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Stability, Observability and Invariance*,†

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Observability-type considerations and the invariance principle are shown to complement each other in the study of the asymptotic behavior and stability of differential equations. Applications to stabilization and identification follow.

It was my privilege to receive the guidance and encouragement of Joe LaSalle since I began working in differential equations. He has my respect and admiration for both his mathematical achievements and his personality. This paper is, in particular, influenced by LaSalle's ideas, and it is with great pleasure that I dedicate it to him.

1. INTRODUCTION

Stability in the sense of Liapunov and boundedness of solutions can effectively be studied with the aid of Liapunov functions, namely, mappings which do not increase along solutions. For asymptotic stability and other attractivity properties the negative semidefiniteness of the derivative of the Liapunov functions is not enough. Negative definite derivatives in the appropriate regions would suffice; however, this stronger property is not always possessed by the natural candidate for the Liapunov function, namely, the energy. The damped harmonic oscillator is the standard example.

An effective tool to overcome the difficulty was developed by LaSalle, and is usually referred to as the invariance principle. (We recall a version of it at the end of this introduction.) By combining information on level sets of the Liapunov function with the topological dynamics of the solution flow, it is often possible to detect the ω-limit set and thus establish attractivity or stability properties. The technique is applicable to a variety of dynamical

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systems, and is well documented in the literature. A nonexhaustive list of surveys, extensions and applications is Ball [7, 8], Ball and Peletier [9], Dafermos [13–17], Hale [20, 21], Kushner [26], LaSalle [29–32], Rouche *et al.* [41], Slemrod [46, 47], Slemrod and Infante [48], Wakeman [50].

A different approach emerged mainly in control and system theory. By using observability-type arguments, or exciteness considerations, it is often possible to show that the Liapunov function strictly decreases if enough time has passed, although locally it might only nonincrease. A list of references where such a technique was used, sometimes implicitly, is Anderson [1], Anderson and Moore [2, 3], Kalman [24], Kwon and Pearson [28], Morgan and Narendra [38, 39], Narendra and Kudva [40], Yuan and Wonham [52].

In this work we intend to piece together the two approaches, and show how they complement each other. Observability-type considerations help to detect the maximal invariant sets needed in the invariance principle, and add quantitative information, namely, rates of convergence to the ω-limit sets. On the other hand, the topological considerations in the invariance principle clarify, and enable us to detect, the asymptotic behavior when only partial exciteness holds, e.g., in nonmatchable problems of identification.

The paper is organized as follows: In Section 2 we analyze in detail the simplest equation, namely, linear with constant coefficients. The simplicity enables us to illustrate the ideas clearly and show the similarity and relations between the invariance principle and observability. An example of an electrical circuit is provided.

In Section 3 we set the framework for the analysis of time-varying ordinary differential equations. Only one aspect of the observability considerations is actually needed, and we term this property as noticeability. (For linear systems they are equivalent.) We then phrase the invariance principle in terms of the unnoticeable solutions of the limiting equations.

In Section 4 we examine the relations between the noticeability of the solutions and stability properties of the equations. We get a characterization of uniform stability in terms of uniform noticeability.

In Section 5 we provide a tool for checking uniform noticeability, by reducing it to checking mere noticeability of solutions of the limiting equations. As a consequence we get conditions for asymptotic stability and eventual asymptotic stability in terms of properties of the limiting systems.

In Section 6 we examine general linear equations. Here the noticeability is equivalent to observability, and uniform noticeability follows from the uniform observability, a notion introduced by Kalman. Thus the abstract conditions for nonlinear systems have a more concrete form.

Two applications are discussed in Sections 7 and 8. In the first we show how the considerations help to construct feedback, bounded, stabilization schemes for linear time-varying equations. In the final section we show how the considerations of the invariance principle help to complement an adaptive identification process, when exciteness properties are not met, or when the plant is not matchable.

The invariance principle is a way of approaching the analysis of a system. For the benefit of those unfamiliar with this mode of thought, we present here a typical statement. We refer to it in the discussions throughout the paper. (See LaSalle [29–32] for the original, and more complete, statements.)

Consider an ordinary differential equation $\dot{x} = f(x)$, for $x \in R^n$. Let $V: R^n \to R$ be smooth, and let $\dot{V}(x) = \operatorname{grad} V(x) \cdot f(x)$. If $\dot{V}4(x) \leqslant 0$ then V is a Liapunov function. We assume that this is the case. Denote by E the set $\{x: \dot{V}(x) = 0\}$ and by E_c the set $\{x \in E: V(x) = c\}$. Let M be the largest invariant set in E. Then any bounded solution $x(t)$ of the equation converges to the intersection of M with a set E_c for a certain constant c.

2. Linear Autonomous Equations

We demonstrate the basic ideas by addressing the simplest, and well-understood, equation—namely, a linear system with constant coefficients. Consider

$$\dot{x} = Ax \tag{2.1}$$

with $x \in R^n$ and A being an $n \times n$ matrix. Much of what we say is available elsewhere; see, in particular, Russell [42, p. 104], and Miller and Michel [35].

Let us recall the concept of observability of Eq. (2.1). Wonham [51] is a good reference. Let H be a $k \times n$ matrix and for each solution $x(t)$ of (2.1) consider the *output* or the observation $y(t) = Hx(t)$. The *unobservable space* of the pair (A, H) is the linear space of vetors x_0 such that $He^{At}x_0 \equiv 0$, namely, those initial states which generate a trivial output. The unobservable space is invariant with respect to the equation, and can be characterized as the kernel of $C'C$ with $C' = [H', (HA)', \ldots, (HA^{n-1})']$. (Here and throughout, prime denotes transposition.) The range of $C'C$ is the *observable space* of (A, H), and since $C'C$ is symmetric the observable and unobservable spaces are orthogonal to each other. The observable space is also invariant with respect to the equation. The pair (A, H) is *observable* if the observable space is the entire space.

As we note later, the first assertion of the following result is a particular case of the invariance principle displayed in the introduction. The second statement follows from Miller and Michel [35].

THEOREM 2.1. *Suppose that G is symmetric and positive definite and that*

$$A'G + GA = -H'H \tag{2.2}$$

with H being a $k \times n$ *matrix. Then every solution* $x(t)$ *of* (2.1) *converges to the intersection of the unobservable space of* (A, H) *with a set of the form*

$\{x: x'Gx = constant\}$. *Equation* (2.1) *is asymptotically stable if and only if* (A, H) *is observable.*

Proof. Define $V(x) = x'Gx$. Then (2.2) amounts to $\dot{V}(x) = -x'H'Hx$; hence (2.1) is stable. The general solution of a stable linear system is a sum of periodic solutions and solutions which converge to zero. The observable space does not contain a periodic solution, since if $x(t)$ is periodic and $\dot{V}(x) \leqslant 0$ it follows that $\dot{V}(x(t)) \equiv 0$; hence $x(t)$ is contained in the unobservable space. Therefore observability implies asymptotic stability. Nonobservability implies existence of a nontrivial solution $x(t)$ with $Hx(t) \equiv 0$, i.e., $\dot{V}(x(t)) \equiv 0$, i.e., $V(x(t)) \equiv$ constant. In particular this $x(t)$ does not converge to zero. This completes the proof of the second statement. If $x_0(t)$ is a solution it has a representation $x_0(t) = x_1(t) + x_2(t)$ with $x_1(t)$ a solution contained in the observable space of (A, H) and $x_2(t)$ contained in the unobservable space of (A, H). this follows from the invariance of the two spaces, and since they span the entire space. We showed already that $x_1(t) \to 0$ and $V(x_2(t)) \equiv$ constant. This completes the proof.

Discussion. We mentioned already that the previous result is a particular case of the invariance principle. Indeed, as can easily be verified, the unobservable space of the pair (A, H) is exactly the largest invariant set in $\{x: \dot{V}(x) = 0\}$ needed in the invariance principle. What we gain by phrasing the theorem in terms of observability is an operational tool for detecting this largest invariant set. Indeed, the unobservable space is the kernel of the matrix $C'C$ mentioned previously. In this simple linear situation we can use the algebra and find explicitly the asymptotic behavior of a solution through an initial state, say, x_0. Indeed, $e^{At}x_0$ converges to $e^{At}x_2$, where x_2 is the projection of x_0 on the unobservable space. There are formulas for this projection, e.g.,

$$x_2 = (I - D^{\#}D)\, x_0$$

with $D = C'C$ and $D^{\#}$ being the Moore–Penrose generalized inverse of D.

We note that H in the previous result and discussion can be replaced, whenever convenient, by another observation provided the unobservable space is maintained. See the discussion and examples in Miller and Michel [35].

Another byproduct of the technique is a rate of convergence to the ω-limit set. It is well known that a bounded solution of (2.1) converges exponentially to its ω-limit set, with a rate estimated by $-\mathrm{Re}\,\lambda$, where λ is the eigenvalue of A having largest, yet negative, real part. The observability technique yields an estimate of the rate of decreasing of $V(x(t)) = x'(t)\, Gx(t)$. Indeed, let

$$W = \int_0^1 e^{At} GG' e^{A't}\, dt.$$

Then W is symmetric. Let α be the smallest, yet positive, eigenvalue of W. It is easy to see that $V(x(t))$ decreases exponentially to a constant, with a rate equal at least to α.

Another concept, borrowed from systems theory, is of relevance. The pair (A, H) is *detectable* if $e^{At}x_0$ converges to zero as $t \to \infty$ whenever x_0 belongs to the unobservable space of (A, H). For a more algebraic definition, and characterizations, see Wonham [51, Chap. 3]. Here, and throughout, inequalities between matrices are meant in the positive semidefiniteness sense.

THEOREM 2.2. *Suppose G is symmetric and $G > 0$. Suppose*

$$A'G + GA \leqslant -H'H. \tag{2.3}$$

Then each solution converges to the intersection of the unobservable space of (A, H) with a set of the form $\{x: x'Gx = \text{constant}\}$. The equation is asymptotically stable if and only if (A, H) is detectable.

Proof. Let $Q = A'G + GA$, and $Q^{1/2}$ be such that $Q^{1/2}Q^{1/2} = Q$. The unobservable space of $(A, Q^{1/2})$ is contained in that of (A, H). Thus the first statement follows from Theorem 2.1. Asymptotic stability implies detectability in a trivial manner. Detectability implies that every solution in the unobservable space converges to 0. Every solution in the observable space tends to 0 by the invariance principle. Hence detectability implies asymptotic stability, and this completes the proof.

The previous considerations apply if a Liapunov function is available. This is the case with many conservative and dissipative systems. We provide here an example, concerning an electrical circuit, which is an extension of the example in Russell [42, p. 104] and Miller and Michel [35].

EXAMPLE 2.3. Consider the electrical circuit given in Fig. 1. There are k LRC loops, with the same resistor sharing all the loops. (An exercise: Find the mechanical analog of this system.) If $I_j(t)$ denotes the current in the jth loop and $I_0(t)$ the current through the resistor, then by Kirchoff's law applied to each loop we get the differential equation

$$L_j \frac{d^2}{dt^2} I_j(t) + R \frac{d}{dt} I_0(t) + \frac{1}{C_j} I_j(t) = 0, \tag{2.4}$$

for $j = 1,..., k$. The currents are related by $I_0(t) \equiv I_1(t) + \cdots + I_k(t)$. If we denote by x the transpose of $(I_1, \dot{I}_1,..., I_k, \dot{I}_k)$, i.e., a vector with $2k$ dimensions, then $x(t)$ solves a linear equation of the type (2.1) with

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & & 0 \\ \frac{-1}{L_1C_1} & \frac{-R}{L_1} & 0 & \frac{-R}{L_1} & 0 & \cdots & & \frac{-R}{L_1} \\ 0 & 0 & 0 & 1 & 0 & \cdots & & 0 \\ 0 & \frac{-R}{L_2} & \frac{-1}{L_2C_2} & \frac{-R}{L_2} & 0 & \cdots & & 0 \\ \vdots & & & & & & & \\ 0 & 0 & & \cdots & & & 0 & 1 \\ 0 & \frac{-R}{L_k} & 0 & \frac{-R}{L_k} & 0 & \cdots & \frac{-1}{L_kC_k} & \frac{-R}{L_k} \end{pmatrix}.$$

Consider now the natural energy function $V(x)$, which is the quadratic form $x'Gx$ generated by the diagonal matrix G whose diagonal is $(C_1^{-1}, L_1, C_2^{-1}, L_2, \dots, C_k^{-1}, L_k)$. A straightforward calculation shows that

$$A'G+GA=-\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & R & 0 & R & \cdots & R \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & R & 0 & R & \cdots & R \\ & \vdots & & & & \\ 0 & R & 0 & R & \cdots & R \end{pmatrix}.$$

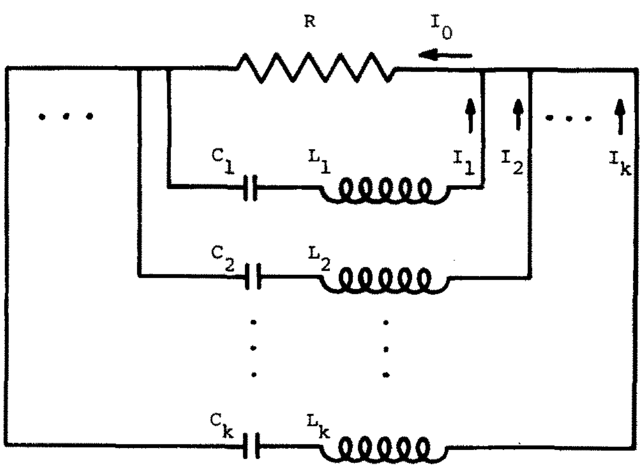

FIGURE 1

In an equivalent way, $A'G + GA = -H'H$ with

$$H = (0, R^{1/2}, 0, R^{1/2}, \ldots, 0, R^{1/2}).$$

Now suppose that (A, H) is not observable. Then a solution $x(t)$ exists with $Hx(t) \equiv 0$; in particular, if $R > 0$, we have $(d/dt) I_0(t) \equiv 0$, or $I_0(t) = \text{constant}$. With these conditions each $I_j(t)$ actually solves the equation

$$L_j \frac{d^2 I_j(t)}{dt^2} + \frac{1}{C_j} I_j(t) = 0.$$

The solutions of these equations are given in a complex notation (and $i = \sqrt{-1}$) by

$$I_j(t) = \alpha_j \exp(i(L_j C_j)^{-1/2} t).$$

This together with $I_0(t) = \text{constant}$ implies $I_0(t) \equiv 0$. Therefore, nonobservability implies the equality

$$\sum_{i=1}^{n} \alpha_j \exp(i(L_j C_j)^{-1/2} t) = 0$$

with coefficients α_j not all 0. Conversely, if such an equality holds, it determines a solution in the unobservable space of (A, H). The known conditions for linear independence of exponents yields the following.

PROPOSITION. *The system in Fig.* 1 *is asymptotically stable if and only if* $L_j C_j \neq L_m C_m$ *whenever* $j \neq m$.

The invariance principle enables us to analyze the circuit even if asymptotic stability fails. Suppose, for instance, that $L_1 C_1 = L_2 C_2$ but $L_1 C_1 \neq L_j C_j$ if $j > 2$. Then part of the initial energy might not be dissipated, and an asymptotic oscillating current would be maintained in the first two loops, avoiding the resistor. The technique explained in the previous discussion helps to find this asymptotic current. The unobservable space in this example, it is easy to see, consists of all the $2k$-vectors of the form

$$(\alpha, \beta, -\alpha, -\beta, 0, \ldots, 0).$$

We conclude, for instance, that if at the initial time there is no energy in the first two loops then all the energy is dissipated. Another example is that if, for instance, the initial conditions are $(2, 3, -4, -1, \ldots)$ then the asymptotics are determined by the projection on the unobservable space, namely, by $(2, 1, -2, -1, 0, \ldots, 0)$; i.e., asymptotically $I_1(t) = \sin(L_1 C_1)^{-1/2} t + 2\cos(L_1 C_1)^{-1/2} t$, $I_2(t) = -I_1(t)$ and $I_j(t) = 0$ for $j > 2$.

3. Unnoticeable Solutions and the Invariance Principle

The purpose of this section is to set the framework for the coming sections and to phrase the invariance principle for time-varying equations in terms of observability-type conditions. We deal, throughout the paper, with ordinary differential equations. The ideas extend to other systems, and we occasionally comment on this possibility. However, in order to cover a large class of ordinary differential equations we allow unordinary limiting equations. We follow but extend the details in LaSalle [32] and Artstein [4]. The geometrical structure is essentially that of Dafermos [13, 16]. The analysis of Ball [8] addresses a considerably relaxed situation.

We examine the differential equation

$$\dot{x} = f(x, t) \tag{3.1}$$

with $x \in R^n$. We assume throughout that f satisfies the Caratheodory conditions (see, e.g., Hale [19, p. 28]). Occasionally we might refer to the following.

Assumption (A). For every compact $K \subset R^n$ there is a nondecreasing function μ_K: $[0, \infty) \to [0, \infty)$, continuous at 0, with $\mu_k(0) = 0$ and such that whenever φ: $[a, b] \to K$ is continuous then $|\int_a^b f(\varphi(s), s)\, ds| \leqslant \mu_K(b-a)$.

Notice that under Assumption (A), solutions with values in K have a common modulus of continuity.

Equation (3.1) is a particular case of the ordinary integral-like operator equation

$$u = Hu \tag{3.2}$$

with H being an operator associating with each R^n-valued mapping φ and a number τ in the domain of φ, a continuous function $(H_\tau \varphi)(\cdot)$ satisfying (i) $H_\tau: C[\tau, b] \to C[\tau, b]$ is continuous in the sup norm, and (ii) $(H_\tau \varphi)(t) = (H_\tau \varphi)(s) + (H_s \varphi)(t)$. A solution of (3.2) is a function $u(\cdot)$ satisfying $u(t) = u(\tau) + (H_\tau u)(t)$. It should be clear how (3.1) is a particular case of (3.2). We say that H is consistent with Assumption (A) if whenever φ has values in K then $(H_\tau \varphi)(\cdot)$ admits μ_K as a modulus of continuity.

A Liapunov function for Eq. (3.2) is a mapping $V(x, t)$: $R^n \times R \to R$ such that $V(x(t), t)$ is nonincreasing whenever $x(t)$ is a solution of (3.2). We shall always demand here that V be continuous in x, but might not require continuity in t. (If V is smooth and the equation is (3.1) then the Liapunov property is checked by the condition $\dot{V}(x, t) \leqslant 0$; see LaSalle [31].)

Definition 3.1. Let V be a Liapunov function for (3.2). A solution $x(\cdot)$ is *unnoticeable by V on* the interval $[t_0, t_1]$ if $V(x(t), t)$ is constant on

$t \in [t_0, t_1]$. A maximally defined solution $x(\cdot)$ is *unnoticeable* if it is unnoticeable by V on any interval in its domain.

We wish to provide a physical interpretation for the previous definition. If the Liapunov function is the energy, then a loss of energy can sometimes be observed physically. For instance, in Example 2.3 a loss of energy is reflected in heating the resistor. An unnoticeable solution in this example is such that no current passes through the resistor.

DEFINITION 3.2. Let V be a Liapunov function for Eq. (3.2). The family of maximally defined and unnoticeable solutions will be denoted by UN, or $UN(H)$ if there is more than one equation. We denote by UN_c the set of solutions $x(\cdot) \in UN$ for which $V(x(t), t) \equiv c$. The set of pairs (x_0, t_0), such that $x(t_0) = x_0$ for a certain $x(\cdot) \in UN$, will be denoted by M, or $M(H)$ if there is more than one equation.

The set M plays the role of the maximal invariant set in the LaSalle invariance principle. It is convenient, however, to phrase the invariance principle in terms of the functions $x(\cdot)$ in UN. To this end we need to consider the family of all continuous noncontinuable functions $\gamma: (\alpha(\gamma), \omega(\gamma)) \to R^n$ so that $\gamma(0)$ is defined and such that γ admits μ_K as a modulus of continuity on each interval for which the values of γ are in K. We denote this collection by Γ and consider a metric $d(\cdot, \cdot)$ on Γ such that $\gamma_k \to \gamma_0$ in the metric d if $\gamma_k(t)$ converges uniformly to $\gamma_0(t)$ on compact intervals of $(\alpha(\gamma_0), \omega(\gamma_0))$. For details see Artstein [4, Definition 4.1, Proposition 4.2].

We now need the concept of a limiting equation of (3.1). Limiting equations were developed and applied by R. Miller and G. Sell in a series of papers; see Sell [44] and the survey by Miller and Sell [36] and references therein. We denote by f^τ the translation of f by τ; i.e., $f^\tau(x, t) = f(x, \tau + t)$. Similarly, $x^\tau(t) = x(t + \tau)$.

DEFINITION 3.3. Let $|t_j| \to \infty$. We say that $\{t_j\}$ generates Eq. (3.2) as a limiting equation of (3.1) if whenever $\varphi_j: [a, b] \to R^n$ is a sequence of continuous functions converging uniformly, say, to φ_0, then

$$\int_a^b f(\varphi_j(s), t_j + s)\, ds \to (H_a \varphi)(b).$$

We say then that (3.2) is a positive (negative) limiting equation if $t_j \to \infty$ (respectively $t_j \to -\infty$).

It is clear that if f satisfies Assumption (A) and (3.2) is a limiting equation then H is consistent with Assumption (A). Then all its solutions are in the family Γ described previously. A continuous dependence argument would show (see Artstein [4]) that if $x^{t_j}(\cdot)$ are solutions of $\dot{x} = f^{t_j}(x, t)$, and $x^{t_j}(\cdot)$ converge in Γ, say, to $u(\cdot)$, then the latter is a solution of (3.2). If stronger conditions than Assumption (A) are satisfied, then the limiting equations are actually ordinary differential equations; see Sell [43].

The following definition arose in a joint work of LaSalle and the author.

DEFINITION 3.4. Equation (3.2) is an exhaustive limiting equation of (3.1), if it is a limiting equation, generated, say, by $\{t_j\}$, and such that whenever $u(\cdot)$ is a solution of (3.2), there is a subsequence $\{t_k\}$ of $\{t_j\}$, and solutions $x^{t_k}(\cdot)$ of $\dot{x} = f^{t_k}(x, t)$, which converge in Γ to $u(\cdot)$.

If the limiting equation has unique solutions for the initial value problem it is automatically exhaustive. Sell [43] introduced the term "regular" to denote equations whose limiting equations have the uniqueness property. Many of the results which employ the regularity can be proved with the weaker notion of exhaustiveness (see results in Artstein [5], Bondi *et al.* [10], D'Anna and Maio [18], Sell [43]).

The following notion is due to Dafermos [13, 16]. We denote by V^τ the function $V(x, \tau + t)$.

DEFINITION 3.5. Suppose that $V(x, t)$ is a Liapunov function for (3.1). The function $V_1(x, t)$ is a limiting Liapunov function of (3.1), generated by $\{t_j\}$, if $V^{t_j}(x, t)$ converges to $V_1(x, t)$ uniformly on compact sets.

LEMMA 3.6. *Suppose that Assumption* (A) *holds and V is a Liapunov function. Let $\{t_j\}$ generate both the limiting equation* (3.2) *and the limiting Liapunov function V_1. Then V_1 is a weak Liapunov function of* (3.2), *meaning that for every (x_0, t_0) there is a solution $u(\cdot)$ of* (3.2) *with $u(t_0) = x_0$ and such that $V_1(u(t), t)$ is not increasing. If* (3.2) *is an exhaustive limiting equation then V_1 is a Liapunov function.*

Proof. If $x^{t_j}(\cdot)$ are solutions of $\dot{x} = f^{t_j}(x, t)$, converging to $u(\cdot)$, then $V_1(u(\cdot), \cdot)$ is a pointwise limit of $V(x^{t_j}(\cdot), \cdot)$, and hence nonincreasing. The second statement follows therefore from the definition (Definition 3.4) of an exhaustive limiting equation. Assumption (A) implies that if $x^{t_j}(\cdot)$ is such a solution with $x^{t_j}(x_0) = t_0$ then a subsequence of it converges in Γ, hence to a solution of (3.2). This completes the first claim.

We say that (3.1) is *precompact* (respectively, *positive precompact*) if any unbounded sequence $\{t_j\}$ (respectively $t_j \to \infty$) has a subsequence which generates a limiting equation. (See Artstein [4] for conditions concerning precompactness.) We say that (3.1) is *exhaustively precompact* if any unbounded sequence has a subsequence generating an exhaustive limiting equation. We say that the Liapunov function $V(x, t)$ is *precompact* if any unbounded sequence has a subsequence which generated a limiting Liapunov function. (If $V(x, t)$ is bounded and uniformly continuous on $K \times R$ for K compact then it is precompact.)

The following results are a combination of the invariance principles of LaSalle [32, Chap. 4, Appendix A] and that of Dafermos [13, 16]. We denote by $\Omega(x(\cdot))$ the ω-limit set of the function $x(\cdot)$, namely, the set $\{z: z = \lim x(t_j)$ for a certain sequence $t_j \to \infty\}$.

THEOREM 3.7. *Suppose that* (3.1) *satisfies Assumption* (A) *and it is positively precompact, and let V be a precompact Liapunov function. Let $x(\cdot)$ be a solution and $x_0 \in \Omega(x(\cdot))$. Then there is a positive limiting system, say,* (3.2) *and a corresponding limiting Liapunov function, such that an unnoticeable solution u of it satisfies $u(0) = x_0$ and $u(t) \in \Omega(x(\cdot))$ for all t in its domain.*

Proof. Let $\{t_j\}$ be such that $x(t_j) \to x_0$ and such that $\{t_j\}$ generates both (3.2) and a limiting Liapunov function, say, V_1. The sequence $x^{t_j}(\cdot)$ has a limit point, say, $u(\cdot)$ in Γ; this by Assumption (A). Clearly $u(\cdot)$ has values in $\Omega(x(\cdot))$ and $u(0) = x_0$ is a solution of (3.2); see Lemma 3.6. Sinc [illegible] e $V(x(t), t)$ is nonincrea since $V_1(u(t), t) = \lim V(x(t_j + t), t_j + t)$ i [illegible] it follows that $V_1(u(t), t)$nt. This completes the proof.

THEOREM 3.8. *Suppose that Assumption* (A) *holds, and let* (3.2) *be a positive limiting equation and V_1 a limiting Liapunov function both generated by the same sequence. Let $x(\cdot)$ be a bounded solution of* (3.1). *Then the family $UN(H)$ of unnoticeable solutions includes one which is contained in $\Omega(x(\cdot))$.*

Proof. Let $x_0 \in \Omega(x(\cdot))$ be a limit point of $x(t_j)$, where $\{t_j\}$ generates the limiting equation. Such a point exists by the boundedness. Assumption (A) implies that $x^{t_j}(\cdot)$ has a limit point in Γ, say, $u(\cdot)$. Clearly $u(\cdot)$ is the desired unnoticeable solution.

The proof of the preceding two results actually established existence of translations of the solution $x(\cdot)$ which converge to the unnoticeable solutions. The following is the general result, and it is the extension of the invariance principle.

THEOREM 3.9. *Suppose that* (3.1) *satisfies Assumption* (A), *is positively precompact, and has a precompact Liapunov function $V(x, t)$. Let $x(\cdot)$ be a bounded solution. Then there is a constant c such that $x^t(\cdot)$ converges in Γ, as $t \to \infty$, to the union of the sets $UN_c(H)$ for the positive limiting systems $\dot{u} = Hu$ and the corresponding limiting Liapunov functions.*

Proof. The result follows by a simple compactness argument from Theorem 3.7.

Here is a simple illustration, which will be elaborated upon in later sections. Consider the system in Example 2.3, but with a time-varying resistance $R = R(t)$. Assume $R(t) \geqslant \varepsilon > 0$ for a certain ε. The quadratic function $V(x) = x'Gx$ is still a Liapunov function. Hence all solutions are bounded. Suppose $L_1C_1 = L_2C_2$ but $L_1C_1 \neq L_jC_j$ if $j > 2$. If $R(t)$ is such that Assumption (A) holds, say, bounded, then the conditions of Theorem 3.9 hold. The unnoticeable solutions of any of the limiting systems are of the form $I_1(t) = \alpha \sin((L_1C_1)^{-1/2}t + \beta)$, $I_2(t) = -I_1(t)$ and $I_j(t) = 0$ for $j > 2$. Here α and β are parameters, and the subset of this family, with solutions of fixed energy identified by fixing α. We conclude therefore that any bounded solution converges to such an oscillating solution. (The linearity enables us to derive more, namely, that the limiting behavior is maintained with β fixed as well.)

4. Noticeability and Stability

In this section we examine the relations between noticeability properties of the Liapunov function, and stability. The unnoticeable solutions were defined in the previous section. Here we shall add quantifiers to describe the degree to which the other solutions are noticeable. Implication to rates of asymptotic stability follow. To a great extent what is done is a translation of the stability, as expressed in the euclidean distance, to properties of the Liapunov function. This is how we manage to get characterizations, namely, necessary and sufficient condition, for stability. It would mean very little if the new conditions were not shown to be more checkable. A step toward this is done in the next section.

The equation we analyze is again (3.1) with the assumption

$$f(0, t) = 0 \qquad \text{for all } t.$$

Then $x(t) \equiv 0$ is a solution, whose stability properties we wish to examine. Our terminology is standard, yet we list it here due to the lack of agreement in the literature.

The equation is *uniformly stable* if for every $\varepsilon > 0$, a $\delta > 0$ exists such that whenever $x(t)$ is a solution, and $|x(t_0)| \leqslant \delta$ for a certain t_0 then $|x(t)| \leqslant \varepsilon$ for all $t \geqslant t_0$. The equation is *asymptotically stable* if it is uniformly stable and a $\delta_0 > 0$ exists such that $x(t_0 + \tau) \to 0$ as $\tau \to \infty$ whenever $x(\cdot)$ is a solution and $|x(t_0)| \leqslant \delta_0$ for a certain t_0. The equation is *uniformly asymptotically stable* if it is asymptotically stable and the rate at which $x(t_0 + \tau)$ converges to 0 in the definition of asymptotic stability does not depend on t_0. Finally, the equation is *exponentially stable* if it is uniformly asymptotically stable and the uniform rate in the definition of the latter is exponential.

Along with (3.1) a Liapunov function $V(x, t)$ is provided. We shall assume that $V(0, t) = 0$ for all t and that $V(x, t) \geqslant v(x)$ with $v: R^n \to R$ continuous, $v(0) = 0$ and $v(x) > 0$ if $x \neq 0$. Then it follows, in a standard way, that the equation is uniformly stable.

The following definitions are concerned with Eq. (3.1) and the given Liapunov function V.

The system is *globally noticeable* if the only unnoticeable solution is $x(t) \equiv 0$. The system is *locally noticeable* if a $\delta_0 > 0$ exists such that whenever $x(\cdot)$ is a solution with $0 < |x(t_0)| \leqslant \delta_0$ for a certain t_0 then $x(\cdot)$ is not unnoticeable. The system is *uniformly* (locally) *noticeable* if there are $\delta_0 > 0$, $\sigma > 0$ and a function $\psi: R \to R$ monotonic, with $\psi(r) > 0$ if $r > 0$, such that whenever $x(\cdot)$ is a solution with $|x(t_0)| \leqslant \delta_0$ then $V(x(t_0), t_0) - V(x(t_0 + \sigma), t_0 + \sigma) \geqslant \psi(|x(t_0)|)$. The system is *uniformly* (locally) *exponentially noticeable* if it is uniformly noticeable and the function ψ in the definition of the latter can be chosen linear.

THEOREM 4.1. *Suppose $V(x, t) \geqslant v(x)$, with $v(x) > 0$ for $x \neq 0$, and that $V(0, t) = 0$. If the system is uniformly noticeable then it is asymptotically stable. If in addition $V(x, t) \leqslant b(x)$, with $b: R^n \to R$ continuous, then uniform noticeability is equivalent to uniform asymptotic stability, and if $V(x) = \varepsilon \|x\|^2$ then exponential noticeability is equivalent to exponential stability.*

Proof. Uniform stability is implied by the properties of V, and let $\delta_0 > 0$ be such that any solution $x(t)$ with $|x(t_0)| \leqslant \delta_0$ exists for all $t \geqslant t_0$. If such a solution does not converge to zero as $t \to \infty$ then for a sequence $t_j \to \infty$ and a certain $\eta > 0$ all $|x(t_j)| \geqslant \eta$. By uniform noticeability

$$V(x(t_j + \sigma), t_j + \sigma)) - V(x(t_j), t_j) \leqslant -\psi(\eta).$$

This, and the fact that V is nonincreasing, implies that $V(x(t), t)$ is negative for large t, a contradiction. Hence $x(t) \to 0$ as $t \to \infty$ and asymptotic stability is established. If the additional boundedness of V holds, then the convergence $V(x(t_0 + t), t)$ to 0 as $t \to \infty$, with a rate which does not depend on t_0, is equivalent to uniform asymptotic stability, and this convergence is with exponential rate (when $V(x) = \varepsilon \|x\|^2$) if and only if the system is exponentially stable. Uniform and exponential noticeability clearly characterize the uniform and exponential convergence of $V(x(t_0 + t), t)$ to 0 as $t \to \infty$. This completes the proof.

The scalar equation $\dot{x} = -t^{-1}x$ (say, for $t \geqslant 1$) with $V(x, t) = (t + 1)x$ shows that for unbounded Liapunov functions uniform noticeability might not imply uniform asymptotic stability.

5. Noticeability and the Limiting Systems

In order to make the result of previous sections effective one ought to develop techniques for checking uniform noticeability. For linear systems the problem is checking uniform observability; some techniques are available and we refer to them in the next section. The observability analysis of nonlinear equations, e.g., in Hermann and Krener [22], might be used for noticeability, but this was not done yet. The techniques of Morgan [37] are concerned with estimations of the derivatives of the Liapunov function and growth estimates on the equation; these are also relevant.

In this section we show how to reduce the problem of checking uniform noticeability to that of checking mere noticeability, however, for the family of all limiting equations. This should help, since noticeability might be checked by local techniques, while uniform noticeability has more of a global character. The analogous reduction of uniform asymptotic stability to asymptotic stability of the limiting systems was established in Artstein [5]. See also the related work of Bondi *et al.* [10] and the papers by Ianiro and Maffei [23], Visentin [49] and D'Anna and Maio [18]. (The latter contains a correction to Theorem A of Artstein [5]. A different way to correct the result is to assume that W is bounded and invariant.) Stability properties via the limiting equations appear already in Sell [43].

In the sequel we consider Eq. (3.1), assuming that $f(0, t) = 0$ and that Assumption (A) holds.

THEOREM 5.1. *Suppose that* (3.1) *is uniformly stable, is precompact and that it has a precompact Liapunov function* $V(x, t)$. *If a* $\delta_0 > 0$ *exists such that no limiting equation and the corresponding limiting Liapunov function have an unnoticeable solution* u *with* $|u(0)| \leqslant \delta_0$, *then* (3.1) *is uniformly noticeable. The condition is also necessary if the equation is exhaustively precompact.*

Proof. If (3.1) with V is not uniformly noticeable, then a sequence of solutions $x_j(\cdot)$, with $0 < \varepsilon \leqslant |x_j(t_j)| \leqslant \delta_0$, for a certain $\varepsilon > 0$, and a sequence $\tau_j \to \infty$ such that $V(x_j(t_j - \tau), t_j - \tau) - V(x_j(t_j + \tau), t_j + \tau)$ converge to zero as $j \to \infty$. Here δ_0 is given by the conditions of the theorem. The uniform stability implies that without loss of generality $|t_j| \to \infty$. If a subsequence $\{t_i\}$ generates a limiting equation and a limiting Liapunov function then a limit in Γ, say, $u(\cdot)$, of $x^{t_i}(\cdot)$ would clearly be unnoticeable and $|u(0)| \leqslant \delta$, a contradiction to the sufficient condition; hence it indeed implies that (3.1) is uniformly noticeable. If exhaustive precompactness holds and $u(\cdot)$ is an unnoticeable solution of a limiting system then $u(\cdot)$ is a limit in Γ of a sequence $x^{t_j}(\cdot)$ of solutions of $\dot{x} = f^{t_j}(x, t)$ and $|t_j| \to \infty$. The definition of the limiting Liapunov function implies that $V(x(t_j - \sigma), t_j - \sigma) -$

$V(x(t_j+\sigma), t_j+\sigma)$ converge to 0, for any fixed σ. If $u(0)$ can be chosen arbitrarily close to 0 this would contradict the uniform noticeability. Hence the condition is necessary if the equation is exhaustively precompact.

Application. Consider Example 2.3 with a time-varying resistance $R(t)$. Suppose $R(t) \geqslant 0$ and bounded. Suppose also that $L_i C_i \neq L_j C_j$ if $i \neq j$. We claim: The equation is uniformly asymptotically stable if and only if

$$\liminf_{\substack{|t_0| \to \infty \\ t \to \infty}} \int_{t_0}^{t} R(s)\, ds > 0. \tag{5.1}$$

Indeed, (5.1) holds if and only if no sequence of the form $R^{t_j}(\cdot)$ (here $R^{\tau}(t) = R(t+\tau)$) converges to 0 in the local weak convergence of, say, the Hilbert space L_2. Hence (5.1) holds if and only if for no limiting equation the (limiting) resistance is identically zero. If the resistance is not identically zero then $L_i C_i \neq L_i C_j$ implies that there are no unnoticeable solutions, and if the resistance is zero all solutions are unnoticeable for the energy function. This completes the argument.

A similar condition was obtained in Artstein [5] for the damped harmonic oscillator; however, the arguments there used some ad hoc considerations, eliminated now by the noticeability technique. The same is true for the examples in Artstein [5, Sect. 6].

6. Linear Time-Varying Systems

For linear equations with quadratic Liapunov functions one has simple expressions for the derivatives of the Liapunov function. In this section we see that the well-known notion of observability applied to the derivative of the Liapunov function plays the role of the noticeability of the Liapunov function. This was used many times in system theory. See, e.g., Anderson and Moore [2, 3], Kalman [24]. In this section we review and modify this technique in light of the preceding sections.

Consider the linear equation

$$\dot{x} = A(t)x \tag{6.1}$$

with $x \in R^n$ and $A(t)$ being measurable and locally integrable $n \times n$ matrix valued mapping. Denote by $\Phi(t, s)$ the fundamental matrix solution; i.e., $x(t) = \Phi(t, t_0)\, x_0$ is the solution satisfying $x(t_0) = x_0$. Along with (6.1) we consider the observations

$$y(t) = H(t)\, x(t) \tag{6.2}$$

with $H(t)$ being a $k \times n$ matrix valued measurable mapping. The pair $(A(t), H(t))$ is *observable* on $[t_0, t_1]$ if $H(t)\,\Phi(t, t_0)\,x_0 \equiv 0$ implies $x_0 = 0$.

A quadratic Liapunov function $V(x, t) = x'G(t)x$, with $G(t)$ symmetric and differentiable, has a quadratic derivative $\dot{V}(x, t) = x'Q(t)x$ given by

$$Q(t) = A'(t)\,G(t) + G(t)\,A(t) + \dot{G}(t).$$

Then V Liapunov means that $Q(t)$ is negative semidefinite. Let us write $Q(t) = -H(t)'H(t)$ with $H(t)$ being $k \times n$, and measurable. (The decomposition is, in general, not unique.)

LEMMA 6.1. *There are no unnoticeable solutions by V on $[t_0, t_1]$ if and only if $(A(t), H(t))$ is observable on $[t_0, t_1]$.*

Proof. Immediate from the equality

$$V(\Phi(t_1, t_0)\,x_0, t_1) - V(x_0, t_0) = x_0' \int_{t_0}^{t_1} \Phi(s, t_0)'Q(t)\,\Phi(s, t_0)\,ds\,x_0.$$

We provide in the next section an application of noticeability when G is not differentiable. However, differentiability enables us to get the information by the analysis of $Q(t) = -H(t)'H(t)$. The *observability matrix* of $(A(t), H(t))$ is given by

$$M(t_1, t_0) = \int_{t_0}^{t_1} \Phi(s, t_0)'H(s)'H(s)\,\Phi(s, t_0)\,ds. \tag{6.3}$$

See Brockett [11], Russel [42]. Then $(A(t), H(t))$ is observable on $[t_0, t_1]$ if and only if $M(t_1, t_0) > 0$, i.e., positive definite. Consider the following conditions:

Condition 6(i). There are $\sigma > 0$ and $k > 0$ such that $kI \leqslant M(t + \sigma, t)$ for all t.

Condition 6(ii). There are $\sigma > 0$ and $h > 0$ such that $M(t + \sigma, t) \leqslant hI$ for all t.

Condition 6(iii). There are $\sigma > 0$ and $\alpha > 0$ such that $\|\Phi(t, \tau)\| \leqslant \alpha$ if $|t - \tau| \leqslant \sigma$.

If a system $(A(t), H(t))$ satisfies the preceding three conditions it is *uniformly observable.* This notion was introduced by Kalman [24, p. 110] and plays a central role in time-varying systems. See, e.g., Anderson [1], Anderson and Moore [2, 3]. Conditions 6(ii) and (iii) are implied by boundedness-type properties. For instance, Assumption (A) implies Condition 6(iii), and if in addition $\int_t^{t+1} \|H(s)\|^2\,ds$ is bounded in t, then Condition 6(ii) is satisfied. At any rate, the following holds.

LEMMA 6.2. *Let $V(x, t) = x'G(t)x$ be a Liapunov function of* (6.1) *with $G(t)$ differentiable, and let $\dot{V}(x, t) = -x'H(t)'H(t)x$. Then the system is uniformly noticeable if and only if Condition* 6(i) *holds.*

Proof. This is implied by the definitions and the estimate in the proof of Lemma 6.1.

The preceding lemma enables us to state the stability statements of Theorem 4.1 in terms of the function $Q(t) = -H(t)'H(t)$. This would overlap, and sometimes extend, results in Anderson [1], Anderson and Moore [2, 3], Conti [12], Morgan and Narendra [38, 39]. We leave out the details.

We wish to make a note about the possible extension of the method of Section 5. Employing the limiting systems for the detection of uniform observability was done in Artstein [6]. Within the framework of the present paper, namely, when $Q(t) = -H(t)'H(t)$ is generated by a Liapunov function, we have the following. (The boundedness assumption is for simplicity, and can be relaxed.) We maintain the preceding notations, and in particular assume that $V(x, t) = x'G(t)x$ is a Liapunov function.

THEOREM 6.3. *Suppose that $A(t)$ and $G(t)$ are bounded. Then $(A(t), H(t))$ is uniformly observable if and only if whenever $(A_1(t), Q_1(t))$ is obtained as a limit of $(A^{t_j}(t), Q^{t_j}(t)) = (A(t_j + t), Q(t_j + t))$ (in the weak-L_2 sense on bounded intervals), and $Q_1(t) = -H_1(t)'H_1(t)$ then $(A_1(t), H_1(t))$ is observable on some interval.*

Proof. Conditions 6(ii), and 6(iii) are implied by the boundedness. Condition 6(i) fails exactly when there are sequences τ_j with $|\tau_j| \to \infty$, $\sigma_j \to \infty$ and x_j with $|x_j| = 1$ such that

$$x_j' M(\tau_j + \sigma_j, \tau_j)\, x_j \to 0 \qquad \text{as} \quad j \to \infty. \tag{6.4}$$

If $\Phi_1(t, s)$ denotes the fundamental matrix of $\dot{x} = A_1(t)x$ then it is easy to see that $\Phi_1(t, s)$ is the uniform limit on bounded intervals of $\Phi(t + t_j, s + s_j)$. Then the observability matrix $M_1(t, s)$ generated by $(A_1(t), H_1(t))$ is the limit of $M(t_j + t, t_j + s)$. If (6.4) holds, then it is easy to see that $M_1(t, s)$ cannot be regular on any interval; hence the "if" part is proved. Conversely, if $M_1(t, s)$ is not regular on any interval, say, on $[0, \sigma_j]$ with $\sigma_j \to \infty$ then taking $t_j = \tau_j$ in (6.4) would show that the latter is satisfied. This completes the proof.

Applying the previous considerations to stability requires computing $Q(t)$. For the cases when only an estimate is available we introduce the following definition and results, motivated by Theorem 2.2. The definition is in the spirit of system theory.

DEFINITION 6.4. The pair $(A, t), H(t))$ is uniformly detectable if there are $\sigma > 0$, $\delta > 0$ and $0 < \alpha < 1$ such that for every x and t either $x'M(t+\sigma, t)x \geqslant \delta \|x\|^2$ or $\|\Phi(t+\sigma, t)x\|^2 \leqslant \alpha \|x\|^2$.

THEOREM 6.5. *Let $G(t) \geqslant \varepsilon I > 0$ be bounded symmetric and differentiable and suppose that*

$$A'(t)\,G(t) + G(t)\,A(t) + \dot{G}(t) \leqslant -H(t)'H(t).$$

Suppose (6.1) *satisfies Assumption* (A). *Then* (6.1) *is exponentially stable if and only if* $(A(t), H(t))$ *is uniformly detectable.*

Proof. The "only if" is trivial since exponential stability implies the validity of the second estimate in Definition 6.4. The "if" part is obtained by noticing that the Liapunov norm $x'G(t)x$ and the euclidean norm $\|x\|^2$ are equivalent. Definition 6.4 implies that they decrease exponentially along solutions.

7. ON STABILIZATION

Consider the linear control system

$$\dot{x} = A(t)x + B(t)u \tag{7.1}$$

with $x \in R^n$, $u \in R^m$, and where $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ measurable matrix-valued mappings. The goal is to stabilize the system by a linear feedback, namely, to find a matrix $K(t)$ such that $u(x, t) = K(t)x$ would produce an exponentially stable equation

$$\dot{x} = (A(t) + B(t)\,K(t))x. \tag{7.2}$$

An additional requirement is that $K(t)$ not blow up in finite time. (Driving the state to 0 in finite time can be done via feedback, but with a singular feedback.) In this section we use the previous considerations to provide, under a uniform controllability condition, two feedback schemes. They generalize the scheme of Lukes and Kleinman (see Lukes [34], Kleinman [25] or Russell [42, p. 104]) for time-invariant systems. A generalization of this scheme for infinite dimensional systems is provided in Slemrod [47].

The controllability matrix of (7.1) is

$$W(t_1, t_0) = \int_{t_0}^{t_1} \Phi(t_0, s)\,B(s)\,B(s)'\Phi(t_0, s)'\,ds \tag{7.3}$$

(see, e.g., Brockett [11], Conti [12] or Russell [42]). System (7.1) is uniformly controllable if the matrix W satisfies the conditions for M in 6(i)–6(iii). (Indeed, it is easy to see that $(A(t), B(t))$ is uniformly controllable if and only if $(A(t)', B(t)')$ is uniformly observable.) See, e.g., Anderson and Moore [2, 3], Kalman [24].

The first scheme that we mention is included in the family of schemes proposed by Kwon and Pearson [28] (the explicit formula, however, is not mentioned in this reference). We therefore only outline the proof.

THEOREM 7.1. *Suppose that* (7.1) *is uniformly controllable with* $\sigma > 0$ *given from condition* 6(i) *for* W. *Then*

$$\dot{x} = (A(t) - B(t)\,B(t)'\,W(t+\sigma, t)^{-1})x \tag{7.4}$$

is exponentially stable; namely, $u = -B(t)'\,W(t+\sigma, t)^{-1}x$ *is a stabilizing feedback.*

Proof. One shows that $W(t+\sigma, t)^{-1}$ generates a quadratic uniformly noticeable Liapunov function for (7.4). It is, however, technically easier to address the adjoint of (7.4) and prove that $W(t+\sigma, t)$ is an anti-Liapunov function for it. Indeed, the derivative of $x'W(t+\sigma, t)x$ along solutions is a quadratic function generated by $B(t)\,B(t)' + \Phi(t, t+\sigma)\,B(t+\sigma)\,B(t+\sigma)'\,\Phi(t, t+\sigma)'$. This implies that 0 is anti-stable for the adjoint equation. If the adjoint equation together with the observation $y = B(t)'x$ is uniformly observable then the exponential anti-stability of the adjoint follows (from Lemma 6.1 and Theorem 5.1). By the duality of uniform observability and uniform controllability, it suffices to show that $(A(t) - B(t)\,B(t)'\,W(t+\sigma, t)^{-1}, B(t))$ is uniformly controllable. This is equivalent (it is easy to see) to the uniform controllability of $(A(t), B(t))$, which is provided as a condition of the theorem.

A possible interpretation of the preceding feedback rule is the following. At each point t, one acts as one wishes to steer the state x to 0 at time $t+\sigma$ with minimum control energy $\int_t^{t+\sigma} u(s)'u(s)\,ds$. This is the receding horizon scheme; see Kwon and Pearon [28]. Another receding horizon scheme is the following. Consider the sequence of times $\{k\sigma\colon k = \pm 1, \pm 2, \ldots\}$ with σ given by the uniform controllability. Then, at each point $t \in [k\sigma, (k+1)\sigma]$ one acts as one wishes to steer x to 0, with minimal control energy, at the time $(k+2)\sigma$. This control rule is

$$u(x, t) = -B(t)'\,W((k+2)\sigma, t)^{-1}x \qquad \text{for} \quad k\sigma \leqslant t < (k+1)\sigma. \tag{7.5}$$

(See Conti [12, Sect. 7.11].)

THEOREM 7.2. *Suppose that* (7.1) *is uniformly controllable, with* $\sigma > 0$ *given by Condition* 6(i) *for* W. *Then*

$$\dot{x} = A(t)x + B(t)\,u(x,t), \tag{7.6}$$

with $u(x,t)$ *given in* (7.5), *is exponentially stable.*

Proof. Consider the function

$$V(x,t) = x' W((k+2)\sigma, t)^{-1} x \qquad \text{for} \quad t \in [k\sigma, (k+1)\sigma).$$

Then $V(x,t)$ is the minimal control energy needed to steer x to 0 along $[t, (k+2)\sigma]$. (See, e.g., Conti [12, Sect. 7].) The construction of the control law implies that $V(x,t)$ is nonincreasing along solutions, namely, a Liapunov function. Uniform stability follows since by Condition 6(ii), for W we have $V(x,t) \geqslant h^{-1}I$. The Liapunov function is discontinuous at the points $k\sigma$, for k an integer. We show now that the jumps at these points are bounded away from 0 for x bounded away from 0. The jump in $V(x,t)$ at the time is the difference in control energies needed to steer x to 0 along $[k\sigma, (k+1)\sigma]$ and $[k\sigma, (k+2)\sigma]$. The controls for which the minimum energies are obtained are given explicitly as $u_1(t) = -B(t)'\Phi((k+1)\sigma, t)\, W((k+1)\sigma, k\sigma)^{-1}x$ (for $t \in [k\sigma, [k+1)\sigma])$ and $u_2(t) = -B(t)'\Phi((k+2)\sigma, t)\, W((k+2)\sigma, k\sigma)^{-1}x$ (on $t \in [k\sigma, (k+2)\sigma])$. See, e.g., Conti [12, (7.11.4)]. For definiteness we set $u_1(t) = 0$ for $t \in [(k+1)\sigma, (k+2)\sigma]$. The norm of $u_1(t) - u_2(t)$ is therefore at least the norm of $u_2(t)$ on $[(k+1)\sigma, (k+2)\sigma]$, and it is bounded away from 0 for x bounded away from 0, as can be easily verified from the uniform controllability. Therefore the norms of $u_1(t)$ and $u_2(t)$ cannot be close; otherwise $\frac{1}{2}(u_1(t) + u_2(t))$ would be a control which steers x to 0 on $[k\sigma, (k+2)\sigma]$ with norm strictly less than that of $u_2(t)$, a contradiction. Therefore the difference of the norms of u_1 and u_2, which is the jump of $V(x,t)$ at $k\sigma$, is bounded away from 0. This shows that the Liapunov function V (though not continuous in t) is uniformly noticeable. Hence (by Theorem 5.1 plus the linearity) Eq. (7.6) is exponentially stable.

8. ON AN IDENTIFICATION SCHEME

In this section we analyze an adaptive identification scheme for linear constant systems due to Narendra and Kudva [40]. The scheme is based on comparing the state output with an output of a reference model whose coefficients are continuously adapted. Modifications and extensions of this scheme were provided by Morgan and Narendra [38, 39], Anderson [1], Yuan and Wonham [52] and Morgan [37]. The conditions used in these references (and in previous literature on the subject, e.g., Lion [33] and

Kushner [27]) imply the uniform asymptotic stability of a certain nonautonomous differential equation, thus guaranteeing complete identification. The conditions that we examine here do not imply the asymptotic stability and the system might be (in Narendra and Kudva [40] terminology) nonmatchable. Yet, with the aid of the invariance principle, we show that the scheme identifies the portion of the coefficients' behavior relevant to the asymptotic behavior of the plant.

For simplicity we consider here the identification of the state equation only. (The references cited before also deal with identifying control parameters.) Consider the equation

$$\dot{x} = A_0 x + f(t) \tag{8.1}$$

with $x \in R^n$, and A_0 being an unknown matrix. The inhomogeneous term $f(t)$ might be generated by noise, or control, but it is assumed to be known. Furthermore, it is assumed that the same forcing term $f(t)$, and the state $x(t)$ of (8.1) can be used in a model plant

$$\dot{y} = -y + (A(t) + 1)\, x(t) + f(t), \tag{8.2}$$

and that $y(t)$, the state of (8.2), can be measured. Denote the "error" between the true and the model states by $e(t) = y(t) - x(t)$, and denote the difference between the model parameters and the true, unknown parameters by $D(t) = A(t) - A_0$. The adaptive scheme for $A(t)$ is provided by

$$A(t) = A(t_0) + \int_{t_0}^{t} -e(s)\, x(s)'\, ds. \tag{8.3}$$

(Here prime denotes transposition; hence $e(s)\,x(s)'$ is, as it should be, a matrix.) If $A(t)$ is generated by (8.3) then it is easy to see that $x(t)$ and $y(t)$ solve (8.1) and (8.2) if and only if $(e(t), D(t))$ is a solution of the nonautonomous system

$$\begin{aligned} \dot{e} &= -e + Dx(t), \\ \dot{D} &= -ex(t)'. \end{aligned} \tag{8.4}$$

(Notice that uniform asymptotic stability of the latter equation implies that $A(t) \to A_0$ as $t \to \infty$, uniformly in the initial conditions.)

Consider the Liapunov function for (8.4)

$$V(e, D) = \tfrac{1}{2}\left(\|e\|^2 + \operatorname{trace} D'D\right).$$

It is easy to see that

$$\dot{V}(e, D) = -\|e\|^2;$$

hence (8.4) is uniformly stable. We shall impose a condition on the state output $x(t)$, for which we need the following concepts. Let Ω be the ω-limit set of $x(\cdot)$ and let $S = \operatorname{span} \Omega$ be the linear space spanned by Ω. For each sequence $\{t_i\}$ with $t_i \to \infty$, and each number $L > 0$ we identify the subset $\Omega(\{t_i\}, L)$ of Ω consisting of all the cluster points of $x(t_i + \tau_i)$, where $0 \leqslant \tau_i \leqslant L$. (In general, $\Omega(\{t_i\}, L)$ is strictly contained in Ω and the linear space spanned by it is strictly contained in S.) We say that S is *minimal* or that S is *uniformly generated* by $x(\cdot)$ if there exists an $L > 0$ such that $S = \operatorname{span} \Omega(\{t_i\}, L)$ for every sequence $\{t_i\}$with $t_i \to \infty$.

If $x(\cdot)$ is asymptotically periodic or almost periodic then S is uniformly generated. Similar conditions are used in the study of minimal ω-limit sets; see Sibirsky [45, Sect. 5]. It is easy to see that the exciteness conditions posed by Narendra and Kudva [40], Morgan and Narendra [38] or Yuan and Wonham [52] imply that the entire space R^n is uniformly generated by $x(\cdot)$.

THEOREM 8.1. *Suppose that $x(t)$ is bounded and uniformly continuous, and that S is uniformly gnerated by $x(\cdot)$. Then $A(t)$ converges to the set of matrices A_1 with the property that $A_0 z = A_1 z$ whenever $z \in S$.*

Proof. (Direct estimates would show that if the conclusion of the theorem fails, the values $V(e(t), D(t))$ tends to $-\infty$. Here we shall use the invariance principle.) A limiting equation of (8.4) has the same structure as (8.4) except when $x(t)$ is replaced by a function $y(t) = \lim x(t_j + t)$ for a certain $t_j \to \infty$. If (e_0, D_0) belongs to the ω-limit set of $(e(\cdot), D(\cdot))$ then an unnoticeable solution $(e_1(t), D_1(t))$, of the limiting equation, passes through it. See Theorem 3.7. Clearly then $e_1(t) \equiv 0$. Therefore $D_1(t) \equiv D_0$ and $D_0 y(t) \equiv 0$. But if $D_0 z = A_1 z - A_0 z$ is not zero for a certain $z \in S$ then, in view of S being uniformly generated, $D_0 y(t)$ is not identically zero. Therefore the conclusion holds.

We see from the statement of the previous theorem in what sense the scheme identifies the parameters. It is the action of A_0 as an operator which is identified, and only the action on the space generated by the ω-limit set. In the general nonmatchable case one can argue that there is no hope that an adaptive scheme would identify more than the operator-action of A_0 on the ω-limit set.

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