameters, i.e. an albedo for single scattering and an indicatrix of scattering. These optical parameters are assumed classically to be constant throughout the layer under consideration. In the present paper we restrict our discussion to the exact solutions of stationary transfer problems of radiation.

The Laplace transform method due to Wiener and Hopf [1] and the Chandrasekhar limiting process [2] are considered to be the local approach for solving exactly the equation of transfer. The above two methods have further been extended by Miss Busbridge [3, 4], Huang [5], and Kourganoff [3], respectively.

Reducing the Milne first integral equation to the auxiliary equation by the idea of linear aggregation, Ambarzumian [1] has elegantly developed a new method giving the exact solution of the transfer equation. Then the extension of Ambarzumian's first method has been done in several transfer problems of current interest in astrophysics: line formation in semi-infinite, coherently and non-coherently scattering atmosphere (Busbridge [4]; Ueno [6]), and line formation in a coherently scattering atmosphere of finite thickness (Busbridge [7, 4]).

Furthermore, in 1943 Ambarzumian [8] initiated a new method based on the principle of invariance. The method is considered to be the global approach, because a nonlinear integral equation in ϕ-function can directly be obtained by physical analysis of the condition of transfer without actually solving the Milne first integral equation for the source function on the way. The extension of Ambarzumian's physical method has been made by Miss

* Now at the RAND Corporation, Santa Monica, and Department of Meteorology, University of California, Los Angeles, California.

Busbridge [9], Stibbs [10], and Busbridge and Stibbs [11] in the problems of line formation in the Milne-Eddington model.

Chandrasekhar [2] has extended ingeniously the invariance method to formulate the four principles of invariance in a finite homogeneous medium. The characteristic feature of this method consists in the deduction of the requisite integral equations for the S- and T-functions from the above set of the invariance principles in conjunction with the equation of transfer. These functions summarizing the global optical characters of the medium represent the diffuse reflection and transmission of radiations. Under such global formulation a number of long outstanding transfer problems were resolved. In more complex problems for finite layers, his generalization is powerful. The application of his invariance method has been made to various kinds of transfer problems; line formation in planetary atmosphere (cf. R.T.), line formation in Schuster model (cf. R.T.), emitting atmosphere (Horak [12], Horak and Lundquist [13]), and molecular absorbing atmosphere (King [14]).

Introducing first into the transfer theory the probability of a photon emergence from the boundary, Sobolev has dealt with various kinds of transfer problems: pure scattering [5], line formation with coherent and non-coherent scattering in semi-infinite homogeneous medium [5], diffuse reflection in a semi-infinite inhomogeneous medium [16], diffuse reflection and transmission in a finite layer [17], line formation in a finite emitting atmosphere [15] and others [15].

Recently, generalizing the methodology of the global approach, Bellman and Kalaba [18] stated the principle of invariant imbedding. By imbedding the original process within the new family of processes and obtaining an invariant process, the functional equations governing the various processes of the class are formulated.

The principle of invariant imbedding was used in the study of a number of the transport problems of current interest in general media, as well as in that of a variety of mathematical questions concerning the relationship between their procedure and the classical analytical techniques: radiative transfer [18, 19], neutron transport [20], random walk and scattering [21], wave propagation [22], and Stefan-type problem [23].

Making explicit use of the collection of transformation in inhomogeneous one-parameter carrier space, Preisendorfer [24] obtained the invariant imbedding relation for radiative transfer and neutron diffusion. Furthermore, Preisendorfer [25] applied it to derive the functional relation for the reflectance and transmittance operators on a nonseparable, plane-parallel layer of finite thickness, allowing for the time-dependence. It is also shown that the polarity of the reflectance and transmittance operators takes place because of the inhomogeneous optical properties in the layer.

Recently, based on a stochastic model of radiative transfer such that multiple scattering of photon is of Markovian property, the probabilistic method has been applied to the study of various transfer problems: Milne's problem [26], line formation in semi-infinite atmospheres with coherent and noncoherent scattering [27, 28], diffuse reflection and transmission in a finite homogeneous and inhomogeneous layer [29, 30], and a Markovian property of radiative transfer [31].

The characteristic feature of the probabilistic method is that, deriving the stochastic integro-differential equation from the Chapman-Kolmogoroff equation, and using the integral transform of the diffusion equation governing the probability distribution of the photon emergence, the angular distributions of emergent radiations can be obtained without actually solving the equation of transfer.

In the present paper we consider the diffuse reflection and transmission of a parallel beam of radiation by a finite plane-parallel inhomogeneous layer. Using the four integral operators, i.e. a scattering-transmission pair for each of the two boundaries of the layer, we get the diffusion operational matrix of two rows and of two columns.

From the viewpoint of the application of the reflectance and transmittance operators, our procedure seems to be somewhat similar to that used by Preisendorfer for the formulation of invariant imbedding relation. The difference concerning the mathematical manipulation between Preisendorfer's method and ours consists in that in the computation of the functional equations for the S- and T-functions, while Preisendorfer's derivation can be patterned after the classical procedure used by Chandrasekhar for the homogeneous case, our method is based on the formulation of the stochastic integro-differential equation.

On the other hand, from the viewpoint of the semi-groups properties, the above two methods are correlated with each other as follows: while the semi-groups relations of the invariant imbedding method are obtained by setting the outward-directed (or inward-directed) flux at the lower (or upper) boundary to be zero, the treatment of the stochastic equation in the probabilistic method is partly similar to that used recently by Feller [32] on boundaries and lateral conditions for the Kolmogoroff differential equation.

II. The Equation of Transfer

Consider the problem of diffuse reflection and transmission by a plane-parallel inhomogeneous atmosphere of optical thickness τ_1 scattering radiation isotropically.

We assume that a parallel beam of radiation of intrinsic flux F_0 is incident on the upper boundary $\tau = 0$ in the direction $-\mu_0$ $(0 < \mu_0 \leqslant 1)$, and that a parallel beam of radiation of intrinsic flux F_1 is similarly falling on the lower boundary $\tau = \tau_1$ in the direction $+\mu_1$ $(0 < \mu_1 \leqslant 1)$. Then we shall find the emergent intensities diffusely reflected and transmitted from the medium.

Following the notation of Chandrasekhar in R.T., we shall use $I_\nu(0, +\mu)$ $(0 < \mu \leqslant 1)$ and $I_\nu(\tau_1, -\mu)$ $(0 < \mu \leqslant 1)$ to denote respectively the angular distribution of the emergent radiation resulting from the diffuse reflection in the direction $+\mu$, and that resulting from the diffuse transmission in the direction $-\mu$ below the lower boundary. Furthermore, let the intensities of the radiations directed towards the surfaces $\tau = 0$ and $\tau = \tau_1$ at level τ respectively be denoted by $I_\nu(\tau, +\mu)$ $(0 < \mu \leqslant 1)$ and $I_\nu(\tau, -\mu)$ $(0 < \mu \leqslant 1)$. In what follows, because of coherency, we suppressed the subscript ν to the various quantities.

The equation of transfer appropriate to the present case is

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \tfrac{1}{2}\varpi(\tau) \int_{-1}^{+1} I(\tau, \mu')\, d\mu', \tag{2.1}$$

together with the boundary conditions

$$I_{\text{inc}}(0, -\mu) = \frac{F_0}{2}\, \delta(\mu - \mu_0) \qquad (0 < \mu_0 \leqslant 1),$$

$$I_{\text{inc}}(\tau_1, +\mu) = \frac{F_1}{2}\, \delta(\mu - \mu_1) \qquad (0 < \mu_1 \leqslant 1). \tag{2.2}$$

In Eq. (2.2) $I_{\text{inc}}(0, -\mu)$ and $I_{\text{inc}}(\tau_1, +\mu)$ represent respectively the intensity incident on the surface $\tau = 0$ in the direction $-\mu$ and that incident on the surface $\tau = \tau_1$ in the direction $+\mu$, and further δ is the Dirac delta-function.

Recalling Eq. (2.1), the equation of transfer in the diffuse radiation field is written in the form (cf. Busbridge [7])

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \mathfrak{I}(\tau), \tag{2.3}$$

where the source function $\mathfrak{I}(\tau)$ is

$$\mathfrak{I}(\tau) = \tfrac{1}{2}\varpi(\tau) \int_{-1}^{+1} I(\tau, \mu')\, d\mu' + \tfrac{1}{4}\varpi(\tau) \left\{ F_0 \exp\left(-\frac{\tau}{\mu_0}\right) + F_1 \exp\left[-\frac{(\tau_1 - \tau)}{\mu_1}\right] \right\}. \tag{2.4}$$

Eq. (2.3) should be solved subject to the boundary conditions

$$I(0, -\mu) = 0 \qquad \text{and} \qquad I(\tau_1, +\mu) = 0 \qquad (0 < \mu \leqslant 1). \tag{2.5}$$

Then the Milne first integral equation appropriate to the present case is

$$\mathfrak{I}(\tau) = \varpi(\tau)\bar{\Lambda}_\tau\{\mathfrak{I}(t)\} + \tfrac{1}{4}\varpi(\tau)\left\{F_0 \exp\left(-\frac{\tau}{\mu_0}\right) + F_1 \exp\left[-\frac{(\tau_1 - \tau)}{\mu_1}\right]\right\} \tag{2.6}$$

where the truncated Hopf operator is

$$\bar{\Lambda}_\tau\{f(t)\} = \tfrac{1}{2}\int_0^{\tau_1} f(t)\, E_1(|\, t - \tau\,|)\, dt. \tag{2.7}$$

In Eq. (2.7) $E_1(\tau)$ is the first exponential integral

$$E_1(\tau) = \int_0^1 \exp\left(-\frac{\tau}{\mu}\right)\frac{d\mu}{\mu}. \tag{2.8}$$

Allowing for the linear property of operator $\bar{\Lambda}$, we have

$$\mathfrak{I}(\tau) = \mathfrak{I}_0(\tau) + \mathfrak{I}_1(\tau), \tag{2.9}$$

where

$$\mathfrak{I}_0(\tau) = \varpi(\tau)\bar{\Lambda}_\tau\{\mathfrak{I}_0(t)\} + \tfrac{1}{4}\varpi(\tau)F_0 \exp\left(-\frac{\tau}{\mu_0}\right), \tag{2.10}$$

$$\mathfrak{I}_1(\tau) = \varpi(\tau)\bar{\Lambda}_\tau\{\mathfrak{I}_1(t)\} + \tfrac{1}{4}\varpi(\tau)F_1 \exp\left[-\frac{(\tau_1 - \tau)}{\mu_1}\right]. \tag{2.11}$$

Then it is of interest to remark that, in a manner similar to that used in the homogeneous case (cf. [7]), the solution of the composite problem can be obtained by the addition of the solutions corresponding to the standard problems in the arbitrary stratification.

III. The Stochastic Integro-Differential Equation

Consider the probability of the photon emergence expressed in terms of a one-dimensional parameter τ and a continuous random variable $\mu(\tau)$. Let $p(\mu; \tau, \tau_1)$ $(0 < \mu \leqslant 1)$ be the probability that a photon absorbed at the level τ will reappear in the direction $+\mu$ in the radiation emerging from the surface $\tau = 0$ after one or more scattering processes. Mathematically, $p(\mu; \tau, \tau_1)d\mu$ is the probability of finding the μ-state in the range $(\mu, \mu + d\mu)$ at the optical depth τ. Similarly, let $p^*(\mu; \tau, \tau_1)$ $(0 < \mu \leqslant 1)$ be the probability for the escape of a photon produced at the level τ from the medium through the surface $\tau = \tau_1$ in the direction $-\mu$. Hence $p(\mu; \tau, \tau_1)$ and

$p^*(\mu;\,\tau,\,\tau_1)$ can respectively be interpreted as the emergence probabilities of a photon towards the upper and the lower boundaries, i.e., $\tau = 0$ and $\tau = \tau_1$. They depend upon the optical properties of the medium, but not upon the emission sources. Therefore, with the aid of the probabilities, the emergent intensities can readily be obtained for any energy source acting in the medium.

From the probabilistic point of view, it is easily shown that the integration of the sum of the emergence probabilities with respect to μ over (0, 1) should fulfil the following condition (cf. [29])

$$\tfrac{1}{2}\int_0^1 [p(\mu;\,\tau,\,\tau_1) + p^*(\mu;\,\tau,\,\tau_1)]\,d\mu \leqslant 1. \tag{3.1}$$

The equality holds only in the case of the pure scattering, i.e. $\varpi = 1$.

Furthermore, let $p(\mu_2|\mu_1;\,\tau_2,\,\tau_1)d\mu_2$ (where $\tau_1 < \tau_2$) be the transition probability that, given $\mu(\tau_1) = \mu_1$, one finds μ in the range $(\mu_2,\,\mu_2 + d\mu_2)$ at the optical depth τ_2. Then $p(\mu;\,\tau,\,\tau_1)$ corresponds to the transition probability of $\mu(\tau)$ when we do not specify the initial condition.

Assuming that the probability distribution is of Markovian property, then it should satisfy the Chapman-Kolmogoroff equation in somewhat modified form

$$p(\mu;\,\tau_1 - \tau - \Delta\tau,\,\tau_1)$$
$$= \int_0^1 \bar{p}(\mu';\,\tau_1 - \tau - \Delta\tau,\,\tau_1 - \Delta\tau)\,p(\mu \mid \mu';\,\tau_1 - \tau,\,\Delta\tau)\,d\mu', \tag{3.2}$$

for all values of $\Delta\tau$ between zero and τ. In Eq. (3.2) $\bar{p}(\mu;\,\tau_1 - \tau - \Delta\tau,\,\tau_1 - \Delta\tau)$ and $\bar{p}(\mu \mid \mu';\,\tau_1 - \tau,\,\Delta\tau)$ are respectively given by

$$\bar{p}\,(\mu;\,\tau_1 - \tau - \Delta\tau,\,\tau_1 - \Delta\tau) = p(\mu;\,\tau_1 - \tau - \Delta\tau,\,\tau_1 - \Delta\tau)$$
$$+ \Delta\tau\int_0^1 p^*\,(\mu';\,\tau_1 - \tau,\,\tau_1)\,R(\mu \mid \mu')\,d\mu', \tag{3.3}$$

and

$$p(\mu \mid \mu';\,\tau_1 - \tau,\,\Delta\tau) = \delta(\mu - \mu'). \tag{3.4}$$

In Eq. (3.3) $R(\mu \mid \mu')d\mu d\tau$ represents the conditional transition probability that, given μ', one finds μ in the range $(\mu,\,\mu + d\mu)$ through a parametric interval $d\tau$ of the optical depth.

Putting

$$R(\mu \mid \mu') = \frac{1}{2\mu'}\,p(\mu;\,\tau_1,\,\tau_1), \tag{3.5}$$

from Eq. (3.3) we have

$$\frac{\partial p(\mu; \tau_1 - \tau, \tau_1)}{\partial \tau} + \frac{\partial p(\mu; \tau_1 - \tau, \tau_1)}{\partial \tau_1}$$
$$= \tfrac{1}{2} p(\mu; \tau_1, \tau_1) \int_0^1 p^*(\mu'; \tau_1 - \tau, \tau_1) \frac{d\mu'}{\mu'} . \tag{3.6}$$

Similarly, another type of the Chapman-Kolmogoroff equation is expressed in the form

$$p^*(\mu; \tau_1 - \tau - \Delta\tau, \tau_1)$$
$$= \int_0^1 p^*(\mu'; \tau_1 - \tau - \Delta\tau, \tau_1 - \Delta\tau)\, p^*(\mu \mid \mu'; \tau_1 - \tau, \Delta\tau)\, d\mu', \tag{3.7}$$

where $p^*(\mu \mid \mu'; \tau_1 - \tau, \Delta\tau)$ is

$$p^*(\mu \mid \mu'; \tau_1 - \tau, \Delta\tau) = R^*(\mu \mid \mu')\, \Delta\tau + \delta(\mu - \mu') \left\{ 1 - \frac{\Delta\tau}{\mu'} \right\} . \tag{3.8}$$

In Eq. (3.8) the conditional transition probability $R^*(\mu \mid \mu')$ is given by

$$R^*(\mu \mid \mu') = \tfrac{1}{2} \frac{p^*(\mu; \tau_1, \tau_1)}{\mu'} . \tag{3.9}$$

Then, from Eq. (3.7) we get

$$\frac{\partial p^*(\mu; \tau_1 - \tau, \tau_1)}{\partial \tau} + \frac{\partial p^*(\mu; \tau_1 - \tau, \tau_1)}{\partial \tau_1} = -\frac{1}{\mu} p^*(\mu; \tau_1 - \tau, \tau_1)$$
$$+ \tfrac{1}{2} p^*(\mu; \tau_1, \tau_1) \int_0^1 p^*(\mu'; \tau_1 - \tau, \tau_1) \frac{d\mu'}{\mu'} . \tag{3.10}$$

Equations (3.6) and (3.10) are the set of the requisite stochastic integro-differential equations. They are equal to those given in the preceding paper [30].

On the other hand, from the probabilistic viewpoint, it is possible to establish the linear integral equation governing $p(\mu; \tau_1 - \tau, \tau_1)$. Let us remark that the probability distribution $p(\mu; \tau_1 - \tau, \tau_1)$ for the escape of a photon from the medium is composed of two components. The first component without suffering scattering process is equal to $\varpi(\tau_1 - \tau) \exp[-(\tau_1 - \tau)/\mu]$. The second component due to successive scatterings can be obtained in such a way that, multiplying the probability of the reabsorption at the level t of a photon produced at the level $\tau_1 - \tau$ by $p(\mu; \tau_1 - t, \tau_1)$ and integrating with respect to t over $(0, \tau_1)$, we get the emergence probability of a photon from the medium after successive scattering processes (after carrying out the transformation of the argument in $\bar{\Lambda}$-operator). Then the photon diffusion equations for $p(\mu;\tau_1-\tau,\tau_1)$ and $p^*(\mu;\tau_1-\tau,\tau_1)$ are written in the forms

$$p(\mu;\tau_1-\tau,\tau_1)=\varpi(\tau_1-\tau)\bar{\Lambda}_\tau\{p(\mu;\tau_1-t,\tau_1)\}+\varpi(\tau_1-\tau)\exp\left[-\frac{(\tau_1-\tau)}{\mu}\right], \tag{3.11}$$

$$p^*(\mu;\tau_1-\tau,\tau_1)=\varpi(\tau_1-\tau)\bar{\Lambda}_\tau\{p(\mu;\tau_1-t,\tau_1)\}+\varpi(\tau_1-\tau)\exp\left(-\frac{\tau}{\mu}\right). \tag{3.12}$$

Equations (3.11) and (3.12) are similar in form to the Milne first integral equations (2.10) and (2.11).

IV. The S- and T-Functions in Terms of the X- and Y-Functions

We shall use respectively $S(\tau_1;\mu,\mu_0)$ and $T(\tau_1;\mu,\mu_0)$ to denote the upward scattering and the downward transmission functions for the monodirectional illumination of the upper boundary $\tau=0$. Similarly, $S^*(\tau_1;\mu,\mu_1)$ and $T^*(\tau_1;\mu,\mu_1)$ are respectively called the downward scattering and the upward transmission functions for the monodirectional illumination of the lower boundary $\tau=\tau_1$.

Let the upward S- and the downward T-functions be

$$S(\tau_1;\mu_0,\mu)=\int_0^{\tau_1}p(\mu;\tau_1-\tau,\tau_1)\exp\left[-\frac{(\tau_1-\tau)}{\mu_0}\right]d\tau, \tag{4.1}$$

$$T(\tau_1;\mu_0,\mu)=\int_0^{\tau_1}p(\mu;\tau_1-\tau,\tau_1)\exp\left(-\frac{\tau}{\mu_0}\right)d\tau. \tag{4.2}$$

The usage of $p(\mu;\tau_1-\tau,\tau_1)$ in Eq. (4.2) may seem to be contradictory, because in the derivation of $T(\tau_1;\mu_0,\mu)$ directed towards the lower boundary $\tau=\tau_1$ we must utilize the probability current directed towards the surface $\tau=\tau_1$. However, by virtue of the principle of reciprocity to be stated below, the above discrepancy vanishes.

Similarly, the downward scattering and the upward transmission functions are defined as follows:

$$S^*(\tau_1;\mu_1,\mu)=\int_0^{\tau_1}p^*(\mu;\tau_1-\tau,\tau_1)\exp\left(-\frac{\tau}{\mu_1}\right)d\tau, \tag{4.3}$$

$$T^*(\tau_1;\mu_1,\mu)=\int_0^{\tau_1}p^*(\mu;\tau_1-\tau,\tau_1)\exp\left[-\frac{(\tau_1-\tau)}{\mu_1}\right]d\tau. \tag{4.4}$$

Equations (4.1)-(4.4) show the polarity of the S- and T-functions.

On multiplying Eq. (3.11) by $p(\mu_0; \tau_1 - \tau, \tau_1)/\varpi(\tau_1 - \tau)$, and integrating with respect to τ over $(0, \tau_1)$, after some minor arguments, we obtain

$$S(\tau_1; \mu, \mu_0) = S(\tau_1; \mu_0, \mu). \tag{4.5}$$

Similarly, making use of Eqs. (3.12) and (3.11), we get

$$S^*(\tau_1; \mu, \mu_1) = S^*(\tau_1; \mu_1, \mu), \tag{4.6}$$

and

$$T(\tau_1; \mu, \mu_0) = T^*(\tau_1; \mu_0, \mu). \tag{4.7}$$

Equations (4.5)-(4.7) represent the principle of reciprocity. Recalling Eqs. (3.11), (3.12), (4.2), and (4.3), and putting $\tau = 0$ in Eqs. (3.6) and (3.10), we have

$$p(\mu; \tau_1, \tau_1) = \varpi(\tau_1) \exp\left(-\frac{\tau_1}{\mu}\right) + \tfrac{1}{2}\varpi(\tau_1) \int_0^1 T(\tau_1; \mu', \mu) \frac{d\mu'}{\mu'}, \tag{4.8}$$

$$p^*(\mu; \tau_1, \tau_1) = \varpi(\tau_1) + \tfrac{1}{2}\varpi(\tau_1) \int_0^1 S^*(\tau_1; \mu', \mu) \frac{d\mu'}{\mu'}. \tag{4.9}$$

Multiply Eq. (3.6) by $\exp(-\tau/\mu_0)$ and integrate with respect to τ from 0 to τ_1. Then, using Eqs. (4.2) and (4.3), and rearranging a term, we get

$$\frac{1}{\mu_0} T(\tau_1; \mu_0, \mu) + \frac{\partial T(\tau_1; \mu_0, \mu)}{\partial \tau_1}$$
$$= p(\mu; \tau_1, \tau_1) + \tfrac{1}{2} p(\mu; \tau_1, \tau_1) \int_0^1 S^*(\tau_1; \mu_0, \mu') \frac{d\mu'}{\mu'}. \tag{4.10}$$

On multiplying Eq. (3.6) by $\exp[-(\tau_1 - \tau)/\mu_0]$, integrating with respect to τ over $(0, \tau_1)$, and using Eqs. (4.1) and (4.3), we obtain

$$\frac{\partial S(\tau_1; \mu_0, \mu)}{\partial \tau_1}$$
$$= p(\mu; \tau_1, \tau_1) \exp\left(-\frac{\tau_1}{\mu_0}\right) + \tfrac{1}{2} p(\mu; \tau_1, \tau_1) \int_0^1 T^*(\tau_1; \mu_0, \mu') \frac{d\mu'}{\mu'}. \tag{4.11}$$

Similarly, from Eq. (3.10), we get

$$\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) S^*(\tau_1; \mu_1, \mu) + \frac{\partial S^*(\tau_1; \mu_1, \mu)}{\partial \tau_1}$$
$$= p^*(\mu; \tau_1, \tau_1) + \tfrac{1}{2} p^*(\mu; \tau_1, \tau_1) \int_0^1 S^*(\tau_1; \mu_1, \mu') \frac{d\mu'}{\mu'}, \tag{4.12}$$

and

$$\frac{1}{\mu} T^*(\tau_1; \mu_1, \mu) + \frac{\partial T^*(\tau_1; \mu_1, \mu)}{\partial \tau_1} = p^*(\mu; \tau_1, \tau_1) \exp\left(-\frac{\tau_1}{\mu_1}\right)$$

$$+ \tfrac{1}{2} p^*(\mu; \tau_1, \tau_1) \int_0^1 T^*(\tau_1; \mu_1, \mu') \frac{d\mu'}{\mu'} . \tag{4.13}$$

Write

$$X(\mu, \tau_1) = 1 + \tfrac{1}{2} \int_0^1 S(\tau_1; \mu, \mu') \frac{d\mu'}{\mu'} , \tag{4.14}$$

$$Y(\mu, \tau_1) = \exp\left(-\frac{\tau_1}{\mu}\right) + \tfrac{1}{2} \int_0^1 T(\tau_1; \mu, \mu') \frac{d\mu'}{\mu'} , \tag{4.15}$$

$$X^*(\mu, \tau_1) = 1 + \tfrac{1}{2} \int_0^1 S^*(\tau_1; \mu, \mu') \frac{d\mu'}{\mu'} , \tag{4.16}$$

and

$$Y^*(\mu, \tau_1) = \exp\left(-\frac{\tau_1}{\mu}\right) + \tfrac{1}{2} \int_0^1 T^*(\tau_1; \mu, \mu') \frac{d\mu'}{\mu'} . \tag{4.17}$$

The X-, Y-, X^*-, and Y^*-functions defined by $S(\tau_1;\ \mu', \mu)$, $T(\tau_1; \mu', \mu)$, $S^*(\tau_1; \mu', \mu)$ and $T^*(\tau_1; \mu', \mu)$ are respectively denoted by $\bar{X}(\mu, \tau_1)$, $\bar{Y}(\mu, \tau_1)$, $\bar{X}^*(\mu; \tau_1)$ and $\bar{Y}^*(\mu; \tau_1)$.

From the reciprocity principle we have

$$\bar{X}(\mu, \tau_1) = X(\mu, \tau_1), \qquad \bar{Y}(\mu, \tau_1) = Y^*(\mu, \tau_1), \tag{4.18}$$

$$\bar{X}^*(\mu, \tau_1) = X^*(\mu, \tau_1), \qquad \bar{Y}^*(\mu, \tau_1) = Y(\mu, \tau_1). \tag{4.19}$$

With the aid of Eqs. (4.8), (4.9), and (4.14)-(4.17), Eqs. (4.10)-(4.13) become respectively

$$\frac{\partial S(\tau_1; \mu_0, \mu)}{\partial \tau_1} = \varpi(\tau_1)\, \bar{Y}(\mu, \tau_1)\, Y^*(\mu_0, \tau_1), \tag{4.20}$$

$$\frac{1}{\mu_0} T(\tau_1; \mu_0, \mu) + \frac{\partial T(\tau_1; \mu_0, \mu)}{\partial \tau_1} = \varpi(\tau_1)\, \bar{Y}(\mu, \tau_1)\, X^*(\mu_0, \tau_1), \tag{4.21}$$

$$\left(\frac{1}{\mu} + \frac{1}{\mu_1}\right) S^*(\tau_1; \mu_1, \mu) + \frac{\partial S^*(\tau_1; \mu_1, \mu)}{\partial \tau_1} = \varpi(\tau_1)\, \bar{X}^*(\mu, \tau_1)\, X^*(\mu_1, \tau_1), \tag{4.22}$$

$$\frac{1}{\mu} T^*(\tau_1; \mu_1, \mu) + \frac{\partial T^*(\tau_1; \mu_1, \mu)}{\partial \tau_1} = \varpi(\tau_1)\, \bar{X}^*(\mu, \tau_1)\, Y^*(\mu_1, \tau_1). \tag{4.23}$$

Then, using Eqs. (4.18)-(4.23), we have

$$S(\tau_1; \mu, \mu_0) = \int_0^{\tau_1} \varpi(\tau)\, Y^*(\mu, \tau)\, Y^*(\mu_0, \tau)\, d\tau, \tag{4.24}$$

$$T(\tau_1; \mu, \mu_0) = \int_0^{\tau_1} \varpi(\tau)\, X^*(\mu, \tau)\, Y^*(\mu_0, \tau) \exp\left[-\frac{(\tau_1 - \tau)}{\mu}\right] d\tau, \tag{4.25}$$

$$S^*(\tau_1; \mu, \mu_1) = \int_0^{\tau_1} \varpi(\tau)\, X^*(\mu, \tau)\, X^*(\mu_1, \tau) \exp\left[-(\tau_1 - \tau)\left(\frac{1}{\mu} + \frac{1}{\mu_1}\right)\right] d\tau, \tag{4.26}$$

$$T^*(\tau_1; \mu, \mu_1) = \int_0^{\tau_1} \varpi(\tau)\, Y^*(\mu, \tau)\, X^*(\mu_1, \tau) \exp\left[-\frac{(\tau_1 - \tau)}{\mu_1}\right] d\tau. \tag{4.27}$$

Substitution of Eqs. (4.24)-(4.27) into Eqs. (4.14)-(4.17) provides respectively

$$X(\mu, \tau_1) = 1 + \tfrac{1}{2}\int_0^1 \frac{d\mu'}{\mu'} \int_0^{\tau_1} \varpi(\tau)\, Y^*(\mu, \tau)\, Y^*(\mu', \tau)\, d\tau, \tag{4.28}$$

$$Y(\mu, \tau_1) = \exp\left(-\frac{\tau_1}{\mu}\right) + \tfrac{1}{2}\int_0^1 \frac{d\mu'}{\mu'} \int_0^{\tau_1} \varpi(\tau)\, X^*(\mu, \tau)\, Y^*(\mu', \tau) \exp\left[-\frac{(\tau_1 - \tau)}{\mu}\right] d\tau, \tag{4.29}$$

$$X^*(\mu, \tau_1) = 1 + \tfrac{1}{2}\int_0^1 \frac{d\mu'}{\mu'} \int_0^{\tau_1} \varpi(\tau)\, X^*(\mu, \tau)\, X^*(\mu', \tau) \times \exp\left[-(\tau_1 - \tau)\left(\frac{1}{\mu} + \frac{1}{\mu'}\right)\right] d\tau, \tag{4.30}$$

$$Y^*(\mu, \tau_1) = \exp\left(-\frac{\tau_1}{\mu}\right) + \tfrac{1}{2}\int_0^1 \frac{d\mu'}{\mu'} \int_0^{\tau_1} \varpi(\tau)\, Y^*(\mu, \tau)\, X^*(\mu', \tau) \exp\left[-\frac{(\tau_1 - \tau)}{\mu'}\right] d\tau. \tag{4.31}$$

Hence a knowledge of X^*- and Y^*-functions gives an another set of X- and Y-functions by means of Eqs. (4.28) and (4.29). Recalling Eqs. (4.1)-(4.4) and (4.7), we get

$$T(\tau_1; \mu_0, \mu) = \exp\left(-\frac{\tau_1}{\mu_0}\right) S(\tau_1; -\mu_0, \mu), \tag{4.32}$$

$$T^*(\tau_1; \mu_1, \mu) = \exp\left(-\frac{\tau_1}{\mu_1}\right) S^*(\tau_1; -\mu_1, \mu), \tag{4.33}$$

$$S(\tau_1; \mu_0, \mu) = \exp\left[-\tau_1\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\right] S^*(\tau_1; -\mu_0, -\mu). \tag{4.34}$$

With the aid of Eqs. (4.14)-(4.17) and (4.32)-(4.34), we have

$$Y(-\mu, \tau_1) = \exp(\tau_1/\mu)\, X(\mu, \tau_1), \tag{4.35}$$

$$Y^*(-\mu, \tau_1) = \exp(\tau_1/\mu)\, X^*(\mu, \tau_1). \tag{4.36}$$

V. The Diffusion Matrix of Radiative Transfer

If the intensities $I_{\text{inc}}(0, -\mu)$ and $I_{\text{inc}}(\tau_1, +\mu)$ incident respectively on the upper and the lower boundaries of the medium are given by Eq. (2.2), the intensities of the radiations at the level τ, i.e., $I(\tau, +\mu)$ and $I(\tau, -\mu)$, are expressed in terms of the diffusion operational matrix whose elements are the reflectance and the transmittance operators in the upward and the downward stochastic processes.

Let the diffusion matrix with two rows and two columns be

$$D_{\mu,\tau} = \begin{pmatrix} R_{\mu,\tau} & T^*_{\mu,\tau} \\ T_{\mu,\tau} & R^*_{\mu,\tau} \end{pmatrix}, \tag{5.1}$$

where

$$R_{\mu,\tau}\{f(\mu')\} = \frac{1}{2\mu}\int_0^1 d\mu' \int_\tau^{\tau_1} p(\mu'; \tau', \tau_1) f(\mu') \exp\left[-\frac{(\tau'-\tau)}{\mu}\right] d\tau', \tag{5.2}$$

$$T_{\mu,\tau}\{f(\mu')\} = \frac{1}{2\mu}\int_0^1 d\mu' \int_0^{\tau} p(\mu'; \tau', \tau_1) f(\mu') \exp\left[-\frac{(\tau-\tau')}{\mu}\right] d\tau', \tag{5.3}$$

$$R^*_{\mu,\tau}\{f(\mu')\} = \frac{1}{2\mu}\int_0^1 d\mu' \int_0^{\tau} p^*(\mu'; \tau', \tau_1) f(\mu') \exp\left[-\frac{(\tau-\tau')}{\mu}\right] d\tau', \tag{5.4}$$

$$T^*_{\mu,\tau}\{f(\mu')\} = \frac{1}{2\mu}\int_0^1 d\mu' \int_\tau^{\tau_1} p^*(\mu'; \tau', \tau_1) f(\mu') \exp\left[-\frac{(\tau'-\tau)}{\mu}\right] d\tau'. \tag{5.5}$$

Equations (5.2)-(5.5) show that the reflectance and the transmittance operators possess polarity. Then, except for the directly transmitted fluxes $\frac{1}{2}F_0 \exp(-\tau/\mu_0)\,\delta(\mu-\mu_0)$ and $\frac{1}{2}F_1 \exp[-(\tau_1-\tau)/\mu_1]\delta(\mu-\mu_1)$ in the directions $-\mu_0$ and $+\mu_1$, respectively, the intensity of the radiation directed towards the surface $\tau = 0$ and that directed towards the surface $\tau = \tau_1$ at each level are provided by

$$\begin{pmatrix} I(\tau, +\mu) \\ I(\tau, -\mu) \end{pmatrix} = D_{\mu,\tau} \begin{pmatrix} \dfrac{F_0}{2}\,\delta(\mu'-\mu_0) \\ \dfrac{F_1}{2}\,\delta(\mu'-\mu_1) \end{pmatrix}. \tag{5.6}$$

With the aid of Eqs. (5.1)-(5.5), from Eq. (5.6) we obtain

$$I(\tau, +\mu) = \frac{F_0}{4\mu} S(\tau, \tau_1; \mu, \mu_0) + \frac{F_1}{4\mu} P(\tau_1, \tau; \mu, \mu_1), \tag{5.7}$$

$$I(\tau, -\mu) = \frac{F_0}{4\mu} T(0, \tau; \mu, \mu_0) + \frac{F_0}{4\mu} S(\tau, 0; \mu, \mu_1), \tag{5.8}$$

where

$$S(\tau, \tau_1; \mu, \mu_0) = \int_0^{\tau_1-\tau} p(\mu_0; \tau_1 - \tau', \tau_1) \exp\left[-\frac{(\tau_1 - \tau - \tau')}{\mu}\right] d\tau', \tag{5.9}$$

$$T(0, \tau; \mu, \mu_0) = \int_{\tau_1-\tau}^{\tau_1} p(\mu_0; \tau_1 - \tau', \tau_1) \exp\left[-\frac{(\tau + \tau' - \tau_1)}{\mu}\right] d\tau', \tag{5.10}$$

$$S(\tau, 0; \mu, \mu_1) = \int_{\tau_1-\tau}^{\tau_1} p^*(\mu_1; \tau_1 - \tau', \tau_1) \exp\left[-\frac{(\tau + \tau' - \tau_1)}{\mu}\right] d\tau', \tag{5.11}$$

$$T(\tau_1, \tau; \mu, \mu_1) = \int_0^{\tau_1-\tau} p^*(\mu_1; \tau_1 - \tau', \tau_1) \exp\left[-\frac{(\tau_1 - \tau - \tau')}{\mu}\right] d\tau'. \tag{5.12}$$

Setting $\tau = 0$ in Eqs. (5.9) and (5.12), $S(0, \tau_1; \mu, \mu_0)$ and $T(\tau_1, 0; \mu, \mu_1)$ reduce respectively to $S(\tau_1; \mu, \mu_0)$ and $T^*(\tau_1; \mu, \mu_1)$ by Eqs. (4.1) and (4.4). Similarly, setting $\tau = \tau_1$ in Eqs. (5.10) and (5.1) and using Eqs. (4.2) and (4.3), $T(0, \tau_1; \mu, \mu_0)$ and $S(\tau_1, 0; \mu, \mu_1)$ reduce respectively to $T(\tau_1; \mu, \mu_0)$ and $S^*(\tau_1; \mu, \mu_1)$.

Putting respectively $\tau = 0$ and $\tau = \tau_1$ in Eqs. (5.7) and (5.8), we have

$$I(0, +\mu) = \frac{F_0}{4\mu} S(\tau_1; \mu, \mu_0) + \frac{F_1}{4\mu} T^*(\tau_1; \mu, \mu_1), \tag{5.13}$$

$$I(\tau_1, -\mu) = \frac{F_0}{4\mu} T(\tau_1; \mu, \mu_0) + \frac{F_1}{4\mu} S^*(\tau_1; \mu, \mu_1). \tag{5.14}$$

The results (5.13) and (5.14) coincide with those given in the preceding paper [30]. When the albedo ϖ is constant throughout the flat layer, the emergent intensities reduce to those yielded by Chandrasekhar in R.T.

Modifying somewhat the above formulation, we obtain another diffusion matrix for the radiation emergence from the medium. The formula is useful for the problem of line formation in finite inhomogeneous layer, provided that the distribution of emission sources acting in the medium is known.

Let the emitting source functions $f_0(\tau)$ and $f_1(\tau_1 - \tau)$ be respectively

$$f_0(\tau) = \tfrac{1}{2}\int_0^1 I_{\text{inc}}(0, -\mu')\exp\left(-\frac{\tau}{\mu'}\right)d\mu' = \frac{F_0}{4}\exp\left(-\frac{\tau}{\mu_0}\right), \tag{5.15}$$

$$\begin{aligned} f_1(\tau_1 - \tau) &= \tfrac{1}{2}\int_0^1 I_{\text{inc}}(\tau_1, +\mu')\exp\left[-\frac{(\tau_1-\tau)}{\mu'}\right]d\mu' \\ &= \frac{F_1}{4}\exp\left[-\frac{(\tau_1-\tau)}{\mu_1}\right], \end{aligned} \tag{5.16}$$

where the intensities of incident radiations $I_{\text{inc}}(0, -\mu)$ and $I_{\text{inc}}(\tau_1, +\mu)$ are given by Eq. (2.2).

Then

$$\begin{pmatrix} I(0, +\mu) \\ I(\tau_1, -\mu) \end{pmatrix} = \mathfrak{D}_\mu \begin{pmatrix} f_0(\tau') \\ f_1(\tau_1 - \tau') \end{pmatrix}, \tag{5.17}$$

where

$$\mathfrak{D}_\mu = \begin{pmatrix} \mathfrak{R}_\mu & \mathfrak{T}^*_\mu \\ \mathfrak{T}_\mu & \mathfrak{R}^*_\mu \end{pmatrix}. \tag{5.18}$$

In the above somewhat modified diffusion matrix the reflectance and the transmittance operators are

$$\mathfrak{R}_\mu\{f_0(\tau')\} = \frac{1}{\mu}\int_0^{\tau_1} p(\mu;\tau',\tau_1)f_0(\tau')\,d\tau', \tag{5.19}$$

$$\mathfrak{T}_\mu\{f_0(\tau')\} = \frac{1}{\mu}\int_0^{\tau_1} p^*(\mu;\tau',\tau_1)f_0(\tau')\,d\tau', \tag{5.20}$$

$$\mathfrak{R}^*_\mu\{f_1(\tau_1-\tau')\} = \frac{1}{\mu}\int_0^{\tau_1} p^*(\mu;\tau',\tau_1)f_1(\tau_1-\tau')\,d\tau', \tag{5.21}$$

$$\mathfrak{T}^*_\mu\{f_1(\tau_1-\tau')\} = \frac{1}{\mu}\int_0^{\tau_1} p(\mu;\tau',\tau_1)f_1(\tau_1-\tau')\,d\tau'. \tag{5.22}$$

The above show that, while the reflectance and transmittance operators for the case of monodirectional illumination of the upper boundary are expressed respectively in terms of the upward and the downward probability distributions of the emergence of quanta, i.e. $p(\mu;\tau,\tau_1)$ and $p^*(\mu;\tau,\tau_1)$, those for the case of monodirectional illumination of the lower boundary are expressed respectively in terms of the downward and the upward probability distributions.

If the optical properties of the medium are constant throughout the layer, the probability distributions $p^*(\mu; \tau, \tau_1)$ and $p^*(\mu; \tau_1 - \tau, \tau_1)$ reduce respectively to $p(\mu; \tau_1 - \tau, \tau_1)$ and $p(\mu; \tau, \tau_1)$. The reflectance operator in the semi-infinite homogeneous medium is equal to that in the preceding paper [26]. The result of Eq. (5.17) reduces to Eqs. (5.13) and (5.14).

Equation (5.17) can be applied to the problem of line formulation in a finite medium of arbitrary stratification. In this case $f(t)dt$ is interpreted as the energy quantity absorbed by an elementary cylinder of the unit cross section in the optical depth range $(\tau, \tau + d\tau)$ per unit time. The radiance is due to the result of action of the internal emission sources as well as the result of scattering of radiation coming from the exterior sources.

VI. Markovian Property of Radiative Transfer

While $p(\mu; \tau_1 - \tau, \tau_1)$ and $p^*(\mu; \tau_1 - \tau, \tau_1)$ represent respectively the probability distributions of emergence of quantum absorbed at the level $\tau_1 - \tau$ from the boundary planes $\tau = 0$ and $\tau = \tau_1$, $p(\mu; \tau, \tau_1)$ and $p^*(\mu; \tau, \tau_1)$ are respectively the probabilities that a quantum absorbed at the level τ will be respectively reemitted in the directions $+\mu$ and $-\mu$ in the radiations emerging from the surfaces $\tau = 0$ and $\tau = \tau_1$.

In the preceding paper [30], starting with the Chapman-Kolmogoroff equation, we established a set of the stochastic integro-differential equations. Following our present notation, the set of equations is written in the form

$$\frac{\partial p(\mu; \tau, \tau_1)}{\partial \tau_1} = \tfrac{1}{2} p(\mu; \tau_1, \tau_1) \int_0^1 p^*(\mu'; \tau, \tau_1) \frac{d\mu'}{\mu'}, \tag{6.1}$$

$$\frac{\partial p(\mu; \tau, \tau_1)}{\partial \tau_1} = -\frac{1}{\mu} p^*(\mu; \tau, \tau_1) + \tfrac{1}{2} p^*(\mu; \tau_1, \tau_1) \int_0^1 p^*(\mu'; \tau, \tau_1) \frac{d\mu'}{\mu'}, \tag{6.2}$$

where

$$p(\mu; \tau, \tau_1) = \varpi(\tau) \bar{\Lambda}_\tau\{p(\mu; t, \tau_1)\} + \varpi(\tau) \exp\left(-\frac{\tau}{\mu}\right), \tag{6.3}$$

$$p^*(\mu; \tau, \tau_1) = \varpi(\tau) \bar{\Lambda}_\tau\{p^*(\mu; t, \tau_1)\} + \varpi(\tau) \exp\left\{-\frac{(\tau_1 - \tau)}{\mu}\right\}. \tag{6.4}$$

On multiplying respectively Eqs. (6.1) and (6.2) with $\exp(-\tau/\mu_0)$ and $\exp[-(\tau_1 - \tau)/\mu_0]$ and integrating with respect to τ over $(0, \tau_1)$, after some minor argument, we get Eqs. (4.20)-(4.23) in which μ_1 is put to be equal to μ_0.

In the present paper, starting with a set of the stochastic integro-differential equations at the level $\tau_1 - \tau$, i.e. Eqs. (3.6) and (3.10), we obtained Eqs. (4.20)-(4.23). Then, it is shown that Eqs. (6.1) and (6.2) are mathematically equivalent to Eqs. (3.6) and (3.10). In other words, the Markovian property of multiple scattering of quanta is symmetric with respect to the boundary planes $\tau = 0$ and $\tau = \tau_1$. Therefore, starting with Eqs. (3.10) and (6.1), the same result such as Eqs. (4.20)-(4.23) can be obtained. The above means that a set of diffusion equations (6.3) and (3.12) in $p(\mu; \tau, \tau_1)$ and $p^*(\mu; \tau_1 - \tau, \tau_1)$ at the levels τ and $\tau_1 - \tau$ are equal also mathematically to that of Eqs. (3.11) and (6.4) in $p(\mu; \tau_1 - \tau, \tau_1)$ and $p^*(\mu; \tau, \tau_1)$ at the levels $\tau_1 - \tau$ and τ.

In the homogeneous medium, recalling Eqs. (3.11), (3.12), (6.3), and (6.4), $p^*(\mu; \tau_1 - \tau, \tau_1)$ and $p^*(\mu; \tau, \tau_1)$ reduce respectively to $p(\mu; \tau, \tau_1)$ and $p(\mu; \tau_1 - \tau, \tau_1)$, because of the constant optical properties of the medium throughout the layer.

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