Isometric decomposition operators, function spaces $E^\lambda_{p,q}$ and applications to nonlinear evolution equations

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Abstract

By using the isometric decomposition to the frequency spaces, we will introduce a new class of function spaces $E^\lambda_{p,q}$, which is a subspace of Gevrey 1-class $G_1(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ for $\lambda > 0$, and we will study the Cauchy problem for the nonlinear Schrödinger equation, the complex Ginzburg–Landau equation and the Navier–Stokes equation. Some well-posed results are obtained for the Cauchy data in $E^0_{2,1}$, and the regularity behavior in $E^{\xi t}_{2,1} \subset G_1(\mathbb{R}^n)$ for the complex Ginzburg–Landau equation and the Navier–Stokes equation is also obtained as time $t \searrow 0$.

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1. Introduction

The main aim of this paper is to study the basic estimates for the semi-group

\[ U(t) := \exp((a + i\alpha)t\Delta), \quad a \geq 0, \quad \alpha \in \mathbb{R}, \quad a + |\alpha| \neq 0 \]

in certain function spaces. It is well known that \( U(t) \) associates with the Cauchy problems for the nonlinear Schrödinger (NLS) equation and the complex Ginzburg–Landau (CGL) equation, see below (1.7) and (1.17). For convenience, we write \( S(t) := \exp(it\Delta) \) and \( G(t) := \exp((1 + i\alpha)t\Delta) \), which are said to be the Schrödinger group and the Ginzburg–Landau group, respectively. In order to solve the NLS and CGL equations, the basic \( L^p - L^r \) estimates of \( S(t) \) and \( G(t) \) play an important role (cf. [4,7,12,30,33,34]):

\[ \|S(t)f\|_p \lesssim t^{-n(1/p' - 1/p)}\|f\|_{p'}, \quad 2 \leq p \leq \infty, \quad 1/p + 1/p' = 1, \quad (1.2) \]

\[ \|G(t)f\|_p \lesssim t^{-n(1/r' - 1/p)}\|f\|_{r'}, \quad 1 \leq r \leq p \leq \infty. \quad (1.3) \]

One sees that the right-hand sides of (1.2) and (1.3) have singularity at \( t = 0 \). In order to control the singularity growth at \( t = 0 \) in (1.2) and (1.3), we generally have a restriction condition \( n(1/r - 1/p) \leq 1 \) (\( r = p' \) for \( S(t) \)). Such a condition essentially arises from Hardy–Littlewood–Sobolev’s singular integral theory, which seems to be one of the main obstacles for us studying the well posedness of solutions of the NLS and CGL equations.

Now, we want to remove the singularity at \( t = 0 \) in certain function spaces. Let \( \{Q_k\}_{k \in \mathbb{Z}^n} \) be a unit-cube decomposition of \( \mathbb{R}^n \), that is \( \bigcup_{k \in \mathbb{Z}^n} Q_k = \mathbb{R}^n \) and \( Q_k \cap Q_j = \emptyset \) if \( k \neq j \). Such a decomposition has been extensively applied by Constantin and Saut [9], Kenig et al. [19,20], Ginibre and Velo [12], and Tao [28]. We now consider the isometric decomposition to the frequency space. We denote \( \Box_k \sim \mathcal{F}^{-1}1_{Q_k} \mathcal{F} \), \( k \in \mathbb{Z}^n \), which is said to be the isometric decomposition operator.\(^1\) One easily sees that

\[ \|\Box_k U(t)f\|_\infty \lesssim \left\| e^{-t(a+i\alpha)|2^k|^2} \hat{1}_{Q_k} \mathcal{F} \right\|_1 \lesssim e^{-ct|k|}\|\Box_k f\|_1, \quad k \in \mathbb{Z}^n, \quad c > 0. \quad (1.4) \]

In (1.4), the singularity disappears at \( t = 0 \) for all \( k \in \mathbb{Z}^n \). This seems to be an interesting fact, which inspires us systematically applying the isometric decomposition operators that associate with a new class of function spaces \( E^{\lambda}_{p,q} \) with the norm

\[ \|f\|_{E^{\lambda}_{p,q}} = \left( \sum_{k \in \mathbb{Z}^n} (2^{|k|}\|\Box_k f\|_p)^q \right)^{1/q}. \quad (1.5) \]

We can show that \( U(t) \) has a very well behavior in \( E^{\lambda}_{p,q} \):

\[ \|U(t)f\|_{E^{\lambda}_{p,q}} \lesssim \|f\|_{E^{0}_{r,q}}, \quad 0 < r \leq 2 \leq p \leq \infty, \quad 0 < q \leq \infty, \quad t \leq T, \quad (1.6) \]

\(^1\)The delicate version of \( \Box_k \) will be given in Section 2, which is a smooth version of \( \mathcal{F}^{-1}1_{Q_k} \mathcal{F} \).
where the singularity on time has been absorbed by the space $E_{r,q}^0$, see Sections 2 and 5.

Next, we will apply (1.6) to study the initial value problem for the NLS equation:

$$iu_t + \Delta u - \mu |u|^{2\kappa} u = 0, \quad u(0, x) = u_0(x),$$

where $u(t, x)$ is a complex-valued function of $(t, x) \in [0, T) \times \mathbb{R}^n$ for some $0 < T \leq \infty$, $i = \sqrt{-1}$, $\Delta = \sum_{i=1}^n \partial_i^2 / \partial x_i^2$, $\mu \in \mathbb{R}$, $\kappa \in \mathbb{N}$, and $u_0$ is a complex-valued function of $x \in \mathbb{R}^n$. We will consider the following problem:

Moreover, consider the case

$$\text{if } 0.$$ Eq. (1.11) is an important model equation in nonlinear optics. Nakamura and Ozawa [24] studied the local well posedness and the global well posedness with small data for Eq. (1.11) in the critical space $H^{n/2}$. In this paper we consider the case $u_0 \in E_{2,1}^0(\mathbb{R}^n)$. We have the following:

**Theorem 1.1.** Let $u_0 \in E_{2,1}^0(\mathbb{R}^n)$, $n \geq 1$, $\mu \in \mathbb{R}$, $\kappa \in \mathbb{N}$. Then there exists $T^* := T^*(\|u_0\|_{E_{2,1}^0}) > 0$ such that (1.7) has a unique solution

$$u \in C([0, T^*); E_{2,1}^0(\mathbb{R}^n)).$$

Moreover, if $T^* < \infty$, then

$$\limsup_{t \nearrow T^*} \|u(t)\|_{E_{2,1}^0} = \infty.$$

Replacing the power nonlinearity in (1.7) by an exponential growth one, say, we consider the following problem:

$$iu_t + \Delta u - \mu (e^{q|u|^2} - 1) u = 0, \quad u(0, x) = u_0(x),$$

where $\mu \in \mathbb{R}$ and $q > 0$. Eq. (1.11) is an important model equation in nonlinear optics. Nakamura and Ozawa [24] studied the local well posedness and the global well posedness with small data for Eq. (1.11) in the critical space $H^{n/2}$. In this paper we consider the case $u_0 \in E_{2,1}^0(\mathbb{R}^n)$. We have the following:

**Theorem 1.2.** Let $u_0 \in E_{2,1}^0(\mathbb{R}^n)$, $n \geq 1$, $\mu \in \mathbb{R}$, $q > 0$. Then there exists $T^* := T^*(\|u_0\|_{E_{2,1}^0}) > 0$ such that (1.11) has a unique solution that satisfies (1.9) and (1.10).
We briefly indicate the ideas in the proof of Theorems 1.1 and 1.2. We will solve the NLS equation by recasting it into the form of the integral equation:

\[
u(t) = S(t)u_0 - i\mu \int_0^t S(t - \tau)(|u|^{2\kappa}u)(\tau) \, d\tau. \tag{1.12}\]

Applying (1.6) and (1.12), we have

\[
\|u(t)\|_{E_{r,1}^0} \lesssim \|u_0\|_{E_{r,1}^0} + \sup_{0 \leq t \leq T} \|(u|^{2\kappa}u)(t)\|_{E_{r,1}^0}, \quad 0 \leq t \leq T. \tag{1.13}\]

In Section 4 we can show that \(E_{r,1}^0\) has the algebra-type structure. By taking \(r = 2/(2\kappa + 1)\), we have

\[
\|u|^{2\kappa}u\|_{E_{r,1}^0} \lesssim \|u\|_{E_{r,1}^0}^{2\kappa+1}. \tag{1.14}\]

Hence, by (1.13) and (1.14),

\[
\|u(t)\|_{C(0,T;E_{r,1}^0)} \lesssim \|u_0\|_{E_{r,1}^0} + T\|u|^{2\kappa+1}_{C(0,T;E_{r,1}^0)}. \tag{1.15}\]

Then one can use a standard contraction method to show the local well posedness of (1.7) in the space \(C(0,T;E_{r,1}^0)\). For the proof of Theorem 1.2, we can use Taylor’s expansion of the nonlinearity,

\[
(e^{|u|^2} - 1)u = \sum_{k=1}^{\infty} \frac{q^k}{k!}|u|^{2\kappa}u, \tag{1.16}\]

and proceed in a similar way as above to get the desired result.

Thirdly, we consider the initial value problem for the CGL equation:

\[
u(t) - (a + i\alpha)\Delta u + (b + i\beta)|u|^{2\kappa}u + \nu u = 0, \quad u(0, x) = u_0(x), \tag{1.17}\]

where \(u(t, x)\) is a complex-valued function of \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+ = [0, \infty)\). \(a > 0, b \geq 0, \kappa \in \mathbb{N}, \) and \(\alpha, \beta, \nu \in \mathbb{R}\). \(u_0\) is a complex-valued function of \(x \in \mathbb{R}^n\). The CGL equation was originally discovered by Ginzburg and Landau for a phase transition in superconductivity [13], which was subsequently derived for instability waves in hydrodynamics [25,27], and for modelling the transition to turbulence in chemical reactions [17] or describing spatial pattern formation and the onset of instabilities in non-equilibrium fluid dynamical systems [5,8].

A large amount of work has been devoted to the study of the global well posedness and the dynamical behavior of the CGL equation (cf. [1,2,11,12,23,31,35]). Applying the compactness method, Ginibre and Velo [11] obtained that if \(u_0 \in L^r, \ |\alpha|<2\sqrt{r}-1, \ r \geq 2\), then (1.17) has a global solution in \(X := L_{\infty, \text{loc}}(\mathbb{R}_+, L^r) \cap L^{2\kappa+r}_{\text{loc}}(\mathbb{R}_+, L^{2\kappa+r}); \) the solutions in \(X\) are uniquely determined by the initial data \(u_0 \in L^r\) if we assume additionally that \(\kappa \leq r/n\); cf. [12]. The Gevrey regularity for the CGL equation in a torus \(\mathbb{T}^n\) was considered in [22,15]. The main tool in these works is the Fourier series theory, which was previously used by Foias and Temam [10], who studied the Gevrey class regularity for the Navier–Stokes equation in \(\mathbb{T}^n\). According to the Fourier series theory, the norms on various function spaces defined in \(\mathbb{T}^n\) have a discrete expression, which seems invalid to the Cauchy problem for the CGL equation.
In this paper we will continuously study the well posedness and the regularity of the solutions of Eq. (1.17) in the case \( u_0 \in E^0_{2,1} \). Let \( s > 0 \). Denote

\[
G_s(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \exists \rho, M > 0 \text{ s.t. } \| f \|_{\dot{H}^s} \leq M \left( \frac{m!}{\rho^m} \right)^s \forall m \in \mathbb{Z}_+ \right\}
\]

and \( G_s(\mathbb{R}^n) \) is said to be the Gevrey s-class.\(^2\) We are interested in the Gevrey 1-class \( G_1(\mathbb{R}^n) \), in which any function is real analytic; cf. [22]. We can show that \( G_1(\mathbb{R}^n) \) is the collection of all \( E^\lambda_{2,1} \), \( \lambda > 0 \), that is \( G_1(\mathbb{R}^n) = \cup_{\lambda > 0} E^\lambda_{2,1} \). We have the following result:

**Theorem 1.3.** Let \( a > 0, \kappa \in \mathbb{N} ; b, \alpha, \beta, \nu \in \mathbb{R} ; u_0 \in E^0_{2,1}(\mathbb{R}^n), n \geq 1 \). Then there exists \( T^* := T^*(\| u_0 \|_{E^0_{2,1}}) > 0 \) such that (1.17) has a unique solution

\[
u \in C_{\text{loc}}([0, T^*); E^0_{2,1}(\mathbb{R}^n)) \tag{1.18}
\]

Moreover, this solution has the Gevrey 1-class regularity effect: there exists \( t_0 > 0 \) such that

\[
\begin{align*}
&\left\{ u(t) \in E^c_{2,1} \subset G_1(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n) \quad \forall t \in [0, t_0], \\
&\left\{ u(t) \in E^{c}_{2,1} \subset G_1(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n) \quad \forall t_0 < t < T^*
\end{align*}
\]

(1.19)

and for any \( T < T^* \),

\[
\sup_{0 \leq t \leq T} \| u(t) \|_{E^0_{2,1}} \leq C(T, \| u_0 \|_{E^0_{2,1}}). \tag{1.20}
\]

Further, if \( b > 0 \) and \( |\alpha| (n\kappa - 2)/a < 2 \sqrt{n\kappa - 1} \), then the above solution is a global one, i.e. \( T^* = \infty \) in (1.18)–(1.20).

It seems that (1.19) and (1.20) exactly describe the disappearing process for the regularity of solutions. Our regularity results (1.19) and (1.20) mainly rely upon the dissipative structure of the Ginzburg–Landau group, see Section 5.

Corresponding to (1.11), we consider the CGL equation with exponential nonlinearity:

\[
u_t - (a + i\alpha)\Delta u + (b + i\beta)(e^{0|u|^2} - 1)u = 0, \quad u(0, x) = u_0(x). \tag{1.21}
\]

Recall that \( L^{n\kappa} (n\kappa > 1) \) is the critical space for Eq. (1.17) with \( b < 0, \alpha, \beta \in \mathbb{R} \), in which the solutions exist for all time if the initial data are small enough; cf. [14,33].

\(^2\)Here the definition of Gevrey s-class is a generalization of the Gevrey s-class defined in a torus \( \mathbb{T}^n \) (cf. [23]):

\[
G_s(\mathbb{T}^n) = \left\{ f : \exists \rho, M > 0 \text{ s.t. } |e^\sigma f(x)| \leq M \left( \frac{\alpha!}{\rho^\alpha} \right)^s \forall x \in \mathbb{Z}^n_+, x \in \mathbb{T}^n \right\} \tag{1.17b}
\]

It is easy to see that \( G_s(\mathbb{T}^n) \) has an equivalent expression similar to (1.17a). But in whole \( \mathbb{R}^n \), \( G_s(\mathbb{R}^n) \) defined in (1.17a) is a subspace of that related to (1.17b).
In view of Taylor’s expansion of (1.16), it is natural to conjecture that \( \cap_{k \in \mathbb{N}} L^{nk} \) is the space so that the CGL equation with exponential nonlinearity is globally well posed with small data. Let \( \{X_k\} \) be a sequence of Banach spaces. We denote by \( \cap_{k \in \mathbb{N}} X_k \) the space \( \{ u : u \in X_k \ \forall \ k \in \mathbb{N} \} \) equipped with the norm \( \|u\|_{\cap_{k \in \mathbb{N}} X_k} = \sup_{k \in \mathbb{N}} \|u\|_{X_k} \). Put

\[
\|u\|_{k,T} = \sup_{0 < t < T} t^{1/\gamma_k} \|u(t)\|_{p_k},
\]

\[
X_T = \left\{ u : \sup_{k \geq 1} \|u\|_{k,T} < \infty \right\}.
\]

We have the following:

**Theorem 1.4.** Let \( n \geq 2, u_0 \in \cap_{k \in \mathbb{N}} L^{nk}(\mathbb{R}^n) \). Let \( a > 0, \ q > 0, \ b, \ z, \beta \in \mathbb{R} \). Then there exists \( T^* > 0 \) such that (1.21) has a unique solution

\[
u \in C \left( 0, T; \cap_{k \in \mathbb{N}} L^{nk}(\mathbb{R}^n) \right) \cap X_T, \quad \forall \ T < T^*.
\]

Moreover, if one of the following conditions is satisfied,

(i) There exists a small \( \delta > 0 \) such that \( \|u_0\|_{\cap_{k \in \mathbb{N}} L^{nk}(\mathbb{R}^n)} \leq \delta \), or

(ii) \( b > 0, \ z = 0 \),

then the solution above is a global one, that is \( T^* = \infty \) in (1.25). Further, if \( u_0 \in E_{2,1}^0 \), then such a solution has the regularity behavior as in (1.19)–(1.20).

Finally, we consider the Cauchy problem for the Navier–Stokes system

\[
\begin{aligned}
&u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \text{div} \ u = 0, \\
&u(0, x) = u_0(x),
\end{aligned}
\]

where \( u = (u_1, \ldots, u_n) \), \( u_i \) and \( p \) are scalar unknown functions of \( (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \) and \( u \cdot \nabla = u_1 \partial_{x_1} + \cdots + u_n \partial_{x_n} \). For our purpose we give another version of (1.26). If \((u, p)\) is a smooth solution of (1.26), taking a divergence in the first equation and applying the second equation in (1.26), we have

\[
\Delta p + \text{div}(u \cdot \nabla u) = 0.
\]

Inserting (1.27) into (1.26), we then get that for \( \mathcal{P} = (-\Delta)^{-1} \text{div}, \)

\[
\begin{aligned}
u_t - \Delta u + (1 + \mathcal{P}) u \cdot \nabla u = 0, \quad u(0, x) = u_0(x).
\end{aligned}
\]

Now our goal is to solve (1.28) by using its integral version:

\[
\begin{aligned}
u(t) &= U_0(t)u_0 - A_0 g(u, \nabla u), \\
g(u, \nabla u) &= (1 + \mathcal{P}) u \cdot \nabla u,
\end{aligned}
\]
where $U_0(t) = \exp(tA)$ and $A_0 = \int_0^t U_0(t - \tau) d\tau$. It is known that $\mathcal{S}(\mathbb{R}^n) \subset E_{2,1}^0$ is dense; cf. Section 3. For convenience, we denote by $[X]^n$ the space $X \times \cdots \times X$. Let

$$[\mathcal{S}(\mathbb{R}^n)]_0^n = \{u \in [\mathcal{S}(\mathbb{R}^n)]^n : \text{div} u = 0\} \quad (1.31)$$

and let $[E_{2,1}^0(\mathbb{R}^n)]_0^n$ be the completion of $[\mathcal{S}(\mathbb{R}^n)]_0^n$ in $[E_{2,1}^0(\mathbb{R}^n)]^n$. For any $u = (u_1, \ldots, u_n)$, we denote $\|D u\|_2 := \sum_{1 \leq i, j \leq n} \|\partial_i u_j\|_2$, and

$$\|u\|_T := \sup_{0 < t \leq T} t^{1/2} \|D u\|_2. \quad (1.32)$$

We have the following result:

**Theorem 1.5.** Let $n \geq 2$, $u_0 \in [E_{2,1}^0(\mathbb{R}^n)]_0^n$. Then there exists $T^* := T^*(\|u_0\|_{E_{2,1}^0}) > 0$ such that (1.29) has a unique solution

$$u \in [C_{\text{loc}}([0, T^*), E_{2,1}^0(\mathbb{R}^n))]^n \cap X, \quad (1.33)$$

where

$$X = \left\{u : \|u(t)\|_T < \infty; \forall T < T^*, \text{ div} u(t) = 0 \forall t \in (0, T^*)\right\}. \quad (1.34)$$

Moreover, there exists $t_0 > 0$ such that

$$u(t) \in E_{2,1}^0([t_0 \wedge 0, t]) \subset G_1(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n) \quad \forall 0 < t < T^*$$

and (1.20) also holds. Further, if $n = 2$, then $T^* = \infty$ in (1.33)–(1.35).

The Gevrey class regularity for the solutions of the Navier–Stokes equation in a torus $\mathbb{T}^n$ was obtained by Foias and Temam [10] in 1989. For the Cauchy problem of the Navier–Stokes equation in three spatial dimensions, the regularity behavior as $t \searrow 0$ of solutions was previously obtained by Lemarie–Rieusset [21] in a different way: If $\sup_{\xi \in \mathbb{R}^3} |\xi|^2 |\hat{u}_0(\xi)| < C$ (that is $\hat{\Delta u}_0 \in L^\infty(\mathbb{R}^3)$), then the solution $u$ of (1.29) satisfies

$$\sup_{t > 0} \sup_{\xi \in \mathbb{R}^3} e^{\sqrt{t}|\xi|} |\xi|^2 |\hat{u}(t, \xi)| < \infty. \quad (1.36)$$

Eq. (1.36) also indicates the disappearing process for the regularity of the solutions. It seems that there is no direct relation between the regularity behavior (1.20) and (1.36), and we will give some further comparison between (1.20) and (1.36) in Section 9, Remark 9.1.

This paper is organized as follows: In Section 2 we introduce the notion of the spaces $E_{p,q}^\lambda$, some properties of such spaces will be discussed in Section 3. In Section 4 we perform the multi-linear estimates in $E_{p,q}^\lambda$. Some basic estimates in $E_{p,q}^\lambda$ for the Ginzburg–Landau group and the Schrödinger group will be given in Section 5. Finally, we show our Theorems 1.1–1.5 in Sections 6–9, respectively.

Throughout this paper, $c < 1$, $C > 1$ will denote positive constants, which can be different at different places. $a \lesssim b$ stands for $a \leq Cb$ for some constant $C > 1$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We write $a \land b = \min(a, b)$ and $a \lor b = \max(a, b)$. We denote by $p'$ the dual number of $p \in [1, \infty]$, i.e. $1/p + 1/p' = 1$. We will
have occasions to use a variety of function spaces: Lebesgue spaces $L^p := L^p(\mathbb{R}^n)$, $\| \cdot \|_p := \| \cdot \|_{L^p}$, Bessel potential spaces $H^s_p = (I - \Delta)^{-s/2}L^p$, $H^s = H^s_2$. Some properties of these function spaces can be found in [3,29]. For any quasi-Banach space $X$, we write $L^q_{\text{loc}}([a, b); X) = \{ u : u \in L^q(a, c; X) \text{ for any } [a, c] \subset [a, b) \}$ and simply write $L^q_{\text{loc}}(a, b; X) = L^q_{\text{loc}}([a, b); X)$. We denote by $B(x, R)$ the ball in $\mathbb{R}^n$ with center $x$ and radius $R$. $\mathcal{F}$ or $\hat{\cdot}$ denotes the Fourier transform; $\mathcal{F}^{-1}$ denotes the inverse Fourier transform of $\mathcal{F}$.

2. Function spaces $E^\lambda_{p,q}$

2.1. Definition of $E^\lambda_{p,q}$

We denote by $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ the Schwartz space and its dual space, respectively. Let $\rho \in S(\mathbb{R}^n)$ and $\rho : \mathbb{R}^n \to [0, 1]$ be a smooth radial bump function adapted to the ball $B(0, \sqrt{2}n)$, say $\rho(\xi) = 1$ as $0 \leq |\xi| \leq \sqrt{n}/2$, and $\rho(\xi) = 0$ as $|\xi| \geq \sqrt{2}n$. Let $\rho_k$ be a translation of $\rho$:

$$\rho_k(\xi) = \rho(\xi - k), \quad k \in \mathbb{Z}^n,$$  

(2.1)

where $k \in \mathbb{Z}^n$ means that $k = (k_1, \ldots, k_n)$, $k_1, \ldots, k_n$ are all integers. Since $\rho_k(\xi) = 1$ in the unit closed cube $Q_k$ with center $k$ and $\{Q_k \}_{k \in \mathbb{Z}^n}$ is a covering of $\mathbb{R}^n$, one has that $\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \geq 1$ for all $\xi \in \mathbb{R}^n$. We write

$$\sigma_k(\xi) = \rho_k(\xi) \left( \sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n.$$  

(2.2)

It is easy to see that

$$\begin{cases}
|\sigma_k(\xi)| \geq c & \forall \xi \in Q_k, \\
\text{supp} \sigma_k \subset \{ \xi : |\xi - k| \leq \sqrt{2}n \}, \\
\sum_{k \in \mathbb{Z}^n} \sigma_k(\xi) \equiv 1 & \forall \xi \in \mathbb{R}^n, \\
|\sigma^{(m)}_k(\xi)| \leq C_m & \forall \xi \in \mathbb{R}^n.
\end{cases}$$  

(2.3)

Hence, the set

$$\tilde{Y} = \{ \{\sigma_k \}_{k \in \mathbb{Z}^n} : \{\sigma_k \}_{k \in \mathbb{Z}^n} \text{ satisfies (2.3)} \}$$  

(2.4)

is non-void. Let $\{\sigma_k \}_{k \in \mathbb{Z}^n} \in \tilde{Y}$ be a function sequence. Define

$$\Box_k := \mathcal{F}^{-1} \sigma_k \mathcal{F}, \quad k \in \mathbb{Z}^n.$$  

(2.5)

For any $k \in \mathbb{Z}^n$, we write $|k| = |k_1| + \cdots + |k_n|$. For any $0 < \lambda < \infty$, $0 < p, q \leq \infty$, we introduce the following function space:

$$E^\lambda_{p,q}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \| f \|_{E^\lambda_{p,q}} := \left( \sum_{k \in \mathbb{Z}^n} (2^{\lambda|k|} \| \Box_k f \|_p)^q \right)^{1/q} < \infty \right\}.$$  

(2.6)
2.2. Comparisons between $B^{\lambda}_{p,q}$ and $E^{\lambda}_{p,q}$

Recall that the Besov space $B^{\lambda}_{p,q}$ is defined as follows. Let $\psi: \mathbb{R}^n \rightarrow [0,1]$ be a smooth radial bump function adapted to the ball $B(0,2)$: $\psi(\xi) = 1$ as $|\xi| \leq 1$ and $\psi(\xi) = 0$ as $|\xi| > 2$. We write $\delta(\cdot) := \psi(\cdot) - \psi(2\cdot)$ and $\delta_k := \delta(2^{-k}\cdot)$ for $k \geq 1$; $\delta_0 := 1 - \sum_{k \geq 1} \delta_k$. Put

$$\Delta_k := \mathcal{F}^{-1}\delta_k \mathcal{F}, \quad k \in \mathbb{Z}_+.$$  \hspace{1cm} (2.7)

Denote

$$B^{\lambda}_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^{\lambda}_{p,q}} := \left( \sum_{k=0}^{\infty} (2^{\lambda k} \|\Delta_k f\|_p)^q \right)^{1/q} < \infty \right\}. \hspace{1cm} (2.8)$$

At first glance one may think that $E^{\lambda}_{p,q}$ is quite similar to Besov spaces $B^{\lambda}_{p,q}$. But we do not forget that the decomposition to the frequency space in the definition of Besov spaces is dyadic, that is $\text{supp} \delta_k \subset \{ \xi : 2^{k-1} \leq |\xi| < 2^{k+1} \}$, which is different from the isometric decomposition to the frequency space (supp $\sigma_k \subset \{ \xi : |\xi - k| \leq \sqrt{2n} \}$) as in the expression of $E^{\lambda}_{p,q}$. If we want to refine the norm on Besov spaces according to the isometric decomposition, it should match with the norm

$$\|f\| = \left( \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{\lambda} \|\square_k f\|_p^q \right)^{1/q}. \hspace{1cm} (2.9)$$

Comparing (2.6) with (2.9), we see that the regularity quantity $2^{\lambda k}$ in (2.6) grows much more rapidly than $|k|^\lambda$ as in (2.9). Hence, $E^{\lambda}_{p,q}$ enjoys much more regularity than $B^{\lambda}_{p,q}$ if $\lambda > 0$.

2.3. Background on $E^{\lambda}_{p,q}(\mathbb{R}^n)$ with $\lambda > 0$

Some ideas on $E^{\lambda}_{p,q}$ have been indicated as in the introduction, see (1.4), (1.5) and (1.6). Another main motivation for us to introduce the spaces $E^{\lambda}_{p,q}$ in the case $\lambda > 0$ is to describe the regularity behavior of solutions for a class of dissipative evolution equations: for instance, we consider the Cauchy problem for the nonlinear heat equation in 1D:

$$u_t - \partial_x^2 u + |u|^4 u = 0, \quad u(x,0) = u_0(x). \hspace{1cm} (2.10)$$

It is known that if $u_0 \in L^2(\mathbb{R})$, then the solution $u$ of (2.10) satisfies $u(t) \in C^\infty(\mathbb{R})$ for any $t > 0$. Hence, $t = 0$ is the singular point for the regularity of solutions if $u_0$ belongs to $L^2(\mathbb{R})$ only. A natural question is how to describe the disappearing process for the regularity of the solutions of Eq. (2.10) as time $t \searrow 0$. In order to answer this question, in [32] we introduce the notion of exponential Besov spaces $B^{(s_k)}_{p,q}(\mathbb{R}^n)$:

$$B^{(s_k)}_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^{(s_k)}_{p,q}} := \left( \sum_{k=0}^{\infty} (2^{s_k k} \|\Delta_k f\|_p)^q \right)^{1/q} < \infty \right\}. \hspace{1cm} (2.11)$$
where \( \lim_{k \to \infty} s_k = \infty \) and \( \triangle_k \) is as in (2.7). \( B^{\{s_k\}}_{p,q} (\mathbb{R}^n) \) is an infinitely smooth space, that is

\[
B^{\{s_k\}}_{p,q} (\mathbb{R}^n) \subset C^\infty (\mathbb{R}^n).
\]  

(2.12)

In [32], we obtain the regularity estimates for the solutions of linear heat equation

\[
\frac{\partial}{\partial t} u - \frac{\partial^2}{\partial x^2} u = 0, \quad u(x, 0) = u_0(x)
\]  

(2.13)

and we have for the solution \( u(t) = \mathcal{F}^{-1} e^{-t \frac{\partial^2}{\partial x^2}} \mathcal{F} u_0 \) of (2.13),

\[
\| \mathcal{F}^{-1} e^{-t \frac{\partial^2}{\partial x^2}} \mathcal{F} u_0 \|_{B^{\{s_k\}}_{2,2} (\mathbb{R})} \lesssim \| u_0 \|_{L^2(\mathbb{R})}, \quad s_k \simeq 2^{2k}/k.
\]  

(2.14)

Eq. (2.14) is a sharp estimate as indicated in [32]. We are expecting that for the solution \( u(t) \) of Eq. (2.10) there holds a similar estimate as in (2.14), that is

\[
\| u(t) \|_{B^{\{s_k\}}_{2,2} (\mathbb{R})} \lesssim \omega(T, \| u_0 \|_{L^2(\mathbb{R})}), \quad 0 < t \leq T, \quad s_k \simeq 2^{2k}/k.
\]  

(2.15)

If (2.15) holds, we see that the regularity behavior of \( u(t) \) at a neighborhood of \( t = 0 \) is very clearly described.

In order to get the estimate as in (2.15), one needs to deal with the nonlinearity \( |u|^4 u \) in exponential Besov spaces. At least, we should prove the following bilinear estimate:

\[
\| uv \|_{E^{\{s_k\}}_{p,q} (\mathbb{R})} \lesssim \| u \|_{E^{\{s_k\}}_{p_1,q} (\mathbb{R})} \| u \|_{E^{\{s_k\}}_{p_2,q} (\mathbb{R})},
\]  

(2.16)

where \( 1/p_1 + 1/p_2 = 1 \). Unfortunately, it seems hard to get the bilinear estimates as in (2.16), see [32]. The main difficulty to realize (2.16) is that the dyadic decomposition to the frequency space seems too rough to match the rapid growth of regularity indices \( s_k \nearrow \infty \). Hence, we are looking for some refined decomposition to the frequency space. Using the isometric decomposition as in (2.5), we then naturally get the space \( E^\lambda_{p,q} \) as in (2.6), in which a bilinear estimate like (2.16) in \( E^\lambda_{p,q} \) can be obtained:

\[
\| uv \|_{E^\lambda_{p,q} (\mathbb{R}^n)} \lesssim \| u \|_{E^\lambda_{p_1,q_1} (\mathbb{R}^n)} \| u \|_{E^\lambda_{p_2,q_2} (\mathbb{R}^n)}.
\]  

(2.17)

Using (2.17), we can get the regularity behavior of solutions like (1.20). Following the ideas as above, one can understand \( E^\lambda_{p,q} \) as a refinement to the exponential Besov spaces, see below, Proposition 3.6.

2.4. Remark on isometric decomposition

An interesting fact is that the inclusions between the spaces \( E^\lambda_{p,q} \) is simpler than the usual (exponential) Besov spaces. Recall that the embedding

\[
B_{p_1,q_1}^{s_1} (\mathbb{R}^n) \subset B_{p_2,q_2}^{s_2} (\mathbb{R}^n), \quad s_1 \geq s_2, \quad s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}
\]  

(2.18)

is based on a scaling inequality:

\[
\| \Delta_k f \|_{p_2} \lesssim 2^{n(1/p_1 - 1/p_2)k} \| \Delta_k f \|_{p_1}, \quad k \in \mathbb{Z}_+, \quad 0 < p_1 \leq p_2 \leq \infty.
\]  

(2.19)
If the dyadic decomposition operator $\triangle_k$ in (2.19) is substituted by the isometric decomposition operator $\square_k$, then we have
\[
\|\square_k f\|_{p_2} \lesssim \|\square_k f\|_{p_1}, \quad k \in \mathbb{Z}^n, \quad 0 < p_1 \leq p_2 \leq \infty.
\] (2.20)

Comparing the right-hand side of (2.19) and (2.20), we see that there is a regularity increment $2^n(1/p_1 - 1/p_2)k$ in (2.19), and this regularity increment disappears in (2.20) as we adopt the isometric decomposition operators. It is easy to see that (2.20) implies that (see Section 3, Proposition 3.5)
\[
E_{p_1,q}^\lambda(\mathbb{R}^n) \subset E_{p_2,q}^\lambda(\mathbb{R}^n), \quad 0 < p_1 \leq p_2 \leq \infty.
\] (2.21)

3. Some properties on $E_{p,q}^\lambda$

For simplicity, we always write $E_{p,q}^\lambda = E_{p,q}^\lambda(\mathbb{R}^n)$. Let $\Omega \subset \mathbb{R}^n$ be a compact set. Define
\[
L^p_\Omega = \{ f \in S(\mathbb{R}^n) : \text{supp } \hat{f} \subset \Omega, \| f \|_p < \infty \}.
\] (3.1)

**Lemma 3.1.** Let $\Omega \subset \mathbb{R}^n$ be a compact set with $\text{diam } \Omega < 2R$, $0 < p \leq q \leq \infty$. Then there exists a constant $C > 0$, which depends on $p, q, R$ such that
\[
\| f \|_q \leq C \| f \|_p \quad \forall f \in L^p_\Omega.
\] (3.2)

**Proof.** Lemma 3.1 is essentially known; cf. [29]. Since we are interested in the case $\Omega = Q_k$, $k \in \mathbb{Z}^n$, it seems necessary to emphasize that the constant $C > 0$ in (3.2) may depend on the size of $\Omega$, but is independent of its positions in frequency spaces. We now sketch the proof. Let $\psi \in S(\mathbb{R}^n)$ satisfy $\text{supp } \hat{\psi} \subset B(0, 2R)$ and $\hat{\psi}|_{B(0,R)} = 1$. In view of $\text{diam } \Omega < 2R$, we can take $\xi_0$ such that $\Omega \subset B(\xi_0, R)$. For any $f \in S(\mathbb{R}^n) \cap L^p_\Omega$, we have $\hat{f} = \hat{f}\hat{\psi}(\cdot - \xi_0)$, that is
\[
f(x) = c \int_{\mathbb{R}^n} f(x - y)e^{i\xi_0 y}\psi(y) dy.
\]
If $1 \leq p \leq \infty$, by Young’s inequality one has that
\[
\| f \|_\infty \leq C_\psi \| f \|_p.
\] (3.2a)

If $0 < p < 1$, then
\[
\| f \|_\infty \leq C_\psi \left( \sup_y |f(x - y)|^{1-p} \right) \int_{\mathbb{R}^n} |f(x - y)|^p dy,
\]
which implies that (3.2a) also holds for $0 < p < 1$. Hence, we have shown (3.2) in the case $q = \infty$. In the general case $0 < p \leq q < \infty$, one can interpolate $L^q$ between $L^p$ and $L^\infty$ and apply (3.2a) to get the result. □
Lemma 3.2 (Bergh and Löfström [3], Hörmander [16], Triebel [29]). Let $\Omega \subset \mathbb{R}^n$ be a compact set, $0 < r \leq \infty$. Define $\sigma_r = n(1/(r \wedge 1) - 1/2)$. Let $s > \sigma_r$. Then there exists a constant $C > 0$ such that
\begin{equation}
\|\mathcal{F}^{-1} \phi \mathcal{F} f\|_r \leq C \|\phi\|_{H^s} \|f\|_r
\end{equation}
holds for all $f \in L^r_\Omega$ and $\phi \in H^s$.

In this section, if there is no explanation, we always assume that $0 \leq \lambda < \infty$, $0 < p, q \leq \infty$. Define
\[ \Lambda = \{ \ell \in \mathbb{Z}^n : B(\ell, \sqrt{2n}) \cap B(0, \sqrt{2n}) \neq \emptyset \}. \]
It is easy to see that $B(0, \sqrt{2n})$ contains $O(n^{n/2})$ unit cubes, which implies that it overlaps $O(n^{n/2})$ balls like $B(\ell, \sqrt{2n})$ with $\ell \in \mathbb{Z}^n$. Hence, $\Lambda$ has at most $O(n^{n/2})$ elements. We give some basic properties on the spaces $E^\lambda_{p,q}$.

Proposition 3.3 (Completeness). $E^\lambda_{p,q}$ is a complete quasi-Banach space. In particular, if $1 \leq p, q \leq \infty$, then $E^\lambda_{p,q}$ is a complete Banach space.

Proof. The proof proceeds in the same way as in [29,32] and we omit the details of the proof. \hfill \Box

Proposition 3.4 (Equivalent norm). Let $\{\sigma_k\}_{k \in \mathbb{Z}^n}, \{\varphi_k\}_{k \in \mathbb{Z}^n} \in \mathcal{Y}$. Then $\{\sigma_k\}_{k \in \mathbb{Z}^n}$ and $\{\varphi_k\}_{k \in \mathbb{Z}^n}$ generate equivalent norms on $E^\lambda_{p,q}$.

Proof. Let us recall the following translation identity:
\begin{equation}
(F^{-1} m \mathcal{F} f)(x) = e^{i \xi k} \left[ F^{-1} m(\cdot + k) \mathcal{F} (e^{-i \xi k} f(y)) \right](x).
\end{equation}
For convenience, we write
\[ \Box_k^\sigma := F^{-1} \sigma_k \mathcal{F}, \quad \Box_k^\varphi := F^{-1} \varphi_k \mathcal{F}. \]
We have from (3.4) that
\begin{equation}
\|\Box_k^\sigma f\|_p \leq \sum_{\ell \in \Lambda} \|\Box_{k+\ell}^\varphi (\Box_k^\sigma f)\|_p.
\end{equation}
By (3.5), it is easy to see that
\begin{equation}
\|\Box_{k+\ell}^\varphi (\Box_k^\sigma f)\|_p = \|F^{-1} \sigma_k (\cdot + k) \varphi_{k+\ell} (\cdot + k) \mathcal{F} (e^{-i \xi k} f(y))\|_p.
\end{equation}
By (3.8) and Lemma 3.2, for $\Omega = \bigcup_{\ell \in \Lambda} B(\ell, \sqrt{2n})$, we have
\begin{equation}
\|\Box_{k+\ell}^\varphi (\Box_k^\sigma f)\|_p \leq \|\sigma_k\|_{H^s} \|\varphi_{k+\ell} f\|_p, \quad s > n(1/(1 \wedge p) - 1/2).
\end{equation}
By the fourth condition in (2.3), we see that $\|\sigma_k\|_{H^s}$ is uniformly bounded on $k \in \mathbb{Z}^n$. Hence, (3.7) and (3.9) imply that
\begin{equation}
\|\Box_k^\sigma f\|_p \leq \sum_{\ell \in \Lambda} \|\Box_{k+\ell}^\varphi f\|_p,
\end{equation}
from which we see that $\|f\|_{E^\lambda_{p,q}} \leq \|f\|_{E^\lambda_{p,q}}$. \hfill \Box
In view of Proposition 3.4, in the sequel we can always assume that \(\{\sigma_k\}_{k \in \mathbb{Z}^n}\) is defined by (2.1)–(2.2) if we only concern the topology property of the norm on \(E^\lambda_{p,q}\). The next proposition is an embedding between \(E^\lambda_{p_1,q_1}\) and \(E^\lambda_{p_2,q_2}\), which describes that the smoothness of \(E^\lambda_{p,q}\) is “monotone” on \(p, q \in (0, \infty]\). This property is quite different from the usual function spaces defined in whole \(\mathbb{R}^n\).

**Proposition 3.5 (Embedding).** Let \(0 \leq \lambda < \infty\), \(0 < p_1 \leq p_2 \leq \infty\), \(0 < q_1 \leq q_2 \leq \infty\). Then we have

\[
E^\lambda_{p_1,q_1}(\mathbb{R}^n) \subset E^\lambda_{p_2,q_2}(\mathbb{R}^n).
\]

**Proof.** Recall that

\[
\|\Box_k f\|_{p_2} \leq \sum_{\ell \in \Lambda} \|F^{-1}\sigma_k F(\Box_{k+\ell} f)\|_{p_2}.
\]

By Lemma 3.1 and similar to (3.9), we see that

\[
\|\Box_k f\|_{p_2} \leq C \sum_{\ell \in \Lambda} \|F^{-1}\sigma_k F(\Box_{k+\ell} f)\|_{p_1} \leq C \sum_{\ell \in \Lambda} \|\Box_{k+\ell} f\|_{p_1}.
\]

Multiplying (3.13) with \(2^\lambda\) and then taking the \(\ell^q\) norm in both sides of (3.13), we have

\[
\|f\|_{E^\lambda_{p_2,q_2}} \leq C \|f\|_{E^\lambda_{p_1,q_2}}.
\]

In view of \(\ell^q_1 \subset \ell^q_2\), we get the result, as desired. \(\square\)

**Proposition 3.6 (Infinite smoothness).** Let \(0 < \lambda < \infty\), \(0 < p, q \leq \infty\). Then we have

\[
E^\lambda_{p,q}(\mathbb{R}^n) \subset B^{[s_k]}_{p,q}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)
\]

if \(s_k \leq \lambda 2^k / 3k\), \(k \in \mathbb{N}\).

**Proof.** In [32], we have shown that \(B^{[s_k]}_{p,q}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)\). So, it suffices to show that \(E^\lambda_{p,q}(\mathbb{R}^n) \subset B^{[s_k]}_{p,q}(\mathbb{R}^n)\). For simplicity, we define \(a_k = (2^{k-1} - \sqrt{2n}) \lor 0\), \(b_k = 2^{k+1} + \sqrt{2n}\). We have from \(\text{supp} \sigma_i \subset \{\xi : |\xi - i| \leq \sqrt{2n}\}\) and \(\text{supp} \delta_k \subset \{\xi : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}\) that \(\Delta_k(\Box_i f) = 0\) if \(|i| \notin [a_k, b_k]\). Moreover, we easily see that the shell \(\{\xi : a_k \leq |\xi| \leq b_k\}\) contains at most \(O(2^{kn})\) unit cubes. Hence,

\[
2^{ks_k} \|\Delta_k f\|_p \leq 2^{ks_k} 2^{nk/p} \sum_{i \in \mathbb{Z}^n, |i| \in [a_k, b_k]} \|\Delta_k(\Box_i f)\|_p.
\]

By (3.4) we have

\[
\|\Delta_k(\Box_i f)\|_p \leq \sum_{\ell \in \Lambda} \|\Delta_k(\Box_{i+\ell} f)\|_p.
\]

Using the translation identity (3.5) and in a similar way as in (3.9),

\[
\|\Delta_k(\Box_{i+\ell} f)\|_p \leq \sup_{k \in \mathbb{Z}_+} \|\delta_k \sigma_{i+\ell}\|_{H^s} \|\Box_i f\|_p \leq C \|\Box_i f\|_p.
\]
Hence, it follows from (3.17) and (3.18) that
\[ 2^{qk_1} \| \Delta_k f \|^q_p \leq C 2^{qk_1} 2^{nqk(1+1/p)} \sum_{i \in \mathbb{Z}^n, |i| \in [a_k,b_k]} \| \Box_i f \|^q_p. \] (3.19)

Summarizing (3.19) over all \( k \in \mathbb{N} \cup \{0\} \), we have
\[ \| f \|_{B^q_{p,2}} \leq C 2^{qk_1} \sum_{k=0}^\infty 2^{qk_1+n+n/p} \sum_{i \in \mathbb{Z}^n, |i| \in [a_k,b_k]} \| \Box_i f \|^q_p. \] (3.20)

For sufficiently large \( k \), we have \((s_k+n+n/p)k < (2^{k-1} - \sqrt{2n})\). (3.20) implies the result, as desired. \( \square \)

In the case \( p = 2 \), we have a simple version for the norm on \( E^\perp_{p,q} \):

**Proposition 3.7 (Equivalent norm on \( E^\perp_{2,q} \)).** We have
\[ \| u \|_{E^\perp_{2,q}^q} \sim \left( \sum_{k \in \mathbb{Z}^n} 2^{qk_1} \| \Box_k f \|_2^q \right)^{1/q}. \] (3.21)

**Proof.** By definition and Plancherel’s identity,
\[ \| \Box_k f \|_2 = \| \sigma_k \hat{f} \|_2 \lesssim \sum_{\ell \in \Lambda} \| \Box_{k+\ell} \hat{f} \|_2, \] (3.22)
\[ \| \Box_k \hat{f} \|_2 \lesssim \| \sigma_k \hat{f} \|_2, \] (3.23)
which implies the result, as desired. \( \square \)

As indicated in the introduction, \( E^\perp_{2,1} \) is of importance for the well posedness of the NLS and CGL equations. We now give a comparison between \( E^\perp_{2,q} \) and \( H^s \), from which we can get that \( H^s \subset E^\perp_{2,1} \) for \( s > n/2 \), and \( H^s \not\subset E^\perp_{2,1} \) for \( s \leq n/2 \).

**Proposition 3.8.** We have
\[ H^s(\mathbb{R}^n) \subset E^\perp_{2,q}(\mathbb{R}^n), \quad s > n(1/q - 1/2), \ 0 < q < 2, \] (3.24)
\[ L^2(\mathbb{R}^n) = E^\perp_{2,2}(\mathbb{R}^n) \ (\text{equivalent norm}), \] (3.25)
\[ E^\perp_{2,q}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n), \quad s < n(1/q - 1/2), \ 2 < q \leq \infty. \] (3.26)

**Proof.** First, we prove (3.24). We see that there exist at most \( O(i^{n-1}) \) many \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \) satisfying \( |k_1| + \cdots + |k_n| = i \). By Proposition 3.7 and Hölder’s
We easily see that
\[
\|f\|_{E_{2,q}}^q \lesssim \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{-3q} \left\langle (1 + |\xi|^2)^{s/2} \mathcal{F}_k f \right\rangle_2^q
\]
\[
\lesssim \left( \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{-2sq/(2-q)} \right)^{(2-q)/2} \left( \sum_{k \in \mathbb{Z}^n} \| \mathcal{F}_k (1 + |\xi|^2)^{s/2} f \|^q_2 \right)^{2/q}
\]
\[
\lesssim \left( \sum_{i=0}^{\infty} \sum_{|k| = i} (1 + |k|)^{-2sq/(2-q)} \right)^{(2-q)/2} \| f \|_{H^s}^q.
\]
(3.27)

Since \( s > n(1/q - 1/2) \), we see that the series \( \sum_{i=0}^{\infty} (1 + i)^{-2sq/(2-q) - 1+n} \) is convergent. Hence, (3.24) holds.

Next, (3.25) is a straightforward consequence of Proposition 3.7 and Plancherel’s identity. Finally, we show (3.26). In an analogous way to (3.27), we have
\[
\|f\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^n} \| \mathcal{F}_k (1 + |\xi|^2)^{s/2} f \|^2_2
\]
\[
\lesssim \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{2s} \| \mathcal{F}_k \|^2_2
\]
\[
\lesssim \left( \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{2sq/(q-2)} \right)^{(q-2)/q} \left( \sum_{k \in \mathbb{Z}^n} \| \mathcal{F}_k \|^q_2 \right)^{2/q}
\]
\[
\lesssim \left( \sum_{i=0}^{\infty} (1 + i)^{2sq/(q-2) - 1+n} \right)^{(q-2)/q} \| f \|_{E_{2,q}}^2.
\]
(3.28)

Due to \( s < n(1/q - 1/2) \), we see that the series \( \sum_{i=0}^{\infty} (1 + i)^{2sq/(q-2) - 1+n} \) is a convergent series. (3.28) implies the result. \( \square \)

**Remark 3.9.** It is natural to ask if (3.24) still holds for \( s \leq n(1/q - 1/2) \). This is not expected. Let us observe the case \( n = 1 \). Let \( \phi \) be a smooth bump function adapted to the interval \([-1/2, 1/2]\), say \( \phi(\xi) = 1 \) if \( |\xi| \leq 1/4 \), and \( \phi(\xi) = 0 \) if \( |\xi| \geq 1/2 \). Put
\[
\mathcal{F}_k (\xi) = \frac{\phi(\xi - k)}{(1 + |k|) |\ln(1 + |k|)|}, \quad k \in \mathbb{Z} \setminus \{0\}, \quad \mathcal{F}_k (\xi) = 0 \forall \xi \in Q_0.
\]
(3.29)

We easily see that
\[
\| \mathcal{F}_k \|^2_2 \approx \frac{1}{(1 + |k| |\ln(1 + |k|)|}, \quad k \in \mathbb{Z} \setminus \{0\}.
\]
(3.30)
Due to the series
\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{(1 + |k|) \ln(1 + |k|)} = \infty,
\] (3.31)
we see that \( f \notin E^0_{2,q} \). On the other hand,
\[
\|(1 + |\xi|^2)^{(1/q-1/2)/2} \hat{f}\|^2_2 = \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{k-1/2}^{k+1/2} (1 + |\xi|^2)^{1/q-1/2} |\hat{f}(\xi)|^2 d\xi
\]
\[
\lesssim \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{(1 + |k|) \ln^2/q (1 + |k|)} \leq C,
\] (3.32)
that is \( f \in H^{1/q-1/2} \).

Thus, we see that (3.24) is a sharp embedding. Since there is no assumption on differentiability in the definition of \( E^0_{2,q} \), one may regard \( E^0_{2,q} \) as the lower regularity version of \( H^s \) \((s > n(1/q - 1/2))\).

Similarly, (3.26) is also a sharp embedding. Indeed, we can take
\[
\hat{f}(\xi) = \frac{\varphi(\xi - k)}{(1 + |k|)^{1/q} \ln^{1/2}(1 + |k|)}, \quad k \in \mathbb{Z} \setminus \{0\}, \quad \hat{f}(\xi) = 0 \quad \forall \xi \in Q_0,
\] (3.29a)
where \( \varphi \) is the same as in (3.29). Analogous to the argument above, one can show that \( f \notin H^{n(1/q-1/2)} \) but \( f \in E^0_{2,q} \) for any \( 2 < q \leq \infty \).

Even though \( H^{n(1/q-1/2)} \notin E^0_{2,q} \) for any \( 0 < q < 2 \), we still have the following:

**Proposition 3.10.** Let \( s > n(1/q - 1/2), \ 0 < q < 2 \). Then we have
\[
\|f\|_{E^0_{2,q}} \leq C \left( 1 + \left[ \ln(1 + \|f\|_{H^s}) \right]^{(1/q-1/2)} \|f\|_{H^{s(n(1/q-1/2))}} \right).
\] (3.33)

**Proof.** We use Brezis and Gallouet’s idea as in [6]. Let \( R > 1 \). By Proposition 3.7,
\[
\|f\|_{E^0_{2,q}}^q \leq \sum_{k \in \mathbb{Z}^a} \|\chi_{Q_k} \hat{f}\|_2^q
\]
\[
= \sum_{|k| \leq R} \|\chi_{Q_k} \hat{f}\|_2^q + \sum_{|k| > R} \|\chi_{Q_k} \hat{f}\|_2^q := I + II.
\] (3.34)
By Hölder’s inequality,

\[ I \lesssim \sum_{|k| \leq R} (1 + |k|)^{-n(1-q/2)} \left\| (1 + |\xi|^2)^{n(1/q - 1/2)}/2 \right\|_2^q \]
\[ \lesssim \left( \sum_{|k| \leq R} (1 + |k|)^{-n} \left\| (1 + |\xi|^2)^{n(1/q - 1/2)}/2 \right\|_2^2 \right)^{q/2} \]
\[ \lesssim \left( \sum_{0 \leq i \leq R} (1 + i)^{-1} \right)^{(2-q)/2} \|
\[ f \|_{H_n(1/q - 1/2)}^q \]
\[ \lesssim \left[ \ln(1 + R) \right]^{(1-q/2)} \|
\[ f \|_{H_n(1/q - 1/2)}^q \]. \tag{3.35} \]

On the other hand, following the proof of Proposition 3.8, one has that

\[ II \lesssim \left( \sum_{i > R} (1 + i)^{-2sq/(2-q) - 1+n} \right)^{(2-q)/2} \|
\[ f \|_{H^s}^q \]
\[ \leq C_s R^{-sq+n(1-q/2)} \|
\[ f \|_{H^s}^q \]. \tag{3.36} \]

Taking

\[ R = \max(1, \| f \|_{H^s}^{1/(s-n(1/q - 1/2))}), \tag{3.37} \]

from (3.35) and (3.36) we immediately get the result. \( \square \)

**Proposition 3.11.** Let \( 0 < q \leq p \wedge 1. \) We have

\[ E_{p,q}^0(\mathbb{R}^n) \subset B_{p,q}^0(\mathbb{R}^n). \tag{3.38} \]

**Proof.** We have

\[ \| f \|_{B_{p,q}^0}^q = \sum_{k=0}^\infty \| \Delta_k f \|_p^q. \tag{3.39} \]

**Case 1:** \( q < 1, \) \( p \geq 1. \) Let \( a_k \) and \( b_k \) be as in the proof of Proposition 3.6. It is easy to see that

\[ \| \Delta_k f \|_p^q \lesssim \left( \sum_{i \in \mathbb{Z}^n, |i| \in [a_k,b_k]} \| \Delta_k(\Box_i f) \|_p \right)^q \lesssim \sum_{i \in \mathbb{Z}^n, |i| \in [a_k,b_k]} \| \Box_i f \|_p^q. \tag{3.40} \]

Eq. (3.40) implies the result, as desired.
Case 2: $q < 1$, $p < 1$. We have from $p < 1$, $q/p \leq 1$ that
\[
\|\triangle_k f\|_p^q \leq \left( \sum_{i \in \mathbb{Z}^n, |i| \in [a_k, b_k]} \int |\triangle_k (\Box_i f)(x)|^p \, dx \right)^{q/p} \leq \sum_{i \in \mathbb{Z}^n, |i| \in [a_k, b_k]} \|\triangle_k (\Box_i f)\|_p^q. \tag{3.41}
\]

By (3.39) and (3.41), we have (3.38).

\[\square\]

Remark 3.12. By (3.37) and $B_{\infty, 1}^0 \subset L^\infty$, we have
\[
E_{\infty, 1}^0 \subset L^\infty. \tag{3.42}
\]
So, by Propositions 3.5 and 3.9,
\[
H^s \subset E_{2, 1}^0 \subset L^\infty, s > n/2. \tag{3.43}
\]

Eq. (3.43) means that $E_{2, 1}^0$ is an intermediate space between $H^s$ ($s > n/2$) and $L^\infty$.

One may ask if Proposition 3.11 still holds for the case $q > p \wedge 1$. Generally speaking, it does not hold; for instance, we consider the case $p = 2 < q$. Let $f \in S(\mathbb{R}^n)$ and $\text{supp} \hat{f} \subset B(0, 1/4)$. We write
\[
\hat{f}_k = e^{i k \cdot x} f(x), \quad k \in \mathbb{Z}^n. \tag{3.44}
\]

Put
\[
\Lambda_j = \left\{ k \in \mathbb{Z}^n : \frac{5}{4} 2^{j-1} \leq |\xi| \leq \frac{3}{4} 2^{j+1} \right\} \tag{3.45}
\]
and define
\[
F_j = \sum_{k \in \Lambda_j} f_k. \tag{3.46}
\]

One can easily verify that $\hat{f}_k = \hat{f}(-k)$ and $\text{supp} \hat{f}_k \subset Q_k$. Hence,
\[
\|F_j\|_{E_{2, q}^0}^q \lesssim \sum_{k \in \Lambda_j} \|Z_{Q_k} \hat{f}_k\|_2^q = (\#\Lambda_j) \|f\|_2^q, \tag{3.47}
\]
where $\#\Lambda_j$ denotes that $\Lambda_j$ has $\#\Lambda_j$ elements. On the other hand, we may assume that $\delta(\tilde{\xi}) > c$ if $5/8 \leq |\tilde{\xi}| \leq 6/4$, which implies that $\delta_j(\tilde{\xi}) \geq c$ if $5 \cdot 2^{j-1}/4 \leq |\tilde{\xi}| \leq 3 \cdot 2^{j+1}/4$. Hence,
\[
\|F_j\|_{B_{2, q}^0}^q \geq \left\| \delta_j \hat{F}_j \right\|_2^q \gtrsim \left\| \sum_{k \in \Lambda_j} \hat{f}_k \right\|_2^q = (\#\Lambda_j)^{q/2} \|f\|_2^q. \tag{3.48}
\]

Eqs. (3.47) and (3.48) imply that
\[
\|F_j\|_{B_{2, q}^0}^q \geq (\#\Lambda_j)^{q/2-1} \|F_j\|_{E_{2, q}^0}^q. \tag{3.49}
\]

Taking notice of $\lim_{j \to \infty} \#\Lambda_j = \infty$, by (3.49) we see that (3.37) does not hold if $q > p = 2$.  

Proposition 3.13. Let $0 < p, q \leq \infty$. Then we have
\[ S(\mathbb{R}^n) \subset E_{p,q}^0 \subset S'(\mathbb{R}^n), \]  
(3.50)
and $S(\mathbb{R}^n)$ is dense in $E_{p,q}^0$ for $0 < p, q < \infty$.

Proof. First, we show that $S(\mathbb{R}^n) \subset E_{p,q}^0$, whose proof follows Triebel [29, Theorem 2.3.3]. In fact, for any $f \in S(\mathbb{R}^n)$, if $L, N \gg 1$, then
\[
\|f\|_{E_{p,q}^0}^q = \sum_{k \in \mathbb{Z}^n} \|\Box_k f\|_p^q
\]
[\leq \sum_{k \in \mathbb{Z}^n} \|\mathcal{F}^{-1}(I - \Delta)^N \sigma_k \hat{f}\|_p^q
\]
[\leq \sum_{k \in \mathbb{Z}^n} \|\mathcal{F}^{-1}(I - \Delta)^N \sigma_k \hat{f}\|_{\infty}^q
\]
[\leq \sum_{k \in \mathbb{Z}^n} \|(I - \Delta)^N \sigma_k \hat{f}\|_1^q
\]
[\leq \sum_{k \in \mathbb{Z}^n} \left(1 + |k|^2\right)^{-Lq} \left(\sup_{\xi \in \mathbb{R}^n} \left(1 + |\xi|^2\right)^L \sum_{|\xi| \leq 2N} |D^\xi \hat{f}(\xi)|\right)^q
\]
[= \left(P_{L,2N}^0(\hat{f})\right)^q \sum_{k \in \mathbb{Z}^n} \left(1 + |k|^2\right)^{-Lq}. \]  
(3.51)
Taking $L$ as sufficiently large, we have
\[
\sum_{k \in \mathbb{Z}^n} \left(1 + |k|^2\right)^{-Lq} \leq 1 + \sum_{i=1}^{\infty} \sum_{|k| \in [2^{i-1}, 2^i)} \left(1 + |k|^2\right)^{-Lq}
\]
\[\leq 1 + \sum_{i=1}^{\infty} 2^{ni} 2^{-Lq(i-1)} \leq C, \]  
(3.52)
which implies that
\[
\|f\|_{E_{p,q}^0} \lesssim P_{L,2N}^0(\hat{f}). \]  
(3.53)
It follows from (3.53) that $S(\mathbb{R}^n) \subset E_{p,q}^0$. For any $f \in E_{p,q}^0$, we see that $f_N := \sum_{k \in \mathbb{Z}^n, |k| \leq N} \Box_k f$ converges to $f$ in $E_{p,q}^0$, $0 < p, q < \infty$. The second inclusion in (3.50) also follows Triebel [29] and we omit the details of the proof. \[\square\]

At the end of this section, we give a relation between Gevrey class and $E_{2,q}^\lambda$ and we show that Gevrey 1-class is the union of all $E_{2,q}^\lambda$ with $\lambda > 0$. 
Proposition 3.14. Let $G_1(\mathbb{R}^n)$ be the Gevrey 1-class. Let $0 < q \leq \infty$. We have

$$G_1(\mathbb{R}^n) = \bigcup_{\lambda > 0} E_{2, q}^{\lambda}(\mathbb{R}^n). \quad (3.54)$$

Proof. Case 1: $q = 1$. First, we show that for any $\lambda > 0$, $E_{2, 1}^{4\lambda}$ is a subset of $G_1$. For any $f \in E_{2, 1}^{4\lambda}$,

$$\| f \|_{\dot{H}^m} \lesssim \| \nabla^m f \|_{E_{2, 1}^{0}} \lesssim \sum_{k \in \mathbb{Z}^n} \| \chi_{Q_k} |\xi|^m \hat{f} \|_2 \leq \sum_{k \in \mathbb{Z}^n} (|k| + \sqrt{n/2})^m \| \chi_{Q_k} \hat{f} \|_2 \lesssim \frac{m!}{(2\lambda)^m} \sum_{k \in \mathbb{Z}^n} (2\lambda)^m |k|^m \frac{m!}{m!} \cdot 2^{4\lambda|k|} \| \chi_{Q_k} \hat{f} \|_2 \lesssim \frac{m!}{(2\lambda)^m} \| f \|_{E_{2, 1}^{4\lambda}}. \quad (3.55)$$

On the other hand, letting $f \in G_1$ and $L > n/2$, we have from Taylor’s expansion and Hölder’s inequality

$$\| f \|_{E_{2, 1}^{4\lambda}} = \sum_{k \in \mathbb{Z}^n} 2^{\lambda|k|} \| \chi_{Q_k} \hat{f} \|_2 = \sum_{m=0}^{\infty} \frac{\lambda \ln 2)^m}{m!} \sum_{k \in \mathbb{Z}^n} |k|^m \| \chi_{Q_k} \hat{f} \|_2 \lesssim \sum_{m=0}^{\infty} \frac{(\lambda \ln 2)^m}{m!} \sum_{k \in \mathbb{Z}^n} C^m |k|^{-L} \| \chi_{Q_k} \nabla^{m+L} f \|_2 \lesssim \sum_{m=0}^{\infty} \frac{(C\lambda)^m}{m!} \| f \|_{\dot{H}^{L+m}} \lesssim \sum_{m=0}^{\infty} \frac{(C\lambda)^m}{m!} (m + L)! \cdot \frac{\rho^m}{m!}. \quad (3.56)$$

Choosing $0 < \lambda \ll 1$, we see that the series $\sum_{m=0}^{\infty} (C\lambda/\rho)^m (m + L)!$ is convergent.

Hence, $f \in E_{2, 1}^{4\lambda}$, from which together with (3.55) we see that (3.54) holds for $q = 1$.

Case 2: For general $q \in (0, \infty]$, from the embedding (see below, (3.58))

$$E_{2, 1}^{4\lambda+\varepsilon} \subset E_{2, q}^{4\lambda} \subset E_{2, 1}^{4\lambda-\varepsilon}, \quad 0 < \varepsilon < \lambda, \quad (3.57)$$

we easily get the result. □
Proposition 3.15. Let $0 < \lambda, \epsilon < \infty$, $0 < p, q_1, q_2 \leq \infty$. We have
\[ E_{\lambda, q_1}^{\lambda + \epsilon}(\mathbb{R}^n) \subset E_{\lambda, q_2}^{\lambda}(\mathbb{R}^n). \tag{3.58} \]

Proof. By definition, we have
\[ \|f\|_{E_{\lambda, q_2}^{\lambda}}^q = \sum_{k \in \mathbb{Z}^n} 2^{\lambda q_2 |k|} \| \square_k f \|_p^q \]
\[ \leq \sum_{k \in \mathbb{Z}^n} 2^{-\epsilon q_2 |k|} \sup_{k \in \mathbb{Z}^n} \left( 2^{(\lambda + \epsilon) q_2 |k|} \| \square_k f \|_p^q \right) \]
\[ \leq \sum_{i=0}^{\infty} 2^{-\epsilon i q_2 |i| - 1} \| f \|_{E_{\lambda, \infty}^{\lambda + \epsilon}}^q \]
\[ \leq \| f \|_{E_{\lambda, q_1}^{\lambda + \epsilon}}^q, \tag{3.59} \]
which is (3.58), as desired. \(\square\)

4. Multi-linear estimate in $E_{\lambda, q}^{\lambda}$

In this section we show a bilinear estimate as in (2.17). The main technique is to extensively use the isometric decomposition to the frequency space. To estimate the norm of $uv$ in $E_{\lambda, q}^{\lambda}$, we can transmit the regularity quantity $2^{\lambda |k|}$ carried by $\| \square_k (uv) \|_p$ to $\| \square_i u \|_{p_1}$ and $\| \square_j v \|_{p_2}$ with the regularity $2^{\lambda |i|}$ and $2^{\lambda |j|}$, $|k - i - j| \leq 6 \sqrt{n}/2$, $1/p_1 + 1/p_2 = 1/p$. If we use exponential Besov spaces $B_{p,q}^{sk}$ as working spaces, then we need to apply the dyadic decomposition to the frequency space. The regularity quantity carried by $\| \Delta_k (uv) \|_p$ should be $2^{\lambda 2^k}$ ($k \in \mathbb{Z}_+$), which grows too fast to be transmitted to $\| \Delta_i u \|_{p_1}$ and $\| \Delta_j v \|_{p_2}$ with the regularity $2^{\lambda 2^i}$ and $2^{\lambda 2^j}$, respectively. We have the following:

Lemma 4.1. Let $0 \leq \lambda < \infty$, $0 < p \leq p_1, p_2 \leq \infty$, $0 < q \leq \infty$. If $1/p = 1/p_1 + 1/p_2$, then we have
\[ \| uv \|_{E_{p,q}^{\lambda}} \leq C 2^{C q^k \| u \|_{E_{p_1,q,1}^{\lambda}}} \| v \|_{E_{p_2,q,1}^{\lambda}}, \tag{4.1} \]
where $C$ is independent of $\lambda, q$; and if $p$ is fixed, then $C$ is also independent of $p_1, p_2$.

Proof. Recall that we denote by $k \in \mathbb{Z}^n$ that $k = (k_1, \ldots, k_n)$, $k_i \in \mathbb{Z}$, $1 \leq i \leq n$ and $|k| = |k_1| + \cdots + |k_n|$. We have
\[ \| uv \|_{E_{p,q}^{\lambda}}^q = \sum_{k \in \mathbb{Z}^n} 2^{\lambda |k| q} \| \square_k (uv) \|_p^q. \tag{4.2} \]
Considering the decomposition of $uv$,
\[ uv = \sum_{i,j \in \mathbb{Z}^n} (\square_i u)(\square_j v), \tag{4.3} \]
we have
\[ \Box_k(uv) = \sum_{i,j \in \mathbb{Z}^n} \Box_k(\Box_i u \Box_j v). \] (4.4)
It is easy to see that \( \mathcal{F}(\Box_i u \Box_j v) = (\sigma_i \hat{u}) \ast (\sigma_j \hat{v}) \) and
\[ \text{supp}(\sigma_i \hat{u}) \ast (\sigma_j \hat{v}) \subset \Omega := \{ \xi : |\xi - i - j| \leq 2\sqrt{n} \}. \] (4.5)
Indeed, in view of \( \text{supp} \sigma_i \subset B(i, \sqrt{2n}) \) we have
\[ (\sigma_i \hat{u}) \ast (\sigma_j \hat{v})(\xi) = \int_{|\eta - i| \leq \sqrt{2n}} (\sigma_i \hat{u})(\eta)(\sigma_j \hat{v})(\xi - \eta) \, d\eta. \] (4.6)
So, it follows from (4.6) that
\[ \text{supp}(\sigma_i \hat{u}) \ast (\sigma_j \hat{v}) \subset \{ \xi : |\xi - \eta - j| \leq \sqrt{2n}, |\eta - i| \leq \sqrt{2n} \}, \] (4.5a)
which implies that (4.5) holds. Hence, we have
\[ \Box_k(\Box_i u \Box_j v) = \mathcal{F}^{-1} \left[ \sigma_k \left( (\sigma_i \hat{u}) \ast (\sigma_j \hat{v}) \right) \chi_{\Omega} \right]. \] (4.7)
Taking notice of \( \text{supp} \sigma_k \subset B(k, \sqrt{2n}) \), we have \( \sigma_k \chi_{\Omega} \equiv 0 \) if \( |k - i - j| \geq 3\sqrt{2n} \). Thus,
\[ \left\| \mathcal{F}^{-1} \left[ \sigma_k \left( (\sigma_i \hat{u}) \ast (\sigma_j \hat{v}) \right) \chi_{\Omega} \right] \right\|_p = \left\| \mathcal{F}^{-1} \left[ \sigma_k \left( (\sigma_i \hat{u}) \ast (\sigma_j \hat{v}) \right) \chi_{\Omega} \right] \right\|_p \chi_{(|k-i-j| \leq 3\sqrt{2n})}. \] (4.8)
Using the fourth condition in (2.3), we see that if \( \text{supp} \hat{f} \subset B(0, 3\sqrt{2n}) \), then it follows from Lemma 3.2 that
\[ \left\| \mathcal{F}^{-1} \sigma_k \mathcal{F} f \right\|_p \leq C \sup_{k \in \mathbb{Z}^n} \| \sigma_k \|_{H^r} \| f \|_p \leq C \| f \|_p. \] (4.9)
By (4.8) and (4.9), together with the translation identity \( \mathcal{F}^{-1} m \cdot (+a) = e^{ixa} \mathcal{F}^{-1} m \), we have
\[ \left\| \mathcal{F}^{-1} \left[ \sigma_k \left( (\sigma_i \hat{u}) \ast (\sigma_j \hat{v}) \right) \chi_{\Omega} \right] \right\|_p \leq C \Box_i u \Box_j v \|_{p} \chi_{(|k-i-j| \leq 3\sqrt{2n})}. \] (4.10)
Case 1: We consider the case \( q \leq 1 \). By (4.2), (4.7), and (4.10), we have
\[ \| uv \|_{E_{p,q}^q}^q \leq \sum_{k \in \mathbb{Z}^n} \sum_{i,j \in \mathbb{Z}^n} \| \Box_k(\Box_i u \Box_j v) \|_p^q \leq \sum_{k \in \mathbb{Z}^n} \sum_{i,j \in \mathbb{Z}^n} 2^{jq} \| \Box_i u \Box_j v \|_p^q \chi_{(|k-i-j| \leq 3\sqrt{2n})}. \] (4.11)
By Hölder’s inequality,
\[ \| \Box_i u \Box_j v \|_p \leq \| \Box_i u \|_{p_1} \| \Box_j v \|_{p_2}. \] (4.12)
Taking notice of (4.12), one has that
\[ 2^{jq}|k| \|\square_i u \square_j v\|_p^q \mathcal{L}_{|k-i-j| \leq 3\sqrt{2n}} \leq 2^{jq}|k-i-j| \mathcal{L}_{|k-i-j| \leq 3\sqrt{2n}} \left(2^{jq|i|} \|\square_i u\|_{p_1}^q \right) \left(2^{jq|j|} \|\square_j v\|_{p_2}^q \right). \]
(4.13)

From Young’s inequality,
\[ \| (a_i) \ast (b_j) \ast (c_k) \|_{\ell^1} \leq \|a_i\|_{\ell^1} \|b_j\|_{\ell^1} \|c_k\|_{\ell^1}, \]
(4.14)
we immediately have from (4.11) and (4.13) that
\[ \sum_{k \in \mathbb{Z}^n} \sum_{i,j \in \mathbb{Z}^n} 2^{jq}|k| \|\square_i u \square_j v\|_p^q \mathcal{L}_{|k-i-j| \leq 3\sqrt{2n}} \leq C 2^{jq} \|u\|_{E_{p_1,q}^\lambda}^q \|v\|_{E_{p_2,q}^\lambda}^q. \]
(4.15)

Eqs. (4.11) and (4.15) imply the result.

Case 2: We consider the case \( q > 1 \). By (4.2), (4.7), and (4.10), using Minkowski’s inequality, we have
\[ \|uv\|_{E_{p,q}^\lambda} \leq \left\{ \sum_{k \in \mathbb{Z}^n} 2^{jq}|k| \left( \sum_{i,j \in \mathbb{Z}^n} \|\square_k (\square_i u \square_j v)\|_p \right) \right\}^{1/q} \]
\[ \leq C \sum_{i,j \in \mathbb{Z}^n} \|\square_i u\|_{p_1} \|\square_j v\|_{p_2} \left( \sum_{k \in \mathbb{Z}^n} 2^{jq}|k| \mathcal{L}_{|k-i-j| \leq 3\sqrt{2n}} \right)^{1/q} \]
\[ \leq C 2^{C\lambda} \sum_{i,j \in \mathbb{Z}^n} 2^{j|i|+j|j|} \|\square_i u\|_{p_1} \|\square_j v\|_{p_2} \]
\[ \leq C 2^{C\lambda} \|u\|_{E_{p_1,1}^\lambda} \|v\|_{E_{p_2,1}^\lambda}, \]
(4.16)
which implies (4.1). Recall that the constant \( C \) in (4.9), and so in (4.1), depends on \( p \). But if \( p \) is a fixed index, then \( C \) in (4.9) is also invariant. Hence, by (4.12) we see that \( C \) in (4.1) is independent of \( p_1, p_2 \). \( \Box \)

In view of Lemma 4.1, we can get a multi-linear estimate in \( E_{p,q}^\lambda \).

**Corollary 4.2.** Let \( 0 \leq \lambda < \infty, 0 < p \leq p_i \leq \infty, i = 1, \ldots, N, 0 < q \leq \infty \). If \( 1/p = 1/p_1 + \cdots + 1/p_N \), then we have
\[ \left\| \prod_{i=1}^N u_i \right\|_{E_{p,q}^\lambda} \leq C N 2^{C \lambda N} \prod_{i=1}^N \|u_i\|_{E_{p_i,q_N}^\lambda}, \]
(4.17)
where \( C > 0 \) is independent of \( \lambda, N \).
We define $D$.

Proof. Let $1/p = 1/p_1 + 1/p^*$. We see that $1/p^* = 1/p_2 + \cdots + 1/p_N$. By Lemma 4.1 we have

$$\left\| \prod_{i=1}^N u_i \right\|_{E_{p,q}^{\lambda}} \leq C 2^{\lambda} q^j \left\| u_1 \right\|_{E_{p_1,q}^{\lambda}} \left\| \prod_{i=2}^N u_i \right\|_{E_{q,q}^{\lambda}}. \quad (4.18)$$

By induction, from (4.18) we easily see that (4.17) holds. \hfill \Box

Remark 4.3. We are expecting that (4.1) can be slightly improved by

$$\|uv\|_{E_{p,q}^{\lambda}} \leq \|u\|_{E_{p_1,q}^{\lambda}} \|v\|_{E_{p_2,q}^{\lambda}}. \quad (4.19)$$

But this is impossible and (4.19) does not hold for $p = q = 2$, $\lambda = 0$. In fact, if (4.19) holds for $p = q = 2$, it follows from Proposition 3.5 that

$$\|uv\|_{E_{p,q}^{0}} \leq \|u\|_{E_{p_1,q}^{0}} \|v\|_{E_{p_2,q}^{0}} \leq \|u\|_{E_{p_1,q}^{0}} \|v\|_{E_{p_2,q}^{0}}. \quad (4.20)$$

By Proposition 3.7, $E_{p,q}^{0}$ is a Banach algebra, but this is impossible.

5. Estimates for the Ginzburg–Landau and Schrödinger groups

Put

$$U_1(t) = \mathcal{F}^{-1} \exp(-t P(\xi)) \mathcal{F}, \quad P(\xi) = 1 + (a + i \alpha) |\xi|^2, \quad a > 0, \alpha \in \mathbb{R}. \quad (5.1)$$

Our aim is to derive the estimate of $U_1(t)$ in the spaces $E_{p,q}^{\lambda}$. We assume $r, p, q$ satisfying

$$0 < r \leq p \leq \infty, \quad 0 < q \leq \infty. \quad (5.2)$$

We define $D_j = \partial_j / \partial x_j$ and, if there is no confusion, we simply write $D_j = D_j$. By the translation identity (3.5), we have for $\gamma = 0, 1$,

$$\left\| \Box_k D_j \gamma U_1(t) f \right\|_p \leq \sum_{\ell \in \Lambda} \left\| \mathcal{F}^{-1} \xi_i^j \sigma_k \sigma_{k+\ell} \exp(-t P(\xi)) \mathcal{F} f \right\|_p \leq \sum_{\ell \in \Lambda} \left\| \mathcal{F}^{-1} (\xi_i + k_i)^j \sigma_0 \sigma_{\ell} \exp(-t P(\xi + k)) \mathcal{F}(e^{-iky} f(y)) \right\|_p. \quad (5.3)$$

Let $\Omega = \cup_{\ell \in \Lambda} B(\ell, \sqrt{2n})$ in Lemmas 3.1 and 3.2. It is easy to see that $\text{supp } \sigma_\ell \subset \Omega$, $\ell \in \Lambda$. By (5.3) and Lemma 3.1, we have

$$\left\| \Box_k D_j \gamma U_1(t) f \right\|_p \leq \sum_{\ell \in \Lambda} \left\| \mathcal{F}^{-1} (\xi_i + k_i)^j \sigma_0 \sigma_{\ell} \exp(-t P(\xi + k)) \mathcal{F}(e^{-iky} f(y)) \right\|_r. \quad (5.4)$$

Taking $L \in \mathbb{N}$, $L > n(1/(1 + r) - 1/2)$, and applying Lemma 3.2, we have from (5.4) that

$$\left\| \Box_k D_j \gamma U_1(t) f \right\|_p \leq \sum_{\ell \in \Lambda} \left\| (\xi_i + k_i)^j \sigma_\ell \exp(-t P(\xi + k)) \right\|_{H^L} \left\| \Box_k f \right\|_r. \quad (5.5)$$
In the following we give the estimate of \( \| (\zeta_i + k_i)^\gamma \sigma \xi e^{-tP(\xi + k)} \|_{H^L} \). In view of the Leibnitz rule we have

\[
D^L \left( \sigma \xi e^{-tP(\xi + k)} \right) \leq C \sum_{L_1 + L_2 = L} \left| D^{L_1} \sigma \xi D^{L_2} e^{-tP(\zeta + k)} \right|
\]

\[
\leq C \sum_{0 \leq s \leq L} \left| D^s e^{-tP(\zeta + k)} \right| \mathcal{A}_\Omega. \quad (5.6)
\]

It is easy to see that

\[
D^s e^Q(\xi) = e^Q(\xi) \left\{ \sum_{q=1}^s \sum_{\Lambda_q^*} C_{\beta} \prod_{i=1}^q D^{\beta_i} Q(\xi) \right\}, \quad (5.7)
\]

where \( \Lambda_q^* = (\beta_1 + \cdots + \beta_q = s, \beta_1, \ldots, \beta_q \geq 1) \), \( \beta = (\beta_1, \ldots, \beta_q) \). Letting \( Q(\xi) = -tP(\xi) \) in (5.7), one easily sees that

\[
|D^{\beta_i} Q(\xi)| \leq Ct(1 + |\xi + k|^2), \quad k \in \mathbb{Z}. \quad (5.8)
\]

Hence, by (5.7) and (5.8),

\[
\left| D^s e^Q(\xi) \right| \leq e^{-ct(1 + |\xi + k|^2)} \sum_{q=1}^s (t(1 + |\xi + k|^2))^q \leq e^{-ct(1 + |\xi + k|^2)}. \quad (5.9)
\]

Combining (5.6) and (5.9), we have

\[
\| \sigma \xi e^{-tP(\xi + k)} \|_{H^L} \leq Ce^{-ct(1 + |k|^2)}. \quad (5.10)
\]

Similarly, if \( \gamma = 1 \), then

\[
D^L \left( (\zeta_i + k_i)\sigma \xi e^{-tP(\xi + k)} \right) \leq \left| (\zeta_i + k_i)D^L \left( \sigma \xi e^{-tP(\zeta + k)} \right) \right| + C \left| D^{L-1} \left( \sigma \xi e^{-tP(\zeta + k)} \right) \right|
\]

\[
\leq Ce(1 + |\xi + k|) e^{-ct(1 + |\xi + k|^2)} \leq Ct^{-1/2} e^{-ct(1 + |k|^2)}. \quad (5.11)
\]

Eq. (5.11) implies that

\[
\| (\zeta_i + k_i)\sigma \xi e^{-tP(\xi + k)} \|_{H^L} \leq Ct^{-1/2} e^{-ct(1 + |k|^2)}. \quad (5.12)
\]

Collecting (5.5), (5.10) and (5.12), we have

\[
e^{-ct(1 + |k|^2)} \| \Box_k D^j U_1(t) f \|_p \leq Ct^{-\gamma/2} \| \Box_k f \|_r, \quad k \in \mathbb{Z}. \quad (5.13)
\]
Taking $\ell^q$ norm in both sides of (5.13), we have
\[
\|D^\gamma U_1(t)f\|_{E^{ct}_{p,q}} \leq Ct^{-\gamma/2}\|f\|_{E^0_{r,q}}, \quad \gamma = 0, 1.
\] (5.14)

So, we have shown the following:

**Proposition 5.1.** Let $U_1(t)$ be as in (5.1) and $r, p, q$ be as in (5.2). Then there exists $c > 0$, $C > 0$ such that (5.14) holds for all $f \in E^0_{r,q}$. In particular,
\[
\|D^\gamma U_1(t)f\|_{E^{ct}_{p,q}} \leq Ct^{-\gamma/2}\|f\|_{E^0_{r,q}}, \quad \gamma = 0, 1.
\] (5.15)

We write
\[
A_1 = \int_0^t U_1(t - \tau)\,d\tau.
\] (5.16)

By (5.13) one has that for $1 \leq p \leq \infty$, $0 < r \leq p$,
\[
\|\Box k D^\gamma A_1 f\|_p \leq C \int_0^t (t - \tau)^{-\gamma/2}e^{-c(t-\tau)(1+|k|^2)}\|\Box k f(\tau)\|_r\,d\tau, \quad k \in \mathbb{Z}.
\] (5.17)

It follows from (5.17) that
\[
e^{ct|k|}\|\Box k D^\gamma A_1 f\|_p \leq C \int_0^t (t - \tau)^{-\gamma/2}e^{ct|k|}\|\Box k f(\tau)\|_r\,d\tau, \quad k \in \mathbb{Z}.
\] (5.18)

Let $q > 1$. Taking $\ell^q$ norm in both sides of (5.18) and using Minkowski’s inequality, we obtain that
\[
\|D^\gamma A_1 f\|_{E^{ct}_{p,q}} \leq C \left( \sum_{k \in \mathbb{Z}} \left( \int_0^t (t - \tau)^{-\gamma/2}e^{ct|k|}\|\Box k f(\tau)\|_r\,d\tau \right)^q \right)^{1/q}
\[
\leq C \int_0^t (t - \tau)^{-\gamma/2}\|f(\tau)\|_{E^{ct}_{r,q}}\,d\tau.
\] (5.19)

Hence, we have

**Proposition 5.2.** Let $U_1(t)$ be as in (5.1) $1 \leq p, q \leq \infty$, $0 < r \leq p$. Then there exists $c > 0$, $C > 0$ such that
\[
\|D^\gamma A_1 f\|_{E^{ct}_{p,q}} \leq C \int_0^t (t - \tau)^{-\gamma/2}\|f(\tau)\|_{E^{ct}_{r,q}}\,d\tau, \quad \gamma = 0, 1.
\] (5.20)

In particular,
\[
\|D^\gamma A_1 f\|_{E^{0}_{p,q}} \leq C \int_0^t (t - \tau)^{-\gamma/2}\|f(\tau)\|_{E^{0}_{r,q}}\,d\tau, \quad \gamma = 0, 1.
\] (5.21)
Let $U(t) = e^{t(a+it)\Delta}$ and $A = \int_0^t U(t-\tau) \, d\tau$ be as in (1.5). We see that $U_1(t) = e^{-t}U(t)$. Hence, in view of Proposition 5.1 and 5.2, we have

**Proposition 5.3.** Let $r, p, q$ be as in Proposition 5.2. Then there exists $c > 0, C > 0$ such that

\[
\| D^{\gamma} U(t) \varphi \|_{E^{r,q}_{p,q}} \leq C e^{t} t^{-\gamma/2} \| \varphi \|_{E^{0,q}_{r,q}}, \quad \gamma = 0, 1. \tag{5.22}
\]

\[
\| D^{\gamma} A f \|_{E^{r,q}_{p,q}} \leq C e^{t} \int_0^t (t-\tau)^{-\gamma/2} \| f(\tau) \|_{E^{r,q}_{p,q}} \, d\tau, \quad \gamma = 0, 1. \tag{5.23}
\]

In particular, (5.22) and (5.23) also hold for $c = 0$.

**Remark 5.4.** In Propositions 5.1 and 5.2, if we replace $t = 0$ by $t = t_0$, then for $\gamma = 0, 1$,

\[
\| D^{\gamma} U_1(t-t_0) \varphi \|_{E^{c(t-t_0)}_{p,q}} \leq C (t-t_0)^{-\gamma/2} \| \varphi \|_{E^{0,q}_{r,q}}, \tag{5.24}
\]

\[
\left\| D^{\gamma} \int_{t_0}^t U_1(t-\tau) f(\tau) \, d\tau \right\|_{E^{c(t-t_0)}_{p,q}} \leq C \int_{t_0}^t (t-\tau)^{-\gamma/2} \| f(\tau) \|_{E^{c(t-t_0)}_{r,q}} \, d\tau, \tag{5.25}
\]

where $C$ is independent of $t_0$.

Next, we consider the estimates of the Schrödinger group $S(t) = e^{it\Delta}$. If $1 \leq r \leq 2 \leq p \leq \infty$, in view of Young’s and Hölder’s inequalities, one has that

\[
\| \Box^k S(t) f \|_p \leq \sum_{\ell \in \Lambda} \| \mathcal{F}^{-1} \sigma_k \sigma_{k+\ell} \exp(-it|\xi|^2) \mathcal{F} f \|_p.
\]

\[
\leq \sum_{\ell \in \Lambda} \| \sigma_{k+\ell} \exp(-it|\xi|^2) \sigma_k \mathcal{F} f \|_{p'}
\]

\[
\leq C \| \sigma_k \mathcal{F} f \|_{p'} \leq C \| \Box^k f \|_r, \tag{5.26}
\]

where $1/p + 1/p' = 1$ and $1/r + 1/r' = 1$. By Lemma 3.1 and (5.26), if $0 < r < 1$, then

\[
\| \Box^k S(t) f \|_p \leq C \| \Box^k f \|_{1} \leq C \| \Box^k f \|_r. \tag{5.27}
\]

Eqs. (5.26) and (5.27) imply that for $0 < r \leq 2 \leq p \leq \infty$,

\[
\| \Box^k S(t) f \|_p \leq C \| \Box^k f \|_r. \tag{5.28}
\]

Taking $\ell^q$ norm in (5.28), we have

\[
\| S(t) f \|_{E^{0}_{p,q}} \leq C \| f \|_{E^{0}_{r,q}}. \tag{5.29}
\]

Hence, we obtain

**Proposition 5.5.** Let $S(t) = \exp(it\Delta)$, $0 < r \leq 2 \leq p \leq \infty$, $0 < q \leq \infty$. Then we have (5.29).
Recall that in the proof of (5.29), we only use the fact that the symbol \( \exp(-it|\xi|^2) \) of \( S(t) \) is bounded for \((t, \xi) \in \mathbb{R} \times \mathbb{R}^n\). Hence, the method above is also useful for some other semi-groups.

6. Local well posedness of the NLS equation in \( E_{2,1}^0 \)

In this section we prove our Theorems 1.1 and 1.2. It seems that the proof of Theorem 1.1 is easier than that of Theorem 1.2 and, so, we only give the details of the proof of Theorem 1.2. As indicated in the introduction, we will solve (1.11) by using its integral form

\[
u(t) = S(t)u_0 - i\mu A_S \left( (e^{\bar{u}}|u| - 1)u \right), \quad (6.1)
\]

where \( A_S := \int_0^t S(t - \tau) d\tau \). By (1.16), we have from Corollary 4.2 that

\[
\| (e^{\bar{u}}|u|^2 - 1)u \|_{E_{2,1}^0} \leq \sum_{k=1}^{\infty} \frac{\varrho^k}{k!} \|u\|_{E_{2,1}^0}^{2k+1} \|u\|_{E_{2,1}^0} \leq \sum_{k=1}^{\infty} \frac{\varrho^k}{k!} C^{2k+1} \|u\|_{E_{(2k+1),1}^0}^{2k+1}. \quad (6.2)
\]

By Proposition 3.5, we have \( E_{2,1}^0 \subset E_{(2k+1),1}^0 \), more precisely,

\[
\| u \|_{E_{(2k+1),1}^0} \leq C \| u \|_{E_{2,1}^0} \quad (6.3)
\]

and the constant \( C \) in (6.3) is independent of \( k \geq 1 \). Indeed, in view of \( E_{2,1}^0 \subset E_{\infty,1}^0 \) and

\[
\| u \|_{E_{\infty,1}^0} \leq C_{\infty} \| u \|_{E_{2,1}^0}, \quad (6.4)
\]

we have for any \( p \in [2, \infty] \),

\[
\| u \|_{E_{p,1}^0} = \sum_{k \in \mathbb{Z}_n} \| \Box_k u \|_p \\
\leq \sum_{k \in \mathbb{Z}_n} \| \Box_k u \|_{E_{\infty,1}^0}^{1-2/p} \| \Box_k u \|_{E_{2,1}^0}^{2/p} \\
\leq \| u \|_{E_{\infty,1}^0}^{1-2/p} \| u \|_{E_{2,1}^0}^{2/p} \leq C_{\infty}^{1-2/p} \| u \|_{E_{2,1}^0}^{2k+1}. \quad (6.5)
\]

So, we see that \( C \) in (6.3) is a universal constant. In view of (6.2) and (6.3),

\[
\| (e^{\bar{u}}|u|^2 - 1)u \|_{E_{2,1}^0} \leq \sum_{k=1}^{\infty} \frac{(C\varrho)^k}{k!} \| u \|_{E_{2,1}^0}^{2k+1}. \quad (6.6)
\]
Proof of Theorem 1.2. We define a metric space as follows:

\[ \mathcal{D} = \{ u : \| u \|_{C(0,T;E^0_{2,1})} \leq M \}, \]  

(6.7)

\[ d(u, v) = \| u - v \|_{C(0,T;E^0_{2,1})}. \]  

(6.8)

By Proposition 5.5,

\[ \| S(t)u_0 \|_{C(0,T;E^0_{2,1})} \leq C \| u_0 \|_{E^0_{2,1}}. \]  

(6.9)

By (6.6) and Proposition 5.5,

\[ \| \mathcal{A}_S \left( (e^{g|u|^2} - 1)u \right) \|_{C(0,T;E^0_{2,1})} \leq CT \sum_{k=1}^{\infty} \frac{(Cg)^k}{k!} \| u \|^{2k+1}_{C(0,T;E^0_{2,1})}. \]  

(6.10)

Let us consider the mapping

\[ \mathcal{T} : u(t) \rightarrow S(t)u_0 - i\mu \mathcal{A}_S \left( (e^{g|u|^2} - 1)u \right). \]  

(6.11)

We show that \( \mathcal{T} : (\mathcal{D}, d) \rightarrow (\mathcal{D}, d) \) is a contraction mapping. Indeed, for any \( u \in \mathcal{D} \), by (6.9) and (6.10) we have

\[ \| \mathcal{T}u \|_{C(0,T;E^0_{2,1})} \leq C \| u_0 \|_{E^0_{2,1}} + CT \sum_{k=1}^{\infty} \frac{(Cg)^k}{k!} M^{2k+1}. \]  

(6.12)

Put \( M = 2C \| u_0 \|_{E^0_{2,1}} \). Since

\[ \sum_{k=1}^{\infty} \frac{(Cg)^k}{k!} M^{2k} \leq C_M \]  

(6.13)

is a convergent series, we have

\[ \| \mathcal{T}u \|_{C(0,T;E^0_{2,1})} \leq M/2 + CT C_M M. \]  

(6.14)

Let \( T \) satisfy \( CT C_M \leq 1/2 \). It follows from (6.14) that \( \mathcal{T}u \in \mathcal{D} \). Similarly,

\[ \| \mathcal{T}u - \mathcal{T}v \|_{C(0,T;E^0_{2,1})} \leq \frac{1}{2} \| u - v \|_{C(0,T;E^0_{2,1})}. \]  

(6.15)

Hence, by Banach’s contraction mapping principle, we see that \( \mathcal{T} \) has a fixed point \( u \in \mathcal{D} \), which is a solution of Eq. (6.1). We can extend this solution step by step and finally find a maximal \( T^* > 0 \) satisfying (1.9) and (1.10). The uniqueness of such solutions can also be shown in a standard way; cf. [12]. This finishes the proof of Theorem 1.2. \( \square \)

7. Uniqueness and regularity of the solutions for the CGL equation

We consider the Cauchy problem for the complex Ginzburg–Landau equation (CGL). For our purpose we rewrite the CGL equation in the following way:

\[ u_t + (1 - (a + i\alpha)\Delta)u + (b + i\beta)|u|^{2k}u + \mu u = 0, \quad u(0, x) = u_0(x), \]  

(7.1)
where $a > 0$, $b > 0$, $\alpha$, $\beta$, $\mu \in \mathbb{R}$, $\kappa \in \mathbb{N}$. Let us consider the integral version of (7.1):

$$u(t) = U_1(t)u_0 - \mathcal{A}_1 f(u), \quad (7.2)$$

where $\mathcal{A}_1$ is defined by (5.16) and $f(u) := (b + i\beta)|u|^{2\kappa}u + \mu u$.

**Proof of Theorem 1.3.** Since the estimates of the Ginzburg–Landau group is much better than that of the Schrödinger group, we see that the results of Theorem 1.1 also hold for Eq. (7.2), i.e. (7.2) has a unique solution,

$$u \in C_{\text{loc}}([0, T^*); E_{2,1}^0), \quad (7.3)$$

satisfying (1.10).

Next, we show the regularity of the solutions. For any $0 < T < T^*$, we write

$$M_0 := \sup_{0 \leq t \leq T} \|u(t)\|_{E_{2,1}^0}.$$

Using Proposition 5.1, we have for any $t_0 \in (0, T)$,

$$\sup_{0 \leq t \leq t_0} \|U_1(t)u_0\|_{E_{2,1}^T} \leq C \|u_0\|_{E_{2,1}^0} \leq CM_0. \quad (7.4)$$

By Proposition 5.2, we see that

$$\|\mathcal{A}_1 f(u)\|_{E_{2,1}^T} \leq C \int_0^T (\|u\|^{2\kappa}u(\tau)\|_{E_{r,1}^T} + \|u(\tau)\|_{E_{2,1}^T}) \, d\tau. \quad (7.5)$$

By Corollary 4.2, for $r = 2/(2\kappa + 1)$,

$$\|u\|^{2\kappa}u\|_{E_{r,1}^T} \leq C 2^C \|u\|^{2\kappa+1}_{E_{2,1}^T}. \quad (7.6)$$

Hence, it follows from (7.5) and (7.6) that

$$\sup_{0 \leq t \leq t_0} \|\mathcal{A}_1 f(u(t))\|_{E_{2,1}^T} \leq C t_0 \sup_{0 \leq t \leq t_0} \left(2^C \|u(t)\|_{E_{2,1}^T}^{2\kappa+1} + \|u(t)\|_{E_{2,1}^T} \right). \quad (7.7)$$

Applying in the same way as above, we can show that for some $t_0 \in (0, T]$, $\mathcal{T}$ is a contraction mapping from

$$\mathcal{D}_1 = \left\{ u : \sup_{0 \leq t \leq t_0} \|u(t)\|_{E_{2,1}^T} \leq M \right\}, \quad d(u, v) = \sup_{0 \leq t \leq t_0} \|u(t) - v(t)\|_{E_{2,1}^T}, \quad (7.8)$$

into itself. Hence, the solution $u$ satisfies

$$\sup_{0 \leq t \leq t_0} \|u(t)\|_{E_{2,1}^T} \leq 2CM_0. \quad (7.9)$$

Now let $[0, T] = \bigcup_{k=1}^K [(k - 1)t_0, kt_0]$. Using Remark 5.4 and noticing that $M_0$ is the upper bound of $\|u(t)\|_{E_{2,1}^T}$ in $[0, T]$, we can repeat the procedure above from $[0, t_0]$ to $[t_0/2, 3t_0/2]$ and get that

$$\sup_{t_0/2 \leq t \leq 3t_0/2} \|u(t)\|_{E_{2,1}^T} \leq 2CM_0, \quad (7.10)$$

which implies that

$$\sup_{t_0 \leq t \leq 3t_0/2} \|u(t)\|_{E_{2,1}^T} \leq 2CM_0. \quad (7.11)$$
We can continuously repeat the argument above from time interval \([t_0/2, 3t_0/2]\) to \([t_0, 2t_0]\), \([3t_0/2, 5t_0/2]\), ..., and finally show that

\[ \|u(t)\|_{L^{2,1}_r} \leq 2CM_0, \quad t_0/2 \leq t \leq T. \]  

(7.12)

Eqs. (7.9) and (7.12) imply the regularity result.

Finally, we show that the solution is a global one. Let \(T < T^*\). We have \(u(T) \in E^c_{2,1}\) for some \(c_0 > 0\). Due to \(E^c_{2,1} \subset E^{c_0}_{r,1} \subset B^{K}_{r,1} \subset H^K_r\) for some sequence \(s_k \to \infty\) and for all \(r > 2, K \in \mathbb{N}\), we see that \(u(T) \in H^K_r\) for any \(K \in \mathbb{N}\). We now rewrite (7.2) in the following form:

\[ u(t) = U_1(t - T)u(T) - \int_T^t U_1(t - \tau)f(u(\tau))d\tau. \]  

(7.13)

Taking \(r = n\kappa\), we see that \(L^r\) is just the critical space for Eq. (7.1); cf. Ginibre and Velo [12].

**Case 1**: we consider the case \(r = n\kappa \geq 2\). Recall that Ginibre and Velo [12] showed that if \(u(T) \in L^r, \ |x|(r - 2)/a < 2\sqrt{r - 1}\), then (7.13) has a unique solution

\[ u \in C_{loc}([T, \infty); L^r) \cap L^{r+2\kappa}_{loc}(T, \infty; L^{r+2\kappa}). \]  

(7.14)

Using the standard regularity technique as in [32], we see that \(u(T) \in H^K_r\) implies that

\[ u \in C_{loc}([T, \infty); H^K_r) \cap L^{r+2\kappa}_{loc}(T, \infty; H^{r+2\kappa}). \]  

(7.15)

Further, since \(E^c_{2,1} \subset H^s\) for any \(s > 0\), we have \(u(T) \in H^s\). By (7.13),

\[ \|u(t)\|_{H^s} \leq \|u(T)\|_{H^s} + \int_T^t \|f(u(\tau))\|_{H^s}d\tau. \]  

(7.16)

We may assume \(s \in \mathbb{N}\) and \(s > n/2\). It is easy to see that

\[ \|f(u)\|_{H^s} \leq C\|u\|_{H^s}(1 + \|u\|_{H^K_r}^{2\kappa}), \quad K \gg 1. \]  

(7.17)

Collecting (7.15)–(7.17) and using Gronwall’s inequality, we have

\[ u \in C_{loc}([T, \infty); H^s). \]  

(7.18)

In view of \(H^s \subset E^0_{2,1}\), we have \(u \in C_{loc}([T, \infty); E^0_{2,1})\). This means that \(T^* = \infty\).

**Case 2**: \(r = n\kappa < 2\). Ginibre and Velo [12] also showed that if \(u(T) \in L^2\), then (7.13) has a unique solution

\[ u \in C_{loc}([T, \infty); L^2) \cap L^{2+2\kappa}_{loc}(T, \infty; L^{2+2\kappa}). \]  

(7.19)

So, we can use the same way as in Case 1 to get the result. \(\square\)

8. CGL equation with exponential nonlinearity

In this section we prove our Theorem 1.4, whose proof is based on the \(L^p - L^r\) estimate of \(U(t) = e^{(a+13)\alpha} \), \(a > 0\). For any \(1 \leq r \leq p \leq \infty\), we have

\[ \|U(t)f\|_p \leq Ct^{-n(1/r-1/p)/2}\|f\|_r, \]  

(8.0)
where $C > 0$ is a universal constant that is independent of $p, r$, and $t$, cf. [14]. For the global existence with small data, it seems necessary to choose a working space so that the exponential nonlinearity has an estimate that is independent of $t > 0$. If the nonlinearity is of the form $\pm |u|^{2k}u$, the working space can be chosen as $X_k := \{ u : sup_{t>0} t^{-n(1/nk-1/pk)/2} \| u(t) \|_{pk} \leq \delta \}$, $nk \vee (1+2k) < pk < nk(1+2k)$, see, e.g. [14]. If we consider the exponential nonlinearity, by Taylor’s expansion of $(e^{0|u|^2} - 1)u$, it seems natural to take $X = \cap_{k \in \mathbb{N}} X_k$ as a working space. For technical reasons, we need to further restrict $pk = kp1$. We now give the proof of Theorem 1.4.

**Proof of Theorem 1.4.** For convenience, we write $rk = nk$. It is easy to see that

$$r_1 \lor 3 < p_1 < 3r_1, \quad pk = kp1, \quad \frac{2}{\gamma_k} = n \left( \frac{1}{rk} - \frac{1}{pk} \right). \tag{8.1}$$

From (8.1) we see that

$$rk \lor (1 + 2k) < pk < rk(1 + 2k). \quad \tag{8.2}$$

We consider the following integral equation:

$$v(t) = U(t)u_0 - \mu A \left( (e^{0|u|^2} - 1)u \right). \tag{8.3}$$

Applying Taylor’s expansion of $(e^{0|u|^2} - 1)u$ and (8.0), we have

$$\| v \|_{k,T} \leq C \| u_0 \|_{rk} + C \sum_{\ell=1}^{\infty} \frac{g_{\ell}}{\ell!} \| A(|u|^{2\ell}u) \|_{k,T}. \tag{8.4}$$

In view of the basic $L^p - L^r$ estimate of $U(t)$,

$$\| A(|u|^{2\ell}u) \|_{k,T} \leq C sup_{0 < t < T} t^{1/\gamma_k} \int_{0}^{t} |t - \tau|^{-n(1/\mu_k - 1/pk)/2} \| u \|_{pk}^{2\ell} \mu_k \| u \|_{pk} d\tau, \tag{8.5}$$

where we take

$$\frac{1}{\mu_k} = 1 + 2k \quad \frac{1}{pk}. \tag{8.6}$$

By (8.1) we have $1/\mu_k = 1/pk + 2\ell/pk$. From Hölder’s inequality it follows that

$$\| u \|^{2\ell} \mu_k \leq \| u \|_{pk}^{2\ell} \| u \|_{pk}. \tag{8.7}$$

Collecting (8.5) and (8.7), we have

$$\| A(|u|^{2\ell}u) \|_{k,T} \leq C \| u \|_{k,T} \mu_k \sup_{0 < t < T} t^{1/\gamma_k} \int_{0}^{t} |t - \tau|^{-n\ell/p_k - 2\ell/\gamma_k} \| u \|_{pk} \| u \|_{pk} d\tau. \tag{8.8}$$

Since $1/\gamma_k < 1/(1 + 2k)$, $\ell/\gamma_\ell = 1/\gamma_1 < 1/3$ and

$$\frac{n\ell}{p_\ell} + \frac{2\ell}{\gamma_\ell} = 1, \tag{8.9}$$

we have

$$\| A(|u|^{2\ell}u) \|_{k,T} \leq C \| u \|_{k,T}^{2\ell} \| u \|_{k,T}. \tag{8.10}$$
where $C$ in (8.9) is a universal constant that is independent of $\ell, k \in \mathbb{N}$. Hence, (8.4) and (8.10) imply that

$$
\|v\|_{k,T} \leq (C\|u_0\|_{r_k}) \wedge \|U(t)u_0\|_{k,T} + C \sum_{\ell=1}^{\infty} \frac{q^\ell}{\ell!} \|u\|_{2\ell,T}^2 \|u\|_{k,T}.
$$

(8.11)

Put

$$
\mathcal{D} = \{u : \|u\|_{k,T} \leq \delta \text{ if } k \leq K; \quad \|u\|_{k,T} \leq M \text{ if } k > K\},
$$

(8.12)

$$
d(u_1, u_2) = \sup_{k \geq 1} \|u_1 - u_2\|_{k,T}.
$$

(8.13)

We show that the mapping $T : u \rightarrow v$ is a contraction mapping from $(\mathcal{D}, d)$ into itself for some $\delta, M > 0$ and $K \in \mathbb{N}$. We can take

$$
M = 2C \sup_{k \geq 1} \|u_0\|_{r_k},
$$

(8.14)

$K \in \mathbb{N}$ verifying

$$
C \sum_{\ell > K} \frac{q^\ell}{\ell!} M^{2\ell} \leq 1/4
$$

(8.15)

and $0 < \delta < M \wedge 1$ satisfying

$$
C \delta^2 \sum_{\ell \leq K} \frac{q^\ell}{\ell!} \leq 1/4.
$$

(8.16)

By (8.11), for $k \leq K$,

$$
\|v\|_{k,T} \leq \|U(t)u_0\|_{k,T} + C \sum_{\ell=1}^{\infty} \frac{q^\ell}{\ell!} \|u\|_{2\ell,T}^2 \|u\|_{k,T}.
$$

(8.17)

Taking notice of (see [13])

$$
\lim_{T \to 0} \|U(t)u_0\|_{k,T} = 0, \quad k = 1, \ldots, K,
$$

(8.18)

we can find some $T > 0$, such that

$$
\|U(t)u_0\|_{k,T} \leq \delta/2, \quad k = 1, \ldots, K.
$$

(8.19)

Collecting (8.15)–(8.17), and (8.19), one sees that if $u \in \mathcal{D}$, then $\|v\|_{k,T} \leq \delta$, $k \leq K$. On the other hand, for $k \leq K$, by (8.15) and (8.16),

$$
\|v\|_{k,T} \leq C \|u_0\|_{r_k} + C \left( \sum_{\ell=1}^{K} + \sum_{\ell > K} \frac{q^\ell}{\ell!} \|u\|_{2\ell,T}^2 \|u\|_{k,T} \right)
$$

$$
\leq M/2 + M(1/4 + 1/4) = M.
$$

(8.20)

Similarly,

$$
d(Tu_1, Tu_2) \leq \frac{1}{2} d(u_1, u_2).
$$

(8.21)
Then we can repeat the procedure as in the proof of Theorem 1.3 and get the local existence and uniqueness of the solution in $X_T$.

Let $\mu_k$ be as in (8.6). Using the same method as in (8.5), (8.8) and (8.10), one has that

$$\|A(|u|^{2\ell}u)\|_{C(0,T;L^\infty)} \leq C \sup_{0 < t < T} \int_0^t |t - \tau|^{-n(1/\mu_k - 1/p_k)/2} \|u|^{2\ell}u\|_{\mu_k} \, d\tau$$

$$\leq C \sup_{0 < t < T} \int_0^t |t - \tau|^{1/\gamma_k - n\ell/p_k} \|u\|_{p_k}^{2\ell} \|u\|_{p_k} \, d\tau$$

$$\leq C \|u\|_{L^\infty,T}^{2\ell} \|u\|_{p_k,T} \sup_{0 < t < T} \int_0^t |t - \tau|^{1/\gamma_k - n\ell/p_k} \tau^{-2\ell/\gamma_k - 1/\gamma_k} \, d\tau$$

$$\leq C \|u\|_{L^\infty,T}^{2\ell} \|u\|_{p_k,T}.$$  \hspace{1cm} (8.10a)

Combining the integral equation (8.3) with (8.10a), we obtain that

$$\|u\|_{p_k,T} \leq C \|u_0\|_{L^\infty} + C \sum_{\ell=1}^\infty \frac{\ell!}{\ell!} \|u\|_{p_k,T}^{2\ell} \|u\|_{p_k,T},$$  \hspace{1cm} (8.11a)

which implies that $u \in C(0,T;L^\infty)$, $k = 1, 2, \ldots$. By a standard argument we can extend the solution step by step and finally obtain the desired local well posedness of solutions.

Next, we consider the small initial data. By (8.11), we have

$$\sup_{k \geq 1} \|u\|_{k,T} \leq C \|u_0\|_{\cap_{k \geq 1} L^\infty} + C \sum_{\ell=1}^\infty \frac{\ell!}{\ell!} \|u\|_{p_k,T}^{2\ell} \sup_{k \geq 1} \|u\|_{k,T}.$$  \hspace{1cm} (8.22)

Taking $T = \infty$ and $M = \delta = 2C\|u_0\|_{\cap_{k \geq 1} L^\infty}$ in (8.12), (8.13), and (8.22), and using the same method as above, we can get the global well posedness of solutions if $\|u_0\|_{\cap_{k \geq 1} L^\infty} \leq \delta/2C$, $\delta$ is sufficiently small.

Now, we consider the case $b > 0$, $x = 0$. For any $r > 2$, we can get a priori bound of the solution in $L^r$, which is independent of $t > 0$. In fact, (1.21) is multiplied by $|u|^{r-2\alpha}$ and then we consider the real part of the result identity; it follows that

$$\|u(t)\|_r \leq \|u_0\|_r.$$  \hspace{1cm} (8.23)

Hence, we obtained that if $u_0 \in \cap_{k \geq 1} L^\infty$, then

$$\|u(t)\|_{r_k} \leq \|u_0\|_{r_k} \leq \|u_0\|_{\cap_{k \geq 1} L^\infty}.$$  \hspace{1cm} (8.24)

Eq. (8.23) also implies that

$$\|u(t)\|_{p_k} \leq \|u_0\|_{\cap_{k \geq 1} L^\infty}.$$  \hspace{1cm} (8.25)

In view of (8.24) and (8.25) we claim that the solution must be a global one.
The regularity of solutions follows the proof of Theorem 1.3. In fact, in view of Proposition 5.3, we have for \(0 < t \leq t_0 < 1\),
\[
\|u(t)\|_{E^{\infty}_{2,1}} \leq C \|u_0\|_{E^{0}_{2,1}} + C \sum_{\ell=1}^{\infty} \frac{q^\ell}{\ell!} \int_0^t \|u\|_{E^{\infty}_{2,1}} \|u(\tau)\|_{E^{\infty}_{2,1}}^2 d\tau. \tag{8.26}
\]
By Corollary 4.2,
\[
\|u\|_{E^{\infty}_{2,1}} \leq (C^2 C_\tau)^{2\ell+1} \|u(\tau)\|_{E^{\infty}_{2,1}}^{2\ell+1}. \tag{8.27}
\]
Hence, for \(0 \leq t \leq t_0 < 1\),
\[
\|u(t)\|_{E^{\infty}_{2,1}} \leq C \|u_0\|_{E^{0}_{2,1}} + Ct_0 \sum_{\ell=1}^{\infty} \frac{q^\ell}{\ell!} C^{2\ell+1} \sup_{0 \leq t \leq t_0} \|u(t)\|_{E^{\infty}_{2,1}}^{2\ell+1}. \tag{8.28}
\]
For any \(M > 0\), noting
\[
\sum_{\ell=1}^{\infty} \frac{q^\ell}{\ell!} C^{2\ell+1} M^{2\ell+1} \tag{8.29}
\]
is a convergent series, we can repeat the procedure as in the proof of Theorem 1.3 to show the regularity results (1.19) and (1.20) and we omit the details. \(\square\)

9. Navier–Stokes equation

The proof of Theorem 1.5 follows a similar idea as in Theorem 1.3. Since the non-linearity in the Navier–Stokes equations contains derivative terms, we need to modify the metric space defined in Theorem 1.3 and add the upper bound of \(\|u\|_T\) in the definition of \(\mathcal{D}\), this technique seems standard, cf. e.g. [14,18], for the Navier–Stokes equation, and Strauss [26] for the nonlinear Schrödinger equation.

**Proof of Theorem 1.5.** For simplicity, we write \(\|(u_1, \ldots, u_n)\|_X = \|u_1\|_X + \cdots + \|u_n\|_X\). Put
\[
\mathcal{D} = \{u : \|u\|_{C(0,T;E^0_{2,1})} \leq M, \quad \|u\|_T \leq M\}, \tag{9.1}
\]
where
\[
\|u\|_T = \sup_{0 < t \leq T} t^{1/2} \|Du(t)\|_{E^0_{2,1}} \tag{9.2}
\]
and, for any \(u, v \in \mathcal{D}\), we denote
\[
d(u, v) = \|u - v\|_{C(0,T;E^0_{2,1})} + \|u - v\|_T. \tag{9.3}
\]
We will show that
\[
\mathcal{T} : u(t) \to U_0(t)u_0 - A_0g(u, \nabla u) \tag{9.4}
\]
is a contraction mapping. In fact, for any $u \in D$, in view of $\mathcal{P} : L^2 \to L^2$, we have
\[
\|T u\|_{C(0,T; E^0_{2,1})} \leq C\|u_0\|_{E^0_{2,1}} + C e^T \int_0^T \|(1 + \mathcal{P})u \cdot \nabla u\|_{E^0_{2,1}} \, d\tau
\]
\[
\leq C\|u_0\|_{E^0_{2,1}} + C e^T \int_0^T \|u(\tau)\|_{E^0_{2,1}} \|Du(\tau)\|_{E^0_{2,1}} \, d\tau
\]
\[
\leq C\|u_0\|_{E^0_{2,1}} + C e^T T^{1/2} \|u\|_{C(0,T; E^0_{2,1})} \|Du\|_T
\]
\[
\leq C\|u_0\|_{E^0_{2,1}} + C e^T T^{1/2} M^2. \tag{9.5}
\]

Similarly, by Proposition 5.3,
\[
\|T u\|_T \leq C\|u_0\|_{E^0_{2,1}} + C e^T \sup_{0 < t \leq T} t^{1/2} \int_0^t (t - \tau)^{-1/2}\|u \cdot \nabla u\|_{E^0_{1,2}} \, d\tau
\]
\[
\leq C\|u_0\|_{E^0_{2,1}} + C e^T \|u\|_{C(0,T; E^0_{2,1})} \|Du\|_T \sup_{0 < t \leq T} t^{1/2} \int_0^t (t - \tau)^{-1/2} \tau^{-1/2} \, d\tau
\]
\[
\leq C\|u_0\|_{E^0_{2,1}} + C e^T T^{1/2} M^2. \tag{9.6}
\]

Taking
\[
M = 2C\|u_0\|_{E^0_{2,1}} \tag{9.7}
\]
and letting $0 < T < 1$ verifying $CeT^{1/2}M \leq 1/4$, we see that
\[
\|T u\|_{C(0,T; E^0_{2,1})} + \|T u\|_T \leq M, \tag{9.8}
\]
that is $T : D \to D$. Moreover,
\[
d(T u, T v) \leq \frac{1}{2} d(u, v). \tag{9.9}
\]

Hence, $T$ has a fixed point $u \in D$, which solves (1.29). The uniqueness of such a solution follows from the proof of Theorem 1.3.

We can extend this solution step by step, say from $[0, T]$ to $[T/2, T_1]$ ($T_1 > T$), $[T_1/2, T_2]$ ($T_2 > T_1$), ..., and finally find a maximal $T^* > 0$ such that
\[
u \in [C_{\text{loc}}([0, T^*); E^0_{2,1})]^n \cap X_{T^*}, \tag{9.10}
\]
where
\[
X_{T^*} = \{u : \|u(t)\|_{T} < \infty \ \forall \ T < T^*\}. \tag{9.11}
\]

Now we show the regularity of the solutions. For any $T < T^*$, we have
\[
\sup_{0 \leq t \leq T} \|u(t)\|_{E^0_{2,1}} \leq M_0 < \infty. \tag{9.12}
\]
For convenience, we write for \( D^\gamma := \partial_{x_i}^\gamma \), \( D^\gamma v = (\partial_{x_i}^\gamma v_1, \ldots, \partial_{x_i}^\gamma v_n) \), \( \gamma = 0, 1 \),

\[
R_\gamma(v) = \sup_{0 < t \leq t_0} t^{\gamma/2} \| D^\gamma v \|_{L^2_t(E^{2,1})}. \tag{9.13}
\]

Following a similar way as in (9.5) and (9.6), in view of Proposition 5.3 we have

\[
R_\gamma(Tu) \leq CM_0 + C e^{t_0/2} R_0(u) R_1(u), \quad \gamma = 0, 1. \tag{9.14}
\]

Similar to Theorem 1.3, by contraction methods, we see that the solution satisfies

\[
R_\gamma(u) \leq 2CM_0, \quad \gamma = 0, 1 \tag{9.15}
\]

if \( t_0 \) is sufficiently small. Repeating the procedure above from \([0, t_0] \) to \([t_0/2, 3t_0/2], [t_0, 2t_0], \ldots\), we obtain the regularity results, as desired.

In the case \( n = 2 \), by Kato’s result we see that \( u_0 \in L^2(\mathbb{R}^2) \) can guarantee the global well posedness of solutions in \( L^2(\mathbb{R}^2) \) of (1.28). By standard regularity result (see e.g. [14]) we have \( u \in C_{loc}([T, \infty); H^k(\mathbb{R}^2)) \) for any \( k \in \mathbb{N} \) and \( T > 0 \). This leads to \( u \in C_{loc}([T, \infty); E_{2,1}(\mathbb{R}^2)) \). □

**Remark 9.1.** We now give a comparison between the regularity behavior (1.20) and (1.36). First, let us compare the initial conditions \( u_0 \in E_{0,1}^2(\mathbb{R}^3) \) and \( \hat{u}_0 \in L^\infty(\mathbb{R}^3) \). We claim that they have no inclusion relation. Indeed, putting

\[
\hat{\hat{f}}(\xi) = \sum_{k_1 \in 2^\mathbb{N}} \frac{\chi_{Q(k_1,0,0)}(\xi)}{k_1}, \tag{9.16}
\]

we have \( \| f \|_{E_{2,1}^0} \leq 1 \), but

\[
\| \hat{\hat{\Delta}} f \|_\infty \geq \sup_{k_1 \in 2^\mathbb{N}} |k_1|^2 \sup_\xi |\hat{\hat{\Delta}}(\xi)| \chi_{Q(k_1,0,0)}(\xi) = \infty. \tag{9.17}
\]

That is \( f \in E_{2,1}^0 \), but \( \hat{\hat{\Delta}} f \notin L^\infty(\mathbb{R}^3) \). On the other hand, taking

\[
\hat{\hat{g}}(\xi) = \sum_{k \in \mathbb{Z}^3 \backslash \{0\}} \frac{\chi_{Q(\xi)}(\xi)}{|k|^2}, \tag{9.18}
\]

we see that \( \hat{\hat{\Delta}} g \in L^\infty(\mathbb{R}^3) \), but

\[
\| g \|_{E_{2,1}^0} = \sum_{k \in \mathbb{Z}^3 \backslash \{0\}} \| \chi_{Q(\xi)} \hat{\hat{\Delta}} g \|_2 = \sum_{k \in \mathbb{Z}^3 \backslash \{0\}} \frac{1}{|k|^2} = \infty. \tag{9.19}
\]

Hence, \( g \notin E_{2,1}^0(\mathbb{R}^3) \), but \( \hat{\hat{\Delta}} g \in L^\infty(\mathbb{R}^3) \).
Next, we can rewrite (1.36) and (1.20), respectively, as
\[
\sup_{k \in \mathbb{Z}^n} 2^{c \sqrt{t}|k|^2} \| \hat{u}(t) \|_{L^\infty(Q_k)} < \infty, 
\]
(9.20)
\[
\sum_{k \in \mathbb{Z}^n} 2^{c (t \wedge t_0)|k|} \| \hat{u}(t) \|_{L^2(Q_k)} < \infty. 
\]
(9.21)
Comparing (9.20) and (9.21), we see that these two kinds of regularity behavior have advantages. At the end of this section, we conjecture that the regularity behavior like (2.14) still holds for the CGL and Navier–Stokes equations, namely,
\[
\sum_{k \in \mathbb{Z}^n} 2^{c|k|^2} \| \hat{u}(t) \|_{L^2(Q_k)} \leq C(T), \quad 0 < t \leq T. 
\]
(9.22)
To reach the regularity quantity \(2^{c|k|^2}\) instead of \(2^{c|k|}\), we still have some difficulty in the bilinear estimates as in Section 4.

**Remark 9.2.** There is no essential difficulty to develop the theory of this paper to the higher-order semi-groups, say, to the Cahn–Hilliard group \(e^{t(\Delta - \Delta^2)}\) and the Cahn–Hilliard equation
\[
\frac{d}{dt} u - \Delta u + \Delta^2 u + \Delta f(u) = 0, 
\]
(9.23)
and we do not perform it in this paper.

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**References**


