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Counterexamples to a Conjecture of Tamaschke*

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Suppose that G is a finite group, and H is any subgroup of G. In a recent article [*1*], Tamaschke defined "$(G : H)$-classes" and "$(G : H)$-characters." He conjectured (in a remark following the proof of Theorem 2.1 in [*1*]) that the numbers $Cl(G : H)$, of $(G : H)$-classes, and $Ch(G : H)$, of $(G : H)$-characters, always coincide:

$$Cl(G : H) = Ch(G : H). \tag{1}$$

We construct an infinite family of counterexamples to this conjecture.

The $(G : H)$-classes form a partition of G, which, in view of Lemma 1.2 of [*1*], may be defined as follows: Two elements σ, τ of G lie in the same $(G : H)$-conjugacy class if and only if:

$$\frac{|K \cap H\sigma H|}{|H\sigma H|} = \frac{|K \cap H\tau H|}{|H\tau H|}, \tag{2}$$

for every ordinary conjugacy class K of G.

From the third paragraph of Section 2 of [*1*], it is clear that the number $Ch(G : H)$ of $(G : H)$-characters is less than or equal to the number of ordinary irreducible characters of G; i.e.,

$$Ch(G : H) \leq Cl(G), \tag{3}$$

where $Cl(G)$ is the number of ordinary conjugacy classes of G. Since our groups will all satisfy:

$$Cl(G) < Cl(G : H), \tag{4}$$

we need no further information about $Ch(G : H)$ in order to disprove (1).

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Let p, q be any two primes satisfying:

$$p \equiv 1 \pmod{q} \tag{5}$$

Since p must be odd, there is a unique non-abelian group P of order p^3 and exponent p. It can be presented as having two generators π_1, π_2 subject to the relations:

$$\pi_1^p = \pi_2^p = 1, \qquad [\pi_1, \pi_2] = \pi_1^{-1}\pi_2^{-1}\pi_1\pi_2 \quad \textit{centralizes both } \pi_1 \textit{ and } \pi_2. \tag{6}$$

Condition (5) gives us an integer t satisfying:

$$t^q \equiv 1 \pmod{p}, \qquad t \not\equiv 1 \pmod{p}. \tag{7}$$

It follows readily from the presentation (6) that the group P has an automorphism α of order q defined by:

$$\pi_1^\alpha = \pi_1^t, \qquad \pi_2^\alpha = \pi_2^{t^{-1}}. \tag{8}$$

We define H to be the cyclic group $\langle\alpha\rangle$ of order q and G to be the semi-direct product HP.

The subgroup $Z = \langle[\pi_1, \pi_2]\rangle$ is the centre of P. So it is normal in G. In fact it is central in G, since (6) and (8) imply:

$$[\pi_1, \pi_2]^\alpha = [\pi_1^\alpha, \pi_2^\alpha] = [\pi_1^t, \pi_2^{t^{-1}}] = [\pi_1, \pi_2]^{tt^{-1}} = [\pi_1, \pi_2]. \tag{9}$$

The quotient group $P^* = P/Z$ is elementary of order p^2. We write it additively. Then it is a vector space over the field F_p of integers mod p. The images π_1^*, π_2^* of π_1, π_2, respectively, form a basis for this vector space. The automorphism α of P induces an automorphism α^* of P^*. This α^* is a non-singular linear transformation of P^*.

LEMMA 10. *If* $\pi^* \in P^*$ *and* $\pi^*\alpha^* = \pi^*$, *then* $\pi^* = 0$.

Proof. In view of (8), we have:

$$\pi_1^*\alpha^* = t\pi_1^*, \qquad \pi_2^*\alpha^* = t^{-1}\pi_2^*.$$

Therefore π_1^*, π_2^* is a basis of eigenvectors of α^* with the eigenvalues t, t^{-1}.

If $\pi^* \in P^*$ and $\pi^*\alpha^* = \pi^*$, then either $\pi^* = 0^*$ or π^* is an eigenvector of α^* with eigenvalue 1. However, the only eigenvalues of α^* are t and t^{-1}, and, by (7), neither of these equals 1. Hence 1 is not eigenvalue of α^*, and we must have $\pi^* = 0$ as stated.

LEMMA 11. *Let K be an ordinary conjugacy class of G. Then either* (i) $K \subseteq Z$, *or* (ii) $K \subseteq P - Z$, *or* (iii) $K \subseteq G - P$. *In case* (i), *K consists of a* *single element of* Z. *In case* (ii), K *is the inverse image in* P *of a subset* $\{\pi^*, \pi^*\alpha^*, \pi^*(\alpha^*)^2,..., \pi^*(\alpha^*)^{q-1}\}$, *where* π^* *is some nontrivial element of* P^*. *In this case,* K *has* pq *elements. Finally, in case* (iii), K *has the form* $\{\sigma^\pi \mid \pi \in P\}$, *where* σ *is any element of* K. *Here* K *has* p^2 *elements.*

Proof. Since $Z \subset P \subset G$ is a chain of normal subgroups of G, the first conclusion of the lemma is obvious.

We have already seen (in (9)), that Z lies in the centre of G. Therefore the conclusion in case (i) is correct.

Suppose that K lies in case (ii) and that π is any element of K. Since $K \subseteq P - Z$, the image π^* of π in P^* is nontrivial. Because P^* is abelian, the image K^* of K in P^* is just the set $\{\pi^*, \pi^*\alpha^*,..., \pi^*(\alpha^*)^{q-1}\}$. The inverse image of π^* in P is the coset πZ of Z. This, however, is the conjugacy class of P containing π. Therefore it lies entirely within K. Similarly K contains the inverse image of each element of K^*. Hence K is the full inverse image of K^* in P.

Since q is a prime, Lemma 10 implies that K^* has q elements. The inverse image in P of any element of K^* is a coset of Z, and therefore has p elements. It follows that K has pq elements. So the conclusions of the lemma in case (ii) are correct.

Finally, let K be in case (iii). Choose any element σ in K. Write $\sigma = \rho\tau$, where the order of ρ is a power of q, the order of τ is a power of p, and $\rho\tau = \tau\rho$. Since $\sigma \notin P$, the element ρ is nontrivial. The cyclic group $\langle\rho\rangle$ has order q, and therefore is a q-Sylow subgroup of G. By passing to a G-conjugate, we may assume that $\langle\rho\rangle = H$. Lemma 10 and (9) imply that Z is the centralizer of ρ in P. Therefore $\tau \in Z$. Since τ is central in G the centralizer of $\sigma = \rho\tau$ in G coincides with the centralizer of ρ in G, which is HZ. Hence K has $(G : HZ) = p^2$ elements. Since $G = (HZ) \cdot P$, the G-conjugacy class K is just the P-conjugacy class $\sigma^P = \{\sigma^\pi \mid \pi \in P\}$. This is true for all elements σ of K because it is true for one of them. Therefore the conclusions of the lemma in case (iii) are correct.

COROLLARY 12. *The number of ordinary conjugacy classes of* G *is given by*:

$$Cl(G) = pq + (p + 1)\frac{p-1}{q}.$$

Proof. The number of classes K in case (i) of Lemma 11 is clearly $|Z| = p$. The number in case (ii) is $|P - Z|/pq = (p^2 - 1)/q$. The number in case (iii) is $|G - P|/p^2 = p(q - 1)$. Adding these three numbers gives the above formula.

Now we come to the heart of the matter.

LEMMA 13. *Two elements σ, τ of G lie in the same $(G : H)$-conjugacy class if and only if they satisfy:*

$$H\sigma H = H\tau H. \tag{14}$$

Proof. It is clear from (2) that any two elements σ, τ satisfying (14) lie in the same $(G : H)$-conjugacy class. The problem is to prove the converse.

Since $G = HP$, we may multiply σ by an appropriate element of H and assume:

$$\sigma \in P. \tag{15}$$

This changes neither the hypothesis nor the conclusion of the lemma.

If $\sigma \in Z$, then $K = \{\sigma\}$ is an ordinary conjugacy class of G by Lemma 11, case (i). For this choice of K, the left side of (2) is nonzero. Hence so is the right side; i.e., $K \cap H\tau H$ is non-empty. Since σ is the only element of K, we conclude that $\sigma \in H\tau H$. This implies (14). Hence the lemma is true if $\sigma \in Z$.

Now assume that $\sigma \notin Z$. Let K be the ordinary conjugacy class of G containing σ. As above, (2) implies that $K \cap H\tau H$ is non-empty. Case (ii) of Lemma 11 tells us that K is the union of the H-conjugates of σZ. Hence $\sigma Z \cap H\tau H$ is non-empty. We may therefore replace τ by another element of the double coset $H\tau H$ so that:

$$\tau = \sigma\zeta, \qquad \zeta \in Z. \tag{16}$$

The element $\alpha\sigma$ evidently lies in $H\sigma H$. Now let K be the ordinary conjugacy class of G containing $\alpha\sigma$. As before, $K \cap H\tau H$ is non-empty. Let $\beta_1\tau\beta_2$ be any element of $K \cap H\tau H$, where $\beta_1, \beta_2 \in H$. Then $K \cap H\tau H$ also contains $\beta_2\beta_1\tau = \beta_2(\beta_1\tau\beta_2)\beta_2^{-1}$. We write β for $\beta_2\beta_1$. From case (iii) of Lemma 11, we know that some element π of P exists so that:

$$(\alpha\sigma)^\pi = \beta\tau.$$

This can be written as:

$$\alpha \cdot \pi^{-\alpha} \cdot \sigma \cdot \pi = \beta \cdot \sigma \cdot \zeta, \tag{17}$$

using (16).

We first consider (17) modulo the subgroup P. Since π, σ and ζ lie in P, and P is normal in G, the factors $\pi^{-\alpha} \cdot \sigma \cdot \pi$ and $\sigma \cdot \zeta$ all lie in P. We are left with:

$$\alpha \equiv \beta \pmod{P}.$$

But α and β both lie in H and $H \cap P = \{1\}$. Hence $\alpha = \beta$. Cancelling α on the left in (17), we obtain:

$$\pi^{-\alpha} \cdot \sigma \cdot \pi = \sigma \cdot \zeta. \tag{18}$$

We now consider (18) modulo Z. Since $\zeta \in Z$ we obtain the equation in P^*:

$$(-\pi^*)\,\alpha^* + \sigma^* + \pi^* = \sigma^*,$$

where, of course, σ^*, π^* are the images of σ, π, respectively. This gives us:

$$\pi^*\alpha^* = \pi^*.$$

By Lemma 10, π^* must be 0; i.e., π lies in Z. But Z is the center of G. Hence $\pi^{-\alpha} = \pi^{-1}$ and $\pi^{-1}\sigma\pi = \sigma$. So (18) becomes:

$$\sigma = \sigma \cdot \zeta.$$

Or:

$$\zeta = 1.$$

This and (16) prove the lemma.

LEMMA 19. *The number of* $(G : H)$*-conjugacy classes is given by*:

$$Cl(G : H) = p + p(p+1)\frac{p-1}{q}.$$

Proof. By Lemma 13, $Cl(G : H)$ is just the number of (H, H)-double cosets of G. Each such double coset $H\sigma H$ has either q or q^2 elements, since $|H| = q$ is a prime. Furthermore, $|H\sigma H| = q$ if and only if $H\sigma H = \sigma H$ is contained in the normalizer $N_G(H)$ of H in G. From (9) and Lemma 10 it is clear that $N_G(H)$ is just HZ. Hence there are $|HZ|/q = p$ double cosets with q elements and $|G - HZ|/q^2 = (p^3 - p)/q$ double cosets with q^2 elements. Adding these two numbers gives the above formula.

Finally we reach:

THEOREM 20. *Each of our groups G satisfies* (4) *and therefore provides a counterexample to Tamaschke's conjectured equality* (1).

Proof. In view of Corollary 12 and Lemma 19, the inequality (4) is:

$$pq + (p+1)\frac{p-1}{q} < p + p(p+1)\frac{p-1}{q}.$$

Subtracting $p + (p + 1)(p - 1)/q$ from both sides, we obtain the logically equivalent inequality:

$$p(q - 1) < (p^2 - 1)\frac{p - 1}{q}. \tag{21}$$

Condition (5) implies that $q - 1 \leqslant p - 2$ and that $(p - 1)/q \geqslant 1$. Hence (21) is implied by the stronger inequality:

$$p(p - 2) < p^2 - 1. \tag{22}$$

Subtracting p^2 from both sides, we obtain the logically equivalent inequality

$$-2p < -1.$$

This is certainly true, since $p \geqslant 2$. Hence (22) is true. This implies (21), and (21) is logically equivalent to (4). In view of (3), the inequality (4) contradicts (1). That proves the theorem.

Reference

1. Tamaschke, Olaf. A generalization of normal subgroups. *J. Algebra* **11** (1969), 338–352.