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Borel quasi-orderings in subsystems of second-order arithmetic

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Abstract

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We study the provability in subsystems of second-order arithmetic of two theorems of Harrington and Shelah [6] about Borel quasi-orderings of the reals. These theorems turn out to be provable in $\mathbf{ATR}_0$, thus giving further evidence to the observation that $\mathbf{ATR}_0$ is the minimal subsystem of second-order arithmetic in which significant portions of descriptive set theory can be developed. As in [6] considering the lightface versions of the theorems will be instrumental in their proof and the main techniques employed will be the reflection principles and Gandy forcing.

In this paper we pursue the study of the provability of theorems of ordinary mathematics within subsystems of second-order arithmetic which was begun by Friedman [2] and has been developed in the program of *reverse mathematics* (e.g., [5, 4, 15, 1, 17, 3, 18, 16]). During these studies it has become clear that $\mathbf{ATR}_0$ is the minimal subsystem of second-order arithmetic in which fragments of mathematics requiring a decent theory of ordinals can be formalized. In particular a significant number of theorems of classical descriptive set theory (such as Lusin's separation theorem, the perfect set theorem for analytic sets, the determinacy of open games in $\mathbb{N}^\omega$ and Ramsey's theorem for open subsets of $[\mathbb{N}]^\omega$) can be proved in $\mathbf{ATR}_0$. Furthermore, Simpson (unpublished notes, March 1984, to appear in [16]) has shown that $\mathbf{ATR}_0$ and $\boldsymbol{\Pi}^1_1$-$\mathbf{CA}_0$ prove two different

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versions of Silver's theorem about the number of equivalence classes of a coanalytic equivalence relation by formalizing Harrington's proof (unpublished, but see [12] or [13] for proofs using Harrington's ideas in a topological context) and this result has inspired the present paper. We obtain Simpson's theorems in the case of a Borel equivalence relation as Corollaries 5.6 and 5.7.

The goal of the present paper is to show that $\mathbf{ATR}_0$ proves two other theorems in descriptive set theory which deal with Borel quasi-orderings of $\mathbb{R}$ and were originally obtained by Harrington and Shelah [6]. As usual in this field we will prove the lightface or 'effective' versions of the theorems, thus substituting Borel, analytic and coanalytic sets respectively with Δ^1_1, Σ^1_1 and Π^1_1 sets. Recall that the difference between the classical and the lightface sets is that in the definitions of the latter parameters are not allowed; [14] is the basic reference for these and other descriptive set theory notions.

Gandy forcing, in which the forcing conditions are nonempty Σ^1_1 sets, has proved to be (either directly or in its equivalent Baire category formulation in terms of the Gandy–Harrington topology) extremely useful for the proof of results about Borel relations [6, 9, 10, 7]. In this paper we develop within $\mathbf{ACA}_0$ the details of the formalization of Gandy forcing over inner models of second-order arithmetic. Model-theoretic techniques have already been employed within $\mathbf{ATR}_0$ in [4] to prove that every open subset of $[\mathbb{N}]^\omega$ is Ramsey and by Simpson in the above-mentioned work on Silver's theorem.

The language L_2 of second-order arithmetic has variables for natural numbers and for sets of natural numbers. The subsystems of second-order arithmetic we will use are $\mathbf{ACA}_0$, $\mathbf{ATR}_0$, $\mathbf{\Pi}^1_1$-$\mathbf{CA}_0$, $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$, and ATR^l_0. $\mathbf{ACA}_0$ is the subsystem of second-order arithmetic with induction restricted to quantifier-free formulas and comprehension restricted to arithmetical formulas. $\mathbf{ATR}_0$ is obtained by adding to $\mathbf{ACA}_0$ the axiom of arithmetical transfinite recursion $\forall X\, \forall a \in \mathcal{O}^X$ (H^X_a exists), i.e., "there exists a Turing jump hierarchy starting with any set along any countable well-ordering". $\mathbf{\Pi}^1_1$-$\mathbf{CA}_0$ is obtained by adding $\mathbf{\Pi}^1_1$-comprehension (or, equivalently, $\mathbf{\Sigma}^1_1$-comprehension) to $\mathbf{ACA}_0$ and is properly stronger than $\mathbf{ATR}_0$. For the detailed definitions and the basic properties of these systems, see e.g. [5]. $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$ is obtained by adding to $\mathbf{ACA}_0$ to the $\mathbf{\Sigma}^1_1$ axiom of choice and has been used for instance in [3]. The most complete reference for these systems is Simpson's forth-coming monograph [16]. ATR^l_0 is the lightface version of $\mathbf{ATR}_0$, which is obtained by adding to $\mathbf{ACA}_0$ the axiom $\forall a \in \mathcal{O}$ ($H^{\emptyset}_a$ exists), i.e., "there exists a Turing jump hierarchy starting with the empty set along any recursive well-ordering". ATR^l_0 has already been considered by Tanaka in [17] and [18]. One of the basic facts we will use about these systems is that $\mathbf{ATR}_0$ proves the existence of countable ω-models of $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$ plus any true $\mathbf{\Sigma}^1_1$ sentence [16, Theorem VIII.4.20] and in particular of $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0 + \mathrm{ATR}^l_0$.

The plan of the paper is as follows: in Section 1 we prove in $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$ some forms of Lusin's separation theorem and of the reflection principles. In Section 2 we study notions of forcing over models of $\mathbf{ACA}_0$ and in Section 3 we apply the results of Section 2 to the particular case of Gandy forcing. Sections 4 and 5 are devoted to the proofs of the main results which are the following:

Theorem 4.2. (**ATR**$_0$) *If* $(B, \preccurlyeq)$ *is a Borel quasi-ordering on the reals then one of the following is true*:

(a) *there exists a well-ordering* W *and a Borel function* $F: B \to 2^W$ *such that* F *is strongly order preserving*;

(b) *there exists a perfect set* $P \subseteq \mathbb{R}$ *such that* $\forall X, Y \in P\ (X \neq Y \Rightarrow X \perp Y)$.

Theorem 5.1. (**ATR**$_0$) *If* $(B, \preccurlyeq)$ *is a Borel quasi-ordering on the reals then one of the following is true*:

(a) *there exists a sequence* $\{B_n\}$ *of Borel chains such that* $\bigcup_n B_n = B$;

(b) *there exists a perfect set* $P \subseteq \mathbb{R}$ *such that* $\forall X, Y \in P\ (X \neq Y \Rightarrow X \perp Y)$.

1. Separation and reflection

In **ACA**$_0$ we identify a subset of $\mathbb{N}$ with its characteristic function and we call it a real.

Definition. (**ACA**$_0$) A real X is *nice* if $\forall a \in \mathcal{O}$ (H_a^X exists), i.e., if there exists a Turing jump hierarchy starting with X along any recursive well-ordering.

Notice that in $\mathbf{\Sigma}^1_1$-**AC**$_0$ being nice is a Σ^1_1 property. Notice also that if there exists a nice real X then any real recursive in X is nice, and in particular we have ATR_0'.

In **ACA**$_0$ we define a code for a Σ^1_1 (resp. Π^1_1, Δ^1_1) set of reals to be an index of the characteristic function of a recursive code for an analytic (resp. coanalytic, Borel) set of reals as defined in [16, Chapter V]. There is however a difficulty because the notion of membership of a real in (a code for) a Δ^1_1 set B is somewhat fuzzy in **ACA**$_0$: for some real X we may lack the appropriate evaluation map [16, Definition V.3.2] for deciding whether $X \in B$ or $X \notin B$, i.e., there may be no F such that $E(F, X, B)$ holds. However, if X is nice then such an evaluation map exists for every B coding a Δ^1_1 set. Notice also that the formula $X \in B$ is Δ^1_1 (i.e., provably equivalent to both a Σ^1_1 and a Π^1_1 formula) only for those X and B such that $\exists F\, E(F, X, B)$; otherwise it is only Σ^1_1 and $\neg X \in B$ is not equivalent to the Σ^1_1 formula $X \notin B$ which asserts that there is an evaluation map showing that X is not in B. Σ^1_1, Π^1_1 and Δ^1_1 subsets of $\mathbb{R}^n$ are coded in the same way and are subject to the same considerations.

Definition. (**ACA**$_0$) We say that a set of reals is Π^1_1 (resp. Σ^1_1, Δ^1_1)*-on-nice* if it is the intersection of a Π^1_1 (resp. Σ^1_1, Δ^1_1) set with the nice reals. If C is Π^1_1, Σ^1_1 or Δ^1_1-on-nice let $C^c = \{X \mid X \text{ is nice} \wedge X \notin C\}$.

Notice that in $\mathbf{\Sigma}_1^1$-$\mathbf{AC}_0$ the Σ_1^1-on-nice sets are exactly the Σ_1^1 sets which are contained in the nice reals, but that a Π_1^1 (resp. Δ_1^1)-on-nice set is not in general Π_1^1 (resp. Δ_1^1). In $\mathbf{ACA}_0$ we code Π_1^1, Σ_1^1 and Δ_1^1-on-nice sets by the code of the set of which they are the intersection with the nice reals.

In $\mathbf{\Sigma}_1^1$-$\mathbf{AC}_0$ we can prove some forms of Σ_1^1-separation and of the reflection principles if we restrict our attention to sets 'on-nice'.

Lemma 1.1 (Σ_1^1-separation). ($\mathbf{\Sigma}_1^1$-$\mathbf{AC}_0$) *If A and A' are Σ_1^1-on-nice sets such that $A \cap A' = \emptyset$ there exists a Δ_1^1-on-nice set B such that*

$$\forall X\,((X \in A \Rightarrow X \in B) \wedge (X \in A' \Rightarrow X \in B^c)).$$

Proof. If there are no nice reals then the conclusion holds for any B. Otherwise we can assume that all the elements of the Σ_1^1 codes for A and A' are nice. We can repeat the proof of Theorem V.3.9 in [16], which shows the provability of Lusin's separation theorem in $\mathbf{ATR}_0$. From the recursive codes for A and A' we obtain a Δ_1^1 code B by transfinite recursion along a recursive well-ordering starting with a recursive set: this can be done in our case because all the recursive sets are nice. Whenever X is nice, an evaluation map at X exists for all the B_τ constructed in that proof and we can prove that $(X \in A \Rightarrow X \in B) \wedge (X \in A' \Rightarrow X \notin B)$. If we now view B as a code for a Δ_1^1-on-nice set the proof is complete. □

Let $\{R_e\}$ be a fixed enumeration of all recursive sets and define $U_e = \{X \in \mathbb{R} \mid X \text{ is nice} \wedge \forall F\, \exists n\ (X[n], F[n]) \in R_e\}$. $\mathbf{ACA}_0$ proves that $\{U_e\}$ is an enumeration of all Π_1^1-on-nice subsets of $\mathbb{R}$ [16, Lemma V.1.4]. In $\mathbf{ACA}_0$ fix Π_1^1-on-nice codes for these U_e.

Definition. ($\mathbf{ACA}_0$) Define $T_e^X = \{\sigma \mid \forall n < \mathrm{lh}(\sigma)\ (X[n], \sigma[n]) \notin R_e\}$ so that for any e and X, T_e^X is a tree. If T is a tree, let $\mathrm{KB}(T)$ be the Kleene–Brouwer ordering of T [16, Definition V.1.2]. Let $\mathrm{WO}(L)$ assert that L is a countable well-ordering. If L and L' are countable linear orderings let $L \leqslant L'$ stand for the Σ_1^1 formula "there exists an order preserving map from L into L'".

$\mathbf{ACA}_0$ proves $X \in U_e \Leftrightarrow X$ is nice $\wedge\ \mathrm{WO}(\mathrm{KB}(T_e^X))$ (see [16, Lemma V.1.3]). By imitating the proof of the boldface case [16, Lemma V.2.9] one can see that $\mathbf{ACA}_0$ proves the following version of comparability of well-orderings: "if X is nice, L a recursive well-ordering and L' a well-ordering recursive in X then $L \leqslant L' \vee L' + \mathbf{1} \leqslant L$".

Definition. ($\mathbf{ACA}_0$) We say that two codes C and D for Σ_1^1, Π_1^1 or Δ_1^1-on-nice sets are *coextensional* if $\forall X\ (X \in C \Leftrightarrow X \in D)$; we write this $C \equiv D$.

Definition. ($\mathbf{ACA}_0$) A formula $\varphi(C)$ is Π^1_1 *on* Π^1_1*-on-nice* if the set $\{e \mid \varphi(U_e)\}$ is Π^1_1. $\varphi(C)$ is *extensional* if for any Δ^1_1, Σ^1_1 or Π^1_1-on-nice codes C and D such that $C \equiv D$ we have $\varphi(C) \Leftrightarrow \varphi(D)$.

Lemma 1.2 (First reflection principle). *For any formula $\varphi(C)$, $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$ proves that if $\varphi(C)$ is Π^1_1 on Π^1_1-on-nice and extensional and C is a Π^1_1-on-nice set of reals such that $\varphi(C)$ holds then there exists a Δ^1_1-on-nice set $B \subseteq C$ such that $\varphi(B)$ holds.*

Proof. We reason within $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$. If no reals are nice then there is nothing to prove. Thus we can suppose that there exists a nice real. Since $\{e \mid \varphi(U_e)\}$ is Π^1_1 there is n such that $\{e \mid \varphi(U_e)\} \equiv U_n$ (via some identification of natural numbers with recursive, and hence nice, reals). Since C is Π^1_1-on-nice there is m such that $C \equiv U_m$. For any $e \in \omega$ define

$$V_e = \{X \in C \mid \mathrm{KB}(T^e_n) \not\preccurlyeq \mathrm{KB}(T^X_m)\}.$$

Each V_e is Π^1_1-on-nice in a uniform way and there is a recursive function f such that $V_e = U_{f(e)}$. By a generalized form of the recursion theorem [14, Exercise 3H.4] there exists i such that $\exists n\,(X[n], F[n]) \in R_{f(i)} \Leftrightarrow \exists n\,(X[n], F(n)) \in R_i$ and therefore $U_{f(i)} \equiv U_i$, that is $V_i \equiv U_i$.

If $\neg\varphi(U_i)$ holds then $\neg\mathrm{WO}(\mathrm{KB}(T^i_n))$. For every $X \in C$ we have $\mathrm{WO}(\mathrm{KB}(T^X_m))$ and therefore $\mathrm{KB}(T^i_n) \not\preccurlyeq \mathrm{KB}(T^X_m)$. Thus $V_i \equiv C$: since $V_i \equiv U_i$ and $\varphi(C)$ holds the extensionality of φ gives a contradiction.

Therefore $\varphi(U_i)$ holds: let W be the recursive well-ordering $\mathrm{KB}(T^i_n)$. We claim

$$X \in U_i \quad \Leftrightarrow \quad X \text{ is nice} \wedge \mathrm{KB}(T^X_m) + \mathbf{1} \preccurlyeq W.$$

To see this notice that if $X \in U_i$ then $X \in C$ and, since X is nice, $\mathrm{KB}(T^X_m)$ is comparable with W: since $W \not\preccurlyeq \mathrm{KB}(T^X_m)$ we have $\mathrm{KB}(T^X_m) + \mathbf{1} \preccurlyeq W$. The reverse implication follows from the fact that W is a well-ordering and the claim is proved.

The claim shows that U_i is also Σ^1_1-on-nice. By Σ^1_1-separation applied to U_i and U^c_i there is a Δ^1_1-on-nice set B such that $B \equiv U_i$. Since $\varphi(U_i)$ holds, by extensionality of φ we have also $\varphi(B)$. $\square$

Definition. ($\mathbf{ACA}_0$) A formula $\varphi(C, D)$ is Π^1_1 *on* Π^1_1*-on-nice* if the set $\{(e, e') \mid \varphi(U_e, U_{e'})\}$ is Π^1_1. $\varphi(C, D)$ is *extensional* if for any Δ^1_1, Σ^1_1 or Π^1_1-on-nice codes C, D, C' and D' such that $C \equiv C' \wedge D \equiv D'$ we have $\varphi(C, D) \Leftrightarrow \varphi(C', D')$. $\varphi(C, D)$ is *monotonic upward* if for any Δ^1_1, Σ^1_1 or Π^1_1-on-nice codes C, D, C' and D' such that $C \subseteq C' \wedge D \subseteq D'$ we have $\varphi(C, D) \Rightarrow \varphi(C', D')$. $\varphi(C, D)$ is *continuous downward* if for any Δ^1_1 sequences of Δ^1_1, Σ^1_1 or Π^1_1-on-nice sets $\{C_n\}$ and $\{D_n\}$ such that $\forall n\,(C_{n+1} \subseteq C_n \wedge D_{n+1} \subseteq D_n)$ we have $\forall n\,\varphi(C_n, D_n) \Rightarrow \varphi(\bigcap_n C_n, \bigcap_n D_n)$.

Lemma 1.3 (Second reflection principle). *For any formula $\varphi(C, D)$, $\mathbf{\Sigma}^1_1$-$\mathbf{ACA}_0$ proves that if φ is Π^1_1 on Π^1_1-on-nice, extensional, monotonic upward and continuous downward and C is a Π^1_1-on-nice set such that $\varphi(C, C^c)$ holds then there exists a Δ^1_1-on-nice set $B \subseteq C$ such that $\varphi(B, B^c)$ holds.*

Proof. We reason within $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$ and claim that if $B \subseteq C$ is Δ^1_1-on-nice there exists a Δ^1_1-on-nice set B' such that $B \subseteq B' \subseteq C$ and $\varphi(B', B^c)$. To see this let

$$\psi(D) \quad \Leftrightarrow \quad \varphi(D, B^c) \wedge B \subseteq D.$$

ψ is Π^1_1 on Π^1_1-on-nice and extensional. The upward monotonicity of φ implies $\psi(C)$. By the first reflection principle there exists a Δ^1_1-on-nice $B' \subseteq C$ such that $\psi(B')$. This B' proves the claim. Furthermore notice that the procedure for going from (an index for) B to (an index for) B' is uniform because it involves only applications of the recursion theorem and of Σ^1_1-separation.

Applying repeatedly the claim we construct a sequence $\{B_n\}$ of Δ^1_1-on-nice sets as follows. Let B_0 be any Δ^1_1-on-nice subset of C. Given $B_n \subseteq C$ apply the claim to get B_{n+1} such that $B_n \subseteq B_{n+1} \subseteq C$ and $\varphi(B_{n+1}, B^c_n)$. By the uniformity noted above the sequence (of the indices for) $\{B_n\}$ is Δ^1_1. Hence $B = \bigcup_n B_n$ is Δ^1_1-on-nice. By upward monotonicity we have $\forall n\, \varphi(B, B^c_n)$, by downward continuity $\varphi(B, \bigcap_n B^c_n)$, i.e., $\varphi(B, B^c)$. $\square$

2. Forcing

Throughout this section we fix a code for a countable ω-model $\mathfrak{R}$ of $\mathbf{ACA}_0$, i.e., $\mathfrak{R} \subseteq \mathbb{N}$ and the sets of the ω-model $\mathfrak{R}$ have the form $(\mathfrak{R})_k = \{n \mid (n, k) \in \mathfrak{R}\}$ for some k (see [16, Definition VII.2.1]). We define forcing over $\mathfrak{R}$ by adapting to our case the approach of [8, Chapter VII].

Definition. ($\mathbf{ACA}_0$) A *notion of forcing over* $\mathfrak{R}$ is a quasi-ordering $(P, \leqslant)$ which is an element of $\mathfrak{R}$. As usual we will indicate $(P, \leqslant)$ by P.

For the rest of the section we fix a notion of forcing P over $\mathfrak{R}$: all the concepts we will define are dependent on P and this can be made explicit whenever necessary.

Definition. ($\mathbf{ACA}_0$) $D \subseteq P$ is *dense* if $\forall p \in P\, \exists q \in D\; q \leqslant p$. If $p_0 \in P$, D is *dense below* p_0 if $\forall p \leqslant p_0\, \exists q \in D\; q \leqslant p$.

Definition. ($\mathbf{ACA}_0$) $G \subseteq P$ is a *generic filter* over $\mathfrak{R}$ if

(i) $\forall p, q \in G\, \exists r \in G\; (r \leqslant p \wedge r \leqslant q)$,

(ii) $\forall D\; ((D \text{ is definable in } \mathfrak{R} \wedge \mathfrak{R} \models (D \text{ is dense})) \Rightarrow G \cap D \neq \emptyset)$,

(iii) $\forall p \in G\, \forall q \in P\; (p \leqslant q \Rightarrow q \in G)$.

Lemma 2.1. ($\mathbf{ACA}_0$) *If G is a generic filter over $\mathfrak{N}$, $p \in G$, D is definable in $\mathfrak{N}$ and $\mathfrak{N} \models (D$ is dense below $p)$ then $G \cap D \neq \emptyset$.*

Proof. Let $D' = \{q \in P \mid q \in D \vee \forall r \leqslant q \neg r \leqslant p\}$. D' is definable in $\mathfrak{N}$ and $\mathfrak{N} \models$ (D' is dense): if $q \in G \cap D'$ by condition (i) in the definition of generic filter we have $q \in D$. $\square$

Lemma 2.2. ($\mathbf{ACA}_0$) *For any $p_0 \in P$ there exists a generic filter over $\mathfrak{N}$ containing p_0.*

Proof. Since $\mathfrak{N}$ is a countable model there are only countably many sets which are definable in $\mathfrak{N}$. Hence we can enumerate all sets definable in $\mathfrak{N}$ which are dense within $\mathfrak{N}$: let $\{D_n\}$ be such an enumeration. Define by recursion a sequence $\{p_n\}$ as follows. We already have p_0: given p_n pick $p_{n+1} \in D_n$ such that $p_{n+1} \leqslant p_n$. Let $G = \{p \in P \mid \exists n\, p_n \leqslant p\}$. Clearly G is a generic filter over $\mathfrak{N}$ and $p_0 \in G$. $\square$

Definition. ($\mathbf{ACA}_0$) A *P-name* is a set $N \subseteq \mathbb{N} \times P$ such that $\forall (n, p) \in N\, \forall q \leqslant p\, (n, q) \in N$. We use $\mathbb{V}^P$ to denote the class of all P-names.

Definition. ($\mathbf{ACA}_0$) If $N \in \mathbb{V}^P$ and G is a generic filter over $\mathfrak{N}$, we define N^G (the *interpretation* of N under G) by $N^G = \{n \mid \exists p \in G\, (n, p) \in N\}$.

If G is a generic filter over $\mathfrak{N}$, we define a new countable *ω-model* $\mathfrak{N}[G]$ by taking $\{N^G \mid N \in \mathbb{V}^P \wedge N \in \mathfrak{N}\}$ as the collection of the sets of $\mathfrak{N}[G]$.

Definition. ($\mathbf{ACA}_0$) Let $\dot{G}$ and, for every $X \in \mathfrak{N}$, $\check{X}$ be the following elements of $\mathbb{V}^P \cap \mathfrak{N}$:

$$\dot{G} = \{(p, q) \mid p \in P \wedge q \leqslant p\}, \qquad \check{X} = \{(n, p) \mid n \in X \wedge p \in P\}.$$

Lemma 2.3. ($\mathbf{ACA}_0$) *For any G generic filter over $\mathfrak{N}$ we have $G \in \mathfrak{N}[G]$ and $\mathfrak{N} \subseteq \mathfrak{N}[G]$.*

Proof. It is immediate to check that $\dot{G}^G = G$ and for any $X \in \mathfrak{N}$, $\check{X}^G = X$. $\square$

The sets of $\mathfrak{N}[G]$ are exactly the reals arithmetical in finitely many elements of $\mathfrak{N} \cup \{G\}$.

Our next goal is to define the forcing relation $p \Vdash \varphi(n_1, \ldots, n_k, N_1, \ldots, N_h)$ for any formula φ of the language of second-order arithmetic L_2. The first step towards this is to define $\Vdash^*$: $\Vdash$ will then be defined as the relativization of $\Vdash^*$ to the model $\mathfrak{N}$.

Definition. Let $\varphi(n_1, \ldots, n_k, X_1, \ldots, X_h)$ be a formula of L_2 with exactly the free variables shown. By recursion on the complexity of φ we define within $\mathbf{ACA}_0$ a formula $p \Vdash^* \varphi(n_1, \ldots, n_k, N_1, \ldots, N_h)$ with $p, n_1, \ldots, n_k, N_1, \ldots, N_h$

as free variables as follows:

If φ is $t_0(n_1, \dots, n_k) = t_1(n_1, \dots, n_k)$ or $t_0(n_1, \dots, n_k) \leq t_1(n_1, \dots, n_k)$ then

$$p \Vdash^* \varphi(n_1, \dots, n_k) \quad \text{is} \quad p \in P \wedge \varphi(n_1, \dots, n_k).$$

If φ is $t(n_1, \dots, n_k) \in X_1$ then $p \Vdash^* \varphi(n_1, \dots, n_k, N_1)$ is

$$p \in P \wedge N_1 \in \mathbb{V}^P \wedge \{q \mid (t(n_1, \dots, n_k), q) \in N_1\} \text{ is dense below } p.$$

If φ is $\neg\varphi_0(n_1, \dots, n_k, X_1, \dots, X_h)$ then $p \Vdash^* \varphi(n_1, \dots, n_k, N_1, \dots, N_h)$ is

$$\forall q \leq p \, \neg q \Vdash^* \varphi_0(n_1, \dots, n_k, N_1, \dots, N_h).$$

If φ is $\varphi_0(n_{i_1}, \dots, n_{i_{k_1}}, X_{j_1}, \dots, X_{j_{h_1}}) \wedge \varphi_1(n_{l_1}, \dots, n_{l_{k_2}}, X_{m_1}, \dots, X_{m_{h_2}})$ then $p \Vdash^* \varphi(n_1, \dots, n_k, N_1, \dots, N_h)$ is

$$p \Vdash^* \varphi_0(n_{i_1}, \dots, n_{i_{k_1}}, N_{j_1}, \dots, N_{j_{h_1}}) \wedge p \Vdash^* \varphi_1(n_{l_1}, \dots, n_{l_{k_2}}, N_{m_1}, \dots, N_{m_{h_2}}).$$

If φ is $\exists n \, \varphi_0(n_1, \dots, n_k, n, X_1, \dots, X_h)$ then $p \Vdash^* \varphi(n_1, \dots, n_k, N_1, \dots, N_h)$ is

$$\{q \mid \exists n \, q \Vdash^* \varphi_0(n_1, \dots, n_k, n, N_1, \dots, N_h)\} \text{ is dense below } p.$$

If φ is $\exists X \, \varphi_0(n_1, \dots, n_k, X_1, \dots, X_h, X)$ then $p \Vdash^* \varphi(n_1, \dots, n_k, N_1, \dots, N_h)$ is

$$\{q \mid \exists N \in \mathbb{V}^P \, q \Vdash^* \varphi_0(n_1, \dots, n_k, N_1, \dots, N_h, N)\} \text{ is dense below } p.$$

Here t, t_0 and t_1 are numerical terms.

In the clauses of the above definition dealing with negation and existential quantifiers the sets on the right-hand side may not exist. We use this notation for perspicuity: a more formal definition would be, e.g. in the case of the last clause,

$$\forall q \leq p \, \exists r \leq q \, \exists N \in \mathbb{V}^P \, r \Vdash^* \varphi_0(n_1, \dots, n_k, N_1, \dots, N_h, N).$$

It is important to keep track of the complexity of the formulas just defined: if φ is arithmetical then $p \Vdash^* \varphi(n_1, \dots, n_k, N_1, \dots, N_h)$ is arithmetical; in $\boldsymbol{\Sigma}^1_1$-$\mathbf{AC}_0$ if φ is $\boldsymbol{\Sigma}^1_i$ (resp. $\boldsymbol{\Pi}^1_i$) then $p \Vdash^* \varphi(n_1, \dots, n_k, N_1, \dots, N_h)$ is $\boldsymbol{\Sigma}^1_i$ (resp. $\boldsymbol{\Pi}^1_i$).

Lemma 2.4. *Let $\varphi(n_1, \dots, n_k, X_1, \dots, X_h)$ be a formula of L_2 with exactly the free variables shown.* $\mathbf{ACA}_0$ *proves that for any $n_1, \dots, n_k$ and any $N_1, \dots, N_h \in \mathbb{V}^P$ the following are equivalent*:

(1) $p \Vdash^* \varphi(n_1, \dots, n_k, N_1, \dots, N_h)$,
(2) $\forall q \leq p \, q \Vdash^* \varphi(n_1, \dots, n_k, N_1, \dots, N_h)$,
(3) $\{q \mid q \Vdash^* \varphi(n_1, \dots, n_k, N_1, \dots, N_h)\}$ *is dense below* p.

[In (3) the set notation is again used for mere convenience, without implying that the set actually exists.]

Proof. (2) implies (3) is trivial. For (1) implies (2) and (3) implies (1) we proceed by induction on the complexity of φ. For notational convenience we suppress mention of $n_1, \ldots, n_k$ and $N_1, \ldots, N_h$.

If φ is either $t_0 = t_1$ or $t_0 \leq t_1$, all the implications are trivial.

If φ is either $t \in N$ or $\exists n\, \varphi_0(n)$ or $\exists X\, \varphi_0(X)$, (1) implies (2) because if D is dense below p and $q \leq p$ then D is dense below q and (3) implies (1) because if $\{q \mid D \text{ is dense below } q\}$ is dense below p, then D is dense below p.

If φ is $\neg\varphi_0$, (1) implies (2) is trivial. For (3) implies (1) suppose $\{q \mid q \Vdash^* \neg\varphi_0\}$ is dense below p. This means $\forall q \leq p\, \exists r \leq q\, \forall s \leq r\, \neg s \Vdash^* \varphi_0$. By induction hypothesis $\forall q \leq p\, \neg q \Vdash^* \varphi_0$, i.e., $p \Vdash^* \neg\varphi_0$.

If φ is $\varphi_0 \wedge \varphi_1$, the implications follow from the induction hypothesis. $\square$

Definition. For any formula $\varphi(n_1, \ldots, n_k, X_1, \ldots, X_h)$ as above define, within $\mathbf{ACA}_0$, for any $n_1, \ldots, n_k$ and any $N_1, \ldots, N_h \in \mathbb{V}^P \cap \mathfrak{N}$

$$p \Vdash \varphi(n_1, \ldots, n_k, N_1, \ldots, N_h) \quad \Leftrightarrow \quad \mathfrak{N} \models (p \Vdash^* \varphi(n_1, \ldots, n_k, N_1, \ldots, N_h)).$$

Notice that for any φ the formula $p \models \varphi(n_1, \ldots, n_k, N_1, \ldots, N_h)$ is arithmetical. Notice also that Lemma 2.4 holds within $\mathfrak{N}$ and thus it holds also if we replace $\Vdash^*$ with $\Vdash$: in the following most of the references to Lemma 2.4 are to this version.

Our next result is our version of the so-called forcing-equals-truth lemma.

Lemma 2.5. *Let $\varphi(n_1, \ldots, n_k, X_1, \ldots, X_h)$ be a formula with exactly the free variables shown.* $\mathbf{ACA}_0$ *proves that for any G generic filter over $\mathfrak{N}$, any $n_1, \ldots, n_k$ and any $N_1, \ldots, N_h \in \mathbb{V}^P \cap \mathfrak{N}$ the following are equivalent*:

(1) $\mathfrak{N}[G] \models \varphi(n_1, \ldots, n_k, N_1^G, \ldots, N_h^G)$,

(2) $\exists p \in G\, p \Vdash \varphi(n_1, \ldots, n_k, N_1, \ldots, N_h)$.

Proof. By induction on the complexity of φ. Again we suppress explicit mention of $n_1, \ldots, n_k$ and $N_1, \ldots, N_h$.

If φ is $t_0 = t_1$ or $t_0 \leq t_1$, the lemma is immediate.

If φ is $t \in N$, the lemma says: $\exists p \in G\;\; (t, p) \in N$ if and only if $\exists p \in G$ $(\{q \mid (t, q) \in N\}$ is dense below $p)$. This follows from the definition of P-name and Lemma 2.1.

If φ is $\neg\varphi_0$, argue as follows. If $\mathfrak{N}[G] \models \varphi$, define within $\mathfrak{N}$ a set $D = \{p \mid p \Vdash^* \varphi_0 \vee p \Vdash^* \neg\varphi_0\}$. D is dense by the definition of $p \Vdash^* \neg\varphi_0$. Let $p \in G \cap D$: if $p \Vdash \varphi_0$, by induction hypothesis $\mathfrak{N}[G] \models \varphi_0$; hence $p \Vdash \neg\varphi_0$. If $p \in G$ and $p \Vdash \varphi$, we have that $\forall q \leq p\, \neg q \Vdash \varphi_0$. By Lemma 2.4 and condition (i) in the definition of generic filter $\neg\exists p' \in G\, p' \Vdash \varphi_0$, which by induction hypothesis implies $\neg\mathfrak{N}[G] \models \varphi_0$, i.e., $\mathfrak{N}[G] \models \varphi$.

If φ is $\varphi_0 \wedge \varphi_1$, the lemma follows from the induction hypothesis, Lemma 2.4 and (i) in the definition of generic filter.

If φ is $\exists n\, \varphi_0(n)$ and $\mathfrak{N}[G] \models \varphi$, fix n such that $\mathfrak{N}[G] \models \varphi_0(n)$; by induction hypothesis there exists $p \in G$ such that $p \Vdash \varphi_0(n)$. By Lemma 2.4, $\{q \mid q \Vdash \varphi_0(n)\}$ is dense below p and this implies $p \Vdash \exists n\, \varphi_0(n)$. For (2) implies (1) suppose $p \in G$ and $p \Vdash \exists n\, \varphi_0(n)$, i.e., in $\mathfrak{N}$ $\{q \mid \exists n q \Vdash^* \varphi_0(n)\}$ is dense below p. By Lemma 2.1 there exist $q \leqslant p$ and n such that $q \in G$ and $q \Vdash \varphi_0(n)$. By induction hypothesis $\mathfrak{N}[G] \models \varphi_0(n)$ and hence $\mathfrak{N}[G] \models \varphi$.

If φ is $\exists X\, \varphi_0(X)$, we can repeat the argument of the previous case using P-names in place of natural numbers. □

Lemma 2.6. ($\mathbf{ACA}_0$) *Let P be a notion of forcing over $\mathfrak{N}$ and G a P-generic filter over $\mathfrak{N}$. If $\mathfrak{N}$ is an ω-model of $\mathbf{ACA}_0$ then $\mathfrak{N}[G]$ is an ω-model of $\mathbf{ACA}_0$. If $\mathfrak{N}$ is an ω-model of $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$ then $\mathfrak{N}[G]$ is an ω-model of Σ^1_1-AC_0.*

Proof. $\mathfrak{N}[G]$ is an ω-model by construction and hence satisfies the basic axioms (i.e., the usual recursive equations for $+$, $\cdot$ and $\leqslant$) and the induction axiom. Now suppose $\mathfrak{N}$ satisfies $\mathbf{ACA}_0$ and let $\varphi(n, X)$ be an arithmetical formula: we need to show that for any $N \in \mathbb{V}^P \cap \mathfrak{N}$, $\{n \mid \mathfrak{N}[G] \models \varphi(n, N^G)\} \in \mathfrak{N}[G]$ (the case where φ has more than one set parameter is entirely analogous). Within $\mathfrak{N}$ by arithmetical comprehension let $N_0 = \{(n, p) \mid p \Vdash^* \varphi(n, N)\}$. By Lemma 2.5, $N_0^G = \{n \mid \mathfrak{N}[G] \models \varphi(n, N^G)\}$. This proves the first part of the lemma.

Now suppose $\mathfrak{N}[G]$ is a model of $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$. By the first part of the lemma $\mathfrak{N}[G]$ is a model of $\mathbf{ACA}_0$ and it suffices to prove that $\mathfrak{N}[G]$ satisfies

$$\forall X\, (\forall k((X)_k \text{ is a tree with a path}) \Rightarrow \exists Y\, \forall k\, ((Y)_k \text{ is a path through } (X)_k)).$$

Let $N \in \mathbb{V}^P \cap \mathfrak{N}$ be such that $\mathfrak{N}[G] \models \forall k\, ((N^G)_k$ is a tree with a path). For any k let $[N]_k = \{(n, p) \mid ((n, k), p) \in N\}$ so that $([N]_k)^G = (N^G)_k$. By Lemma 2.5 there exists $p_0 \in G$ such that $p_0 \Vdash \forall k\, \exists F$ (F is a path through $[N]_k$). Working through the definition of forcing one sees that this means $\mathfrak{N} \models (\forall k\, \forall p \leqslant p_0\, \exists N' \in \mathbb{V}^P$ $\exists q \leqslant p\, q \Vdash^* (N'$ is a path through $[N]_k))$. By $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$ in $\mathfrak{N}$, there exists $Z \in \mathfrak{N}$ such that $\forall k\, \forall p\, (Z)_{(k,p)} \in \mathbb{V}^P \cap \mathfrak{N}$ and $\mathfrak{N} \models (\forall k\, \forall p \leqslant p_0\, \exists q \leqslant p\, q \Vdash^* ((Z)_{(k,p)}$ is a path through $[N]_k))$. This means that for each k, $D_k = \{q \mid \exists p \geqslant p_0\, (q \leqslant p \wedge q \Vdash^*$ $((Z)_{(k,p)}$ is a path through $[N]_k))\}$ is dense below p_0 in $\mathfrak{N}$. For each k pick $q_k \in D_k \cap G$ and a corresponding p_k. By arithmetical comprehension in $\mathfrak{N}[G]$ the set $Y = \{(n, k) \mid n \in (Z)^G_{(k,p_k)}\}$ is in $\mathfrak{N}[G]$ and by Lemma 2.5, $\mathfrak{N}[G] \models \forall k$ $((Y)_k$ is a path through $(N^G)_k)$. □

Lemma 2.7. *Let $\varphi(n_1, \dots, n_k, X_1, \dots, X_h)$ be a formula with exactly the free variables shown. $\mathbf{ACA}_0$ proves that if for any G generic filter over $\mathfrak{N}$ we have $\mathfrak{N}[G] \models \varphi(n_1, \dots, n_k, N_1^G, \dots, N_h^G)$, then $\forall p \in P$ $p \Vdash \varphi(n_1, \dots, n_k, N_1, \dots, N_h)$.*

Proof. Suppose that there exists $p \in P$ such that $\neg p \Vdash \varphi$. Then by definition of $\Vdash^*$ there exists $p_0 \leqslant p$ such that $p_0 \Vdash \neg\varphi$. By Lemma 2.2 let G be a generic filter containing p_0. Lemma 2.5 now gives a contradiction. □

The following lemma is not concerned with forcing but will be applied in Sections 4 and 5 to the model $\mathfrak{N}[G]$. It asserts that when we deal with codes for $\boldsymbol{\Delta}^1_1$ sets that are such 'in the real world', even a model of $\boldsymbol{\Sigma}^1_1$-$\mathbf{AC}_0$ contains all the appropriate evaluation maps.

Lemma 2.8. ($\mathbf{ACA}_0$) *Let $\mathfrak{M}$ be a countable ω-model of $\boldsymbol{\Sigma}^1_1$-$\mathbf{AC}_0$ and B a (code for a) $\boldsymbol{\Delta}^1_1$ set such that $B \in \mathfrak{M}$. For any $X \in \mathfrak{M}$ there exists in $\mathfrak{M}$ an evaluation map for B at X, and hence $\mathfrak{M} \models (X \in B)$ if and only if $X \in B$. In particular* $\mathfrak{M} \models (\neg X \in B \Leftrightarrow X \notin B)$. [For the difference between $\neg X \in B$ and $X \notin B$ see the remarks at the beginning of Section 1.]

Proof. This is an immediate consequence of Lemma VIII.4.15 of [16]. It can be proved directly by arithmetical transfinite induction along $\mathrm{KB}(B)$. $\square$

3. Gandy forcing

We will now study some specific notions of forcing: the so-called Gandy forcing and some product forcings obtained from it. We fix a countable ω-model $\mathfrak{M}$ of $\mathrm{ATR}^{\prime}_0 + \boldsymbol{\Sigma}^1_1$-$\mathbf{AC}_0$ and a countable ω-model $\mathfrak{N}$ of $\boldsymbol{\Sigma}^1_1$-$\mathbf{AC}_0$ such that $\mathfrak{M} \in \mathfrak{N}$.

Definition. ($\mathbf{ACA}_0$) For any n define $\mathbf{P}^n$ to be the set of all (codes for) Σ^1_1 sets $A \in \mathfrak{M}$ such that $\mathfrak{M} \models (A \subseteq \mathbb{R}^n \wedge A$ is Σ^1_1-on-nice $\wedge\ \exists X\ (X \in A))$. For A, $A' \in \mathbf{P}^n$ define $A \leqslant A'$ if and only if $\mathfrak{M} \models A \subseteq A'$. We write $\mathbf{P}$ in place of $\mathbf{P}^1$.

Note that the code for any $A \in \mathbf{P}^n$ is a natural number and that the formulas defining $\mathbf{P}^n$ and $\leqslant$ are both arithmetical within $\mathfrak{N}$. Therefore Gandy forcing can be considered a notion of forcing in the sense defined in Section 2. One of the key properties of Gandy forcing is given by the following lemma.

Lemma 3.1. ($\mathbf{ACA}_0$) *If G is a $\mathbf{P}^n$-generic filter over $\mathfrak{N}$ then there exists a unique $X \in \mathfrak{N}[G] \cap \mathbb{R}^n$ such that $X \in \bigcap G$, i.e., such that $\forall A \in G\ X \in A$. In this situation we say that X is $\mathbf{P}^n$-generic.*

Proof. For notational convenience we prove the lemma for $n = 1$: the general case is similar. We reason within $\mathfrak{N}[G]$. Pick some $A \in G$ and write

$$X \in A \quad \Leftrightarrow \quad \exists F\ \forall n\ \theta(F[n], X[n])$$

where θ is Σ^0_0. Within $\mathfrak{M}$ for $\sigma \in 2^{<\omega}$ and $\tau \in \mathbb{N}^{<\omega}$ define

$$A_{\sigma,\tau} = \{X \mid X \supset \sigma \wedge \exists F \supset \tau\ \forall n\ \theta(F[n], X[n])\}.$$

The sequence (of codes for) $\{A_{\sigma,\tau}\}$ belongs to $\mathfrak{M}$ by Theorem V.1.7' of [16].

We define two sequences $\{\sigma_n^A\}$ and $\{\tau_n^A\}$ such that $\mathrm{lh}(\sigma_n^A)=\mathrm{lh}(\tau_n^A)=n$ and $A_{\sigma_n^A,\tau_n^A}\in G$ as follows. Let $\sigma_0^A=\tau_0^A=\langle\rangle$. Suppose we have defined σ_n^A and τ_n^A.

$$\mathbf{D}=\{A'\in\mathbf{P}\mid \exists i\,\exists m\, A'\leqslant A_{\sigma_n^{A\frown}\langle i\rangle,\tau_n^{A\frown}\langle m\rangle}\}$$

is dense below $A_{\sigma_n^A,\tau_n^A}$ in $\mathfrak{N}$. Thus there exist i and m such that $A_{\sigma_n^{A\frown}\langle i\rangle,\tau_n^{A\frown}\langle m\rangle}\in G$. Set $\sigma_{n+1}^A=\sigma_n^{A\frown}\langle i\rangle$ and $\tau_{n+1}^A=\tau_n^{A\frown}\langle m\rangle$.

Now let $X^A=\bigcup_n\sigma_n^A$ and $F^A=\bigcup_n\tau_n^A$. It is straightforward to check that $\forall n\,\theta(F^A[n],X^A[n])$ and hence $X^A\in A$. We repeat this construction for every $A\in G$. If $X^A\neq X^{A'}$, take n so that $X^A[n]\neq X^{A'}[n]$: then $\mathfrak{M}\models(A_{\sigma_n^A,\tau_n^A}\cap A_{\sigma_n^{A'},\tau_n^{A'}}=\emptyset)$ and hence $\neg\exists A''\ (A''\leqslant A_{\sigma_n^A,\tau_n^A}\wedge A''\leqslant A_{\sigma_n^{A'},\tau_n^{A'}})$ against (i) in the definition of generic filter. Thus $\forall A\in G\, X^A=X$ for some X and $\forall A\in G\, X\in A$. Clearly this X is unique. $\square$

Definition. ($\mathbf{ACA}_0$) Let $\dot X\in\mathbb{V}^{\mathbf{P}^n}\cap\mathfrak{N}$ be defined within $\mathfrak{N}$ by

$$\begin{aligned}(n,A')\in\dot X\quad&\Leftrightarrow\quad A'\Vdash^*\exists X\,(\forall A\in G\,(X\in A)\wedge n\in X)\\&\Leftrightarrow\quad A'\Vdash^*\forall X\,(\forall A\in G\,(X\in A)\Rightarrow n\in X).\end{aligned}$$

The two formulas on the right-hand side are equivalent because, by Lemmas 3.1 and 2.7, $\{X\mid X\text{ is nice}\}\Vdash^*\exists!X\,\forall A\in G\,(X\in A)$. $\dot X$ is defined within $\mathfrak{N}$ by $\mathbf{\Delta}_1^1$-comprehension, a consequence of $\mathbf{\Sigma}_1^1$-$\mathbf{AC}_0$ [16, Lemma VII.6.6.1].

For any G, $\mathbf{P}^n$-generic filter over $\mathfrak{N}$, we have $\mathfrak{N}[G]\models\bigcap G=\{\dot X^G\}$ so that $\dot X$ is a $\mathbf{P}^n$-name for the X whose existence in $\mathfrak{N}[G]$ is asserted by Lemma 3.1.

We will consider also some product forcings modulo an equivalence relation. In the following let $E\subseteq\mathbb{R}^2$ be a fixed Σ_1^1 set such that $\mathfrak{M}\models(E$ is an equivalence relation).

Definition. ($\mathbf{ACA}_0$) Let

$$\begin{aligned}&\mathbf{P}^n\times\mathbf{P}^m=\{(A,A')\mid A\in\mathbf{P}^n\wedge A'\in\mathbf{P}^m\},\\&\mathbf{P}_E^n=\{A\in\mathbf{P}^n\mid\mathfrak{M}\models(\forall X\in A\,\forall i,j<n\,X_i\,E\,X_j)\},\\&\mathbf{P}_E^n\times_E\mathbf{P}_E^m=\{(A,A')\mid A\in\mathbf{P}_E^n\wedge A'\in\mathbf{P}_E^m\wedge\mathfrak{M}\models(\exists X\in A\,\exists Y\in A'\,X_0\,E\,Y_0)\}.\end{aligned}$$

For (A,A'), $(A_0,A_0')\in\mathbf{P}^n\times\mathbf{P}^m$ let $(A,A')\leqslant(A_0,A_0')$ if and only if $A\leqslant A_0$ and $A'\leqslant A_0'$ (where the last two $\leqslant$ are the orderings of $\mathbf{P}^n$ and $\mathbf{P}^m$). The orderings of $\mathbf{P}_E^n$ and $\mathbf{P}_E^n\times_E\mathbf{P}_E^m$ are the restrictions of those of $\mathbf{P}^n$ and $\mathbf{P}^n\times\mathbf{P}^m$. We abbreviate $\mathbf{P}_E^1\times_E\mathbf{P}_E^1$ by $\mathbf{P}\times_E\mathbf{P}$.

All the above notions of forcing can be considered in the framework developed in Section 2. Moreover the analogues of Lemma 3.1 hold also for these forcings. For example it is easy to check that if G is a $\mathbf{P}_E^n$-generic filter over $\mathfrak{N}$ then $\{A\in\mathbf{P}^n\mid\exists A'\leqslant A\,A'\in G\}$ is $\mathbf{P}^n$-generic over $\mathfrak{N}$ and hence there exists a unique $\mathbf{P}_E^n$-generic $X\in\mathfrak{N}[G]\cap\mathbb{R}^n$ such that $X\in\bigcap G$. The case of $\mathbf{P}_E^n\times_E\mathbf{P}_E^m$ is less trivial and is considered in the next lemmas.

Lemma 3.2. ($\mathbf{ACA}_0$) *If G is a $\mathbf{P}^n_E \times_E \mathbf{P}^m_E$-generic filter over $\mathfrak{N}$ then $G_0 = \{A \mid \exists A' \in \mathbf{P}^m_E \ (A, A') \in G\}$ is a $\mathbf{P}^n_E$-generic filter over $\mathfrak{N}$. Similarly $G_1 = \{A' \mid \exists A \in \mathbf{P}^n_E \ (A, A') \in G\}$ is a $\mathbf{P}^m_E$-generic filter over $\mathfrak{N}$.*

Proof. To avoid a cumbersome notation we prove the lemma for $n = m = 1$ and G_0. Let $\mathbf{D}$ be a $\mathbf{P}$-dense set definable in $\mathfrak{N}$; we need to show that $G_0 \cap \mathbf{D} \neq \emptyset$. Let $\mathbf{D}' = \{(A, A') \in \mathbf{P} \times_E \mathbf{P} \mid A \in \mathbf{D}\}$. We claim that $\mathbf{D}'$ is $\mathbf{P} \times_E \mathbf{P}$-dense. We reason in $\mathfrak{N}$: let $(A, A') \in \mathbf{P} \times_E \mathbf{P}$ and define

$$B' = \{X \in A \mid \exists Y \, (Y \in A' \wedge X \, E \, Y)\} \in \mathbf{P}.$$

Then there exists $B \in \boldsymbol{D}$ such that $B \leqslant B'$; moreover $(B, A') \in \mathbf{P} \times_E \mathbf{P}$ by the definition of B'. This proves the claim.

Therefore there exists $(A, A') \in G \cap \mathbf{D}'$ and hence $A \in G_0 \cap \mathbf{D}$. □

Lemma 3.3. ($\mathbf{ACA}_0$) *If G is a $\mathbf{P}^n_E \times_E \mathbf{P}^m_E$-generic filter over $\mathfrak{N}$ then there exists a unique pair $(X, Y) \in \mathfrak{N}[G] \cap (\mathbb{R}^n \times \mathbb{R}^m)$ such that $\forall (A, A') \in G \ (X \in A \wedge Y \in A')$. In this situation we say that (X, Y) is $\mathbf{P}^n_E \times_E \mathbf{P}^m_E$-generic.*

Proof. This is a consequence of Lemma 3.2 and of the observation immediately preceding it. □

Definition. ($\mathbf{ACA}_0$) Let $\dot{X}, \dot{Y} \in \mathbb{V}^{\mathbf{P}^n_E \times_E \mathbf{P}^m_E} \cap \mathfrak{N}$ be such that for any G, $\mathbf{P}^n_E \times_E \mathbf{P}^m_E$-generic filter over $\mathfrak{N}$, $(\dot{X}^G, \dot{Y}^G)$ is the pair whose existence in $\mathfrak{N}[G]$ is asserted by Lemma 3.3. Explicit definitions of $\dot{X}$ and $\dot{Y}$ can be given similarly to the definition of $\dot{X}$ given after Lemma 3.1.

Another consequence of Lemma 3.2 is that $\mathbf{ACA}_0$ proves that for any (X, Y) which is $\mathbf{P}^n_E \times_E \mathbf{P}^m_E$-generic, X is $\mathbf{P}^n_E$-generic and Y is $\mathbf{P}^m_E$-generic. Along the same lines we can prove the following very useful lemma.

Lemma 3.4. ($\mathbf{ACA}_0$) *If (X, Y) is $\mathbf{P}^n_E \times_E \mathbf{P}^m_E$-generic then (X_i, Y_j) and (Y_j, X_i) are $\mathbf{P} \times_E \mathbf{P}$-generic for any $i < n$, $j < m$.*

Proof. We prove this for $n = m = 2$ and (X_0, Y_1). Let $\mathbf{D}$ be a $\mathbf{P} \times_E \mathbf{P}$-dense set definable in $\mathfrak{N}$. We need to show $\exists (A, A') \in \mathbf{D} \, (X_0 \in A \wedge Y_1 \in A')$. Define

$$\mathbf{D}' = \{(C, C') \in \mathbf{P}^2_E \times_E \mathbf{P}^2_E \mid (\{X \mid \exists Y (X, Y) \in C\}, \{Y \mid \exists X (X, Y) \in C'\}) \in \mathbf{D}\}.$$

We claim that $\mathbf{D}'$ is $\mathbf{P}^2_E \times_E \mathbf{P}^2_E$-dense. We reason in $\mathfrak{N}$: given $(A, A') \in \mathbf{P}^2_E \times_E \mathbf{P}^2_E$ let $\bar{A} = \{X \mid \exists Y \, (X, Y) \in A\}$ and $\bar{A}' = \{Y \mid \exists X \, (X, Y) \in A'\}$. Then $(\bar{A}, \bar{A}') \in \mathbf{P} \times_E \mathbf{P}$. Since $\mathbf{D}$ is dense there exists $(\bar{C}, \bar{C}') \in \mathbf{D}$ such that $(\bar{C}, \bar{C}') \leqslant (\bar{A}, \bar{A}')$. Let $C = \{(X, Y) \mid (X, Y) \in A \wedge X \in \bar{C}\}$ and $C' = \{(X, Y) \mid (X, Y) \in A' \wedge Y \in \bar{C}'\}$. We have $(C, C') \in \mathbf{P}^2_E \times_E \mathbf{P}^2_E$ because $\exists X \in \bar{C} \, \exists Y \in \bar{C}' \, (X \, E \, Y \wedge X \in \bar{A} \wedge Y \in \bar{A}')$. This proves the claim.

Therefore there exists $(C, C') \in \mathbf{D}'$ such that $(X_0, X_1) \in C$ and $(Y_0, Y_1) \in C'$. If $A = \{X \mid \exists Y\, (X, Y) \in C\}$ and $A' = \{Y \mid \exists X\, (X, Y) \in C'\}$ we have $(A, A') \in \mathbf{D}$, $X_0 \in A$ and $Y_1 \in A'$. $\square$

In the proofs of the main theorems we will need to obtain a perfect set of reals which are pairwise generic in some product forcing. The next lemma enables us to obtain this set in a quite general situation.

Definition. ($\mathbf{ACA}_0$) We will use the following notation: if $t \subset 2^{<\omega}$ is a finite tree let L_t be the set of its leaves, i.e., $L_t = \{\sigma \in t \mid \sigma^\frown\langle 0\rangle, \sigma^\frown\langle 1\rangle \notin t\}$; if $\sigma, \tau \in 2^{<\omega}$ let $\sigma \sqcap \tau$ be the greatest common initial segment of σ and τ; similarly if $F, G \in 2^\omega$, $F \neq G$, let $F \sqcap G$ be the greatest common initial segment of F and G; if $T \subseteq 2^{<\omega}$ is a tree, let $[T]$ be the set of the paths through T. A tree T is *perfect* if any of its elements has two incomparable extensions in T.

Lemma 3.5. ($\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$) *Let $\tilde{A} \in \mathbf{P}$ and suppose that for each $A \leqslant \tilde{A}$ we have a Σ^1_1 set $E_A \subseteq \mathbb{R}^2$ such that $\mathfrak{M} \models (E_A$ is an equivalence relation$)$. Moreover, suppose that $A \leqslant A'$ implies $\mathfrak{M} \models \forall X, Y\ (X\, E_A\, Y \Rightarrow X\, E_{A'}\, Y)$ and that for some formula $\varphi(X, Y)$ and for all $A \leqslant \tilde{A}$ it is not the case that $(A, A) \Vdash_{\mathbf{P} \times_{E_A} \mathbf{P}} \neg\varphi(\dot{X}, \dot{Y})$. Then there exists a perfect tree $T \subseteq 2^{<\omega}$, a function $H : T \to \mathbf{P}$ and a $\mathbf{\Sigma}^1_1$ map $F \mapsto X_F$ from $[T]$ to $\mathbb{R}$ such that for all $F_0, F_1 \in [T]$ if $F_0 \neq F_1$ then $(X_{F_0}, X_{F_1}) \in \tilde{A} \times \tilde{A}$ are $\mathbf{P} \times_{E_{H(F_0 \sqcap F_1)}} \mathbf{P}$-generic and if G is the generic filter which gives rise to F_0 and F_1 then $\mathfrak{N}[G] \models \varphi(X_{F_0}, X_{F_1})$.*

Proof. We will use another notion of forcing over $\mathfrak{N}$, whose elements are finite approximations of the perfect tree we want to construct. By arithmetical comprehension let Q be the set of all pairs (t, h) such that $t \subset 2^{<\omega}$ is a nonempty finite tree, $h : t \to \mathbf{P}$ and the following conditions are satisfied:

(1) $h(\langle\rangle) = \tilde{A}$,

(2) $\sigma \subseteq \tau \in t \Rightarrow h(\tau) \leqslant h(\sigma)$,

(3) $\sigma^\frown\langle 0\rangle, \sigma^\frown\langle 1\rangle \in t \Rightarrow (h(\sigma^\frown\langle 0\rangle), h(\sigma^\frown\langle 1\rangle)) \Vdash_{\mathbf{P} \times_{E_{h(\sigma)}} \mathbf{P}} \varphi(\dot{X}, \dot{Y})$,

(4) $\mathfrak{M} \models \exists\{X_\sigma \mid \sigma \in L_t\}[\forall \sigma \in L_t\, (X_\sigma \in h(\sigma)) \wedge \forall \sigma, \sigma' \in L_t\, (X_\sigma\, E_{h(\sigma \sqcap \sigma')}\, X_{\sigma'})]$.

Any $\{X_\sigma \mid \sigma \in L_t\}$ satisfying the condition in brackets in (4) is called a leaves labelling for (t, h).

We define a quasi-ordering (in fact a partial ordering) on Q by:

$$(t, h) \leqslant (t', h') \quad \Leftrightarrow \quad t' \subseteq t \wedge \forall \sigma \in t'\ h(\sigma) = h'(\sigma).$$

$(Q, \leqslant)$ is a notion of forcing over $\mathfrak{N}$: by Lemma 2.2 let G be a Q-generic filter over $\mathfrak{N}$. Clearly $T = \{\sigma \mid \exists (t, h) \in G\ \sigma \in t\}$ is a tree and $H : T \to \mathbf{P}$ defined by $H(\sigma) = h(\sigma)$ for any h such that there exists $(t, h) \in G$ with $\sigma \in t$ is a function which satisfies (1), (2) and (3) in the definition of Q.

Our first claim is that T is perfect. To prove this we show that for any $\tau \in 2^{<\omega}$ the set $D_\tau = \{(t, h) \in Q \mid t$ contains two incomparable extensions of $\tau\}$ is dense below any (t', h') such that $\tau \in t'$. To this end it is clearly enough to show that for any such (t', h') we can find $(t, h) \in D_\tau$ such that $(t, h) \leqslant (t', h')$. Let $\rho \supseteq \tau$ be such that $\rho \in L_{t'}$. Let $A = \{X_\rho \mid \{X_\sigma \mid \sigma \in L_{t'}\}$ leaves labelling for $(t', h')\}$. By (4) in the definition of Q we have $A \in \mathbf{P}$; notice also that $A \leqslant h'(\rho)$. By one of the hypothesis of the lemma there exists $(A_0, A_1) \in \mathbf{P} \times_{E_A} \mathbf{P}$ such that $(A_0, A_1) \leqslant (A, A)$ and $(A_0, A_1) \Vdash_{\mathbf{P} \times_{E_A} \mathbf{P}} \varphi(\dot{X}, \dot{Y})$. Let $t = t' \cup \{\rho^\frown\langle 0\rangle, \rho^\frown\langle 0, 0\rangle, \rho^\frown\langle 0, 1\rangle\}$ and

$$h(\sigma) = \begin{cases} h'(\sigma) & \text{if } \sigma \in t', \\ A & \text{if } \sigma = \rho^\frown\langle 0\rangle, \\ A_i & \text{if } \sigma = \rho^\frown\langle 0, i\rangle \quad (i = 0, 1). \end{cases}$$

It is clear that $(t, h) \in Q$ ((4) follows from the definition of A and from the fact that $\mathfrak{M} \models (X E_A Y \Rightarrow X E_{h(\sigma)} Y)$ for any $\sigma \in t'$, $\sigma \subseteq \rho$), $(t, h) \in D_\tau$ and $(t, h) \leqslant (t', h')$. This completes the proof of the claim.

Now we claim that for any $F_0, F_1 \in [T]$ if $F_0 \neq F_1$, $\tau = F_0 \sqcap F_1$ and $A = H(\tau)$ the set $G' = \{(A_0, A_1) \in \mathbf{P} \times_{E_A} \mathbf{P} \mid \forall i < 2\, \exists \sigma_i \subset F_i\, H(\sigma_i) \leqslant A_i\}$ is $\mathbf{P} \times_{E_A} \mathbf{P}$-generic. If we prove this claim the proof of the lemma is completed by letting, for every $F \in [T]$, $X_F = \bigcap_n H(F[n])$ (this map is $\mathbf{\Sigma}^1_1$ by $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$) and applying Lemma 2.5.

For any $(t, h) \in Q$ let ρ_i^t and σ_i^t $(i = 0, 1)$ be such that $\rho_i^t \subset F_i$, $\rho_i^t \in t$ and $\sigma_i^t = \rho_i^{t\frown}\langle F_i(\mathrm{lh}(\rho_i^t))\rangle \notin t$. Given any $\mathbf{P} \times_{E_A} \mathbf{P}$-dense set $\mathbf{D}$ definable in $\mathfrak{N}$ to show $\mathbf{D} \cap G' \neq \emptyset$ let

$$\begin{aligned} D' = \{(t, h) \in Q \mid\, & \exists \sigma_0, \sigma_1 \in t\, (\sigma_0 \subset F_0 \wedge \sigma_1 \subset F_1 \wedge (h(\sigma_0), h(\sigma_1)) \in \mathbf{D}) \\ & \vee \neg\exists (t', h') \leqslant (t, h)\, (\sigma_0^t \in t' \wedge \sigma_1^t \in t')\}. \end{aligned}$$

It suffices to show that D' is dense below any (t', h') such that $\tau^\frown\langle 0\rangle, \tau^\frown\langle 1\rangle \in t'$ and to this end it is enough to show that for any such (t', h') we can find $(t, h) \in D'$ such that $(t, h) \leqslant (t', h')$. Given (t', h') if the second disjunct in the definition holds of (t', h') we are done, otherwise we can suppose (t', h') is such that $\rho_i^{t'} \in L_{t'}$. Let $A_i = \{X_{\rho_i^{t'}} \mid \{X_\sigma \mid \sigma \in L_{t'}\}$ leaves labelling for $(t', h')\}$. Clearly $(A_0, A_1) \in \mathbf{P} \times_{E_A} \mathbf{P}$ and hence there exists $(A_0', A_1') \in \mathbf{D}$ such that $(A_0', A_1') \leqslant (A_0, A_1)$. Let $t = t' \cup \{\sigma_0^{t'}, \sigma_1^{t'}\}$ and

$$h(\sigma) = \begin{cases} h'(\sigma) & \text{if } \sigma \in t', \\ A_i & \text{if } \sigma = \sigma_i^{t'} \quad (i = 0, 1). \end{cases}$$

The only nontrivial point is to find in $\mathfrak{M}$ a leaves labelling for (t, h), since then it is clear that $(t, h) \in Q$, $(t, h) \in D'$ and $(t, h) \leqslant (t', h')$.

We reason within $\mathfrak{M}$. Suppose X_0 and X_1 are such that $X_0 \in A_0 \wedge X_1 \in A_1 \wedge X_0\, E_A\, X_1$. Let $\{X_\sigma^i \mid \sigma \in L_{t'}\}$ $(i = 0, 1)$ be leaves labellings for (t', h') such that $X^i_{\rho_i^{t'}} = X_i$. Define $\{X_\sigma \mid \sigma \in L_t\}$ by

$$X_\sigma = \begin{cases} X_\sigma^0 & \text{if } \sigma \in L_{t'} \wedge (\tau \not\subseteq \sigma \vee \tau \subset \sigma \sqcap \rho_0^{t'}), \\ X_\sigma^1 & \text{if } \sigma \in L_{t'} \wedge \tau \subset \sigma \sqcap \rho_1^{t'}, \\ X_i & \text{if } \sigma = \sigma_i^{t'} \quad (i = 0, 1). \end{cases}$$

$\{X_\sigma \mid \sigma \in L_t\}$ is a leaves labelling for (t, h) and this completes the proof of the lemma. □

4. Borel order-preserving functions

Definition. ($\mathbf{ACA}_0$) A Borel (resp. Δ^1_1) quasi-ordering of the reals is a pair $(B, \preccurlyeq)$ such that B is a Borel (resp. Δ^1_1) subset of $\mathbb{R}$ and $\preccurlyeq$ is a Borel (resp. Δ^1_1) subset of $B \times B$ which is reflexive and transitive. We write $X \preccurlyeq Y$ instead of $(X, Y) \in \preccurlyeq$.

In the above situation we define the following auxiliary relations:

$$X \prec Y \iff X \preccurlyeq Y \wedge Y \not\preccurlyeq X,$$

$$X \approx Y \iff X \preccurlyeq Y \wedge Y \preccurlyeq X,$$

$$X \perp Y \iff X \not\preccurlyeq Y \wedge Y \not\preccurlyeq X.$$

The first theorem about Borel quasi-orderings we will prove in $\mathbf{ATR}_0$ deals with the possibility of mapping a Borel quasi-ordering into a linear order of the form 2^α for some countable ordinal α. In subsystems of second-order arithmetic we cannot deal with ordinals directly, and hence in place of α we substitute a countable well-ordering W.

Definition. ($\mathbf{ACA}_0$) If W is a countable well-ordering we define 2^W to be the set of all functions from the domain of W to $\{0, 1\}$ with the lexicographic order denoted by $\leqslant_l$. If $(B, \preccurlyeq)$ is a Borel quasi-ordering of the reals and W a countable well-ordering the map $F: B \to 2^W$ is said to be *order preserving* if $\forall X, Y \in B\ (X \preccurlyeq Y \Rightarrow F(X) \leqslant_l F(Y))$. The map F is said to be *strongly* order preserving if it is order preserving and $\forall X, Y \in B\ (F(X) = F(Y) \Rightarrow X \approx Y)$.

Similar definitions of 2^a and (strongly) order preserving map $F: B \to 2^a$ can be given for $a \in \mathcal{O}$ ($\mathcal{O}$ is the set of all notations for recursive well-orderings, as defined e.g. in [16, Section VIII.3]). Moreover we define in the obvious way, i.e., by restricting the quantifiers to range on nice reals, (strongly) order preserving-on-nice maps.

Definition. ($\mathbf{ACA}_0$) Let $(\mathbb{R}, \preccurlyeq)$ be a Δ^1_1 quasi-ordering. Define $\mathcal{F}$ by putting $F \in \mathcal{F}$ if and only if F is a Δ^1_1-on-nice function and

$$\exists a \in \mathcal{O}\ (\mathrm{rng}(F) \subseteq 2^a \wedge F \text{ is order preserving-on-nice}).$$

$\mathcal{F}$ is Π^1_1 in $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$ because Δ^1_1 and $\mathcal{O}$ are Π^1_1. The elements of $\mathcal{F}$ are recursive codes and hence if nice reals exist they are nice and $\mathcal{F}$ is Π^1_1-on-nice.

Definition. ($\mathbf{ACA}_0$) For $F \in \mathcal{F}$ and X and Y nice we write $X E_F Y$ to mean $F(X) = F(Y)$. Then we set $X E Y$ if and only if $\forall F \in \mathcal{F}\, X E_F Y$.

Notice that for $F \in \mathcal{F}$, E_F is a Δ^1_1-on-nice equivalence relation in $\mathbf{ACA}_0$. E is clearly an equivalence relation and is Σ^1_1-on-nice in $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$ because $X E Y$ is equivalent to $\forall F \in \Delta^1_1\, (F \in \mathcal{F} \Rightarrow X E_F Y)$. Moreover it is clear that

$$\forall X, Y \text{ nice } (X \approx Y \Rightarrow X E Y).$$

Lemma 4.1. ($\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$) *If $\mathcal{G} \subseteq \mathcal{F}$ is a Δ^1_1 set there exists $F \in \mathcal{F}$ such that*

$$\forall X, Y \text{ nice } (X E_F Y \Leftrightarrow \forall G \in \mathcal{G}\, X E_G Y).$$

Proof. If there are no nice reals then there is nothing to prove. Otherwise we have ATR^l_0: let $\{G_n\}$ be a Δ^1_1 enumeration of $\mathcal{G}$ and let $\mathrm{rng}(G_n) = 2^{b_n}$. Define a sequence of Δ^1_1-on-nice functions $\{F_n\}$ as follows. Let $F_0 = G_0$; if F_n has been defined set $F_{n+1}(X) = F_n(X)^\frown G_{n+1}(X)$, so that $F_n \in \mathcal{F}$ and $\mathrm{rng}(F_n) = 2^{a_n}$ where $a_n = \sum_{m \leq n} b_m$. The sequence $\{b_n\}$ is Δ^1_1 (each b_n is coded in G_n) and hence by Σ^1_1-boundedness (provable in ATR^l_0 by the lightface version of Lemma V.6.2 of [16]) there exists $b \in \mathcal{O}$ such that $\forall n\, b_n <_{\mathcal{O}} b$. Hence $\sup a_n = a <_{\mathcal{O}} b \cdot \omega$. Define $F \in \mathcal{F}$ with $\mathrm{rng}(F) = 2^a$ by $F(X) = \bigcup_n F_n(X)$. F clearly satisfies the statement. $\square$

We are now ready to prove the main result of this section, which asserts that Theorem 3.1 of [6] is provable in $\mathbf{ATR}_0$.

Theorem 4.2. ($\mathbf{ATR}_0$) *If $(B, \preccurlyeq)$ is a Borel quasi-ordering on the reals then one of the following is true*:

(a) *there exists a well-ordering W and a Borel function $F : B \to 2^W$ such that F is strongly order preserving*;

(b) *there exists a perfect set $P \subseteq \mathbb{R}$ such that $\forall X, Y \in P\ (X \neq Y \Rightarrow X \perp Y)$.*

Proof. We will prove the lightface version of the statement, substituting Δ^1_1 for Borel and recursive well-ordering for well-ordering: the boldface version will follow by relativization. It is clear that without loss of generality we can consider only the case $B = \mathbb{R}$.

Define $\mathcal{F}$ and E as above: since in $\mathbf{ATR}_0$ all sets are nice we have $\approx\, \subseteq E$. The proof splits into two cases.

Case I: $\approx\, = E$.

In this case we will obtain case (a) of the statement because $\mathcal{F}$ contains enough functions to separate any two non-equivalent reals and the reflection principles allow us to obtain a Δ^1_1 set with the same property.

More in detail let $\varphi(\mathcal{G})$ be the following formula

$$\forall X, Y\, (X \not\approx Y \Rightarrow \exists F \in \Delta^1_1\, (F \in \mathcal{G} \wedge F(X) \neq F(Y))).$$

φ is Π^1_1 on Π^1_1-on-nice and extensional and $\varphi(\mathscr{F})$ holds by case hypothesis. By the first reflection principle there exists a Δ^1_1-on-nice set $\mathscr{G}$ such that $\mathscr{G} \subseteq \mathscr{F}$ and $\varphi(\mathscr{G})$, i.e.,

$$\forall X, Y\, (X \not\approx Y \Rightarrow \exists F \in \mathscr{G}\, F(X) \neq F(Y)).$$

Apply Lemma 4.1 to this $\mathscr{G}$ and obtain a function $F \in \mathscr{F}$ such that for all X and Y

$$X \approx Y \quad \Leftrightarrow \quad F(X) = F(X).$$

This F is strongly order preserving and in this case (a) holds.

Case II: $\approx \subsetneq E$.

In this case we will obtain case (b) of the statement. Let

$$\tilde{A} = \{X \mid X \text{ is nice} \wedge \exists Y \text{ nice } (X\, E\, Y \wedge X \not\approx Y)\}.$$

$\tilde{A}$ is Σ^1_1-on-nice and by case hypothesis $\tilde{A} \neq \emptyset$.

Let $\mathfrak{M}$ be a countable ω-model of $\mathbf{\Sigma^1_1}$-$\mathbf{AC}_0 + \exists X\, (X \in \tilde{A})$. Let $\mathfrak{N}$ be a countable ω-model of $\mathbf{\Sigma^1_1}$-$\mathbf{AC}_0$ such that $\mathfrak{M} \in \mathfrak{N}$. Here we use twice Theorem VIII.4.20 of [16]. By Lemma 2.8 we have $\mathfrak{M} \models X \preccurlyeq Y$ if and only if $X \preccurlyeq Y$. In particular this implies $\mathfrak{M} \models (\preccurlyeq$ is a quasi-ordering on the reals). Since $\mathfrak{M} \models \exists X$ (X is nice) we have that $\mathfrak{M}$ is a model of $\mathrm{ATR}_0^{\prime}$.

Notice that while $(\mathfrak{M} \models X\, E\, Y) \Rightarrow X\, E\, Y$ the converse implication is in general false. Nevertheless, since $X\, E\, Y$ means $\forall F \in \mathscr{F}\, F(X) = F(Y)$, it is clear that $\mathfrak{M} \models (E$ is an equivalence relation).

Define $\mathbf{P}^n$ and $\mathbf{P}^n_E \times_E \mathbf{P}^m_E$ as before: we will consider $\mathbf{P} \times_E \mathbf{P}$-generic pairs belonging to $\tilde{A} \times \tilde{A}$ (i.e., such that $(\tilde{A}, \tilde{A})$ belongs to the corresponding generic filter over $\mathfrak{N}$) and show that their elements are incomparable in $\preccurlyeq$: Lemma 3.5 will then give a perfect set of mutually incomparable elements.

Definition. ($\mathbf{ACA}_0$) A set B is *downward closed in each E-class-on-nice* if

$$\forall X \in B\, \forall Y \text{ nice } (Y \preccurlyeq X \wedge X\, E\, Y \Rightarrow Y \in B).$$

B is *upward closed in each E-class-on-nice* if the same holds with $\succcurlyeq$ in place of $\preccurlyeq$.

Lemma 4.3. *The following holds in* $\mathfrak{M}$: *if* B *is a* Δ^1_1*-on-nice set downward (upward) closed in each E-class-on-nice there exists* $F \in \mathscr{F}$ *such that* B *is downward (upward) closed in each* E_F*-class-on-nice.*

Proof. We reason in $\mathfrak{M}$ and prove the statement in the downward case. Let

$$\varphi(\mathscr{G}) \quad \Leftrightarrow \quad \forall X \in B\, \forall Y \in B^{\mathrm{c}}\, (Y \preccurlyeq X \Rightarrow \exists F \in \Delta^1_1\, (F \in \mathscr{G} \wedge F(X) \neq F(Y))).$$

φ is Π^1_1 on Π^1_1-on-nice and extensional and $\varphi(\mathscr{F})$ holds by the hypothesis on B. By the first reflection principle there exists a Δ^1_1-on-nice set $\mathscr{G} \subseteq \mathscr{F}$ such that $\varphi(\mathscr{G})$. Applying Lemma 4.1 to $\mathscr{G}$ we obtain $F \in \mathscr{F}$ such that B is downward closed in each E_F-class-on-nice. $\square$

Lemma 4.4. *Let $A, A' \in \mathbf{P}$. The following holds in $\mathfrak{M}$: if $A \cap A' = \emptyset$ and A is downward closed in each E-class-on-nice (or the same with upward in place of downward) then $\forall X \in A\ \forall Y \in A'\ \neg X\, E\, Y$.*

Proof. We reason in $\mathfrak{M}$ and consider the downward case. Let

$$\varphi(B, C) \quad \Leftrightarrow \quad A' \subseteq B \wedge \forall X, Y \text{ nice } (X\, E\, Y \wedge Y \leqslant X \wedge X \notin B \Rightarrow Y \in C).$$

φ is Π^1_1 on Π^1_1-on-nice, extensional, monotonic upward and continuous downward and $\varphi(A^c, A)$ holds. By the second reflection principle there exists a Δ^1_1-on-nice set $B \supseteq A$ such that $\varphi(B^c, B)$ holds. Thus $B \cap A' = \emptyset$ and B is downward closed in each E-class-on-nice. By Lemma 4.3 there exists $F \in \mathscr{F}$ such that B is downward closed in each E_F-class-on-nice. Define G by:

$$G(X) = \begin{cases} F(X)^\frown\langle 0\rangle & \text{if } X \in B, \\ F(X)^\frown\langle 1\rangle & \text{if } X \in B^c. \end{cases}$$

G is Δ^1_1-on-nice and we claim that G is order preserving-on-nice. If X and Y are nice and $X \leqslant Y$ we have $F(X) \leqslant_l F(Y)$. If $F(X) <_l F(Y)$ we have $G(X) <_l G(Y)$. If $F(X) = F(Y)$ and $Y \in B^c$ then $G(X) \leqslant_l G(Y)$. If $F(X) = F(Y)$ and $Y \in B$ then, since B is downward closed in each E_F-class-on-nice, $X \in B$ and $G(X) = G(Y)$. In any case $G(X) \leqslant_l G(Y)$ and the claim is proved.

Hence $G \in \mathscr{F}$. If $X \in A \subseteq B$ and $Y \in A' \subseteq B^c$ we have $\neg X\, E_G\, Y$ and thus $\neg X\, E\, Y$. $\square$

Lemma 4.5. *If $(A, A') \in \mathbf{P} \times_E \mathbf{P}$ then $\mathfrak{M} \models \exists X \in A\ \exists Y \in A'\ (X\, E\, Y \wedge Y \leqslant X)$.*

Proof. Suppose not, i.e.,

$$\mathfrak{M} \models \forall X \in A\ \forall Y \in A'\ (X\, E\, Y \Rightarrow \neg Y \leqslant X).$$

In $\mathfrak{M}$ define $A_0 = \{X \mid X \text{ is nice } \wedge \exists Y \in A\ (X\, E\, Y \wedge X \leqslant Y)\}$. We can apply Lemma 4.4 to (A_0, A') obtaining $\mathfrak{M} \models (\forall X \in A_0\, \forall Y \in A' \neg X\, E\, Y)$. Since $\mathfrak{M} \models A \subseteq A_0$ we have also $\mathfrak{M} \models (\forall X \in A\ \forall Y \in A' \neg X\, E\, Y)$ which implies $(A, A') \notin \mathbf{P} \times_E \mathbf{P}$. $\square$

Lemma 4.6. *For any $A \in \mathbf{P}$ if $A \leqslant \tilde{A}$ then $\mathfrak{M} \models \exists X, X' \in A\ (X\, E\, X' \wedge X \not\equiv X')$.*

Proof. We reason in $\mathfrak{M}$ and define

$$A^+ = \{Y \mid Y \text{ is nice } \wedge \exists X \in A\ (X\, E\, Y \wedge (X < Y \vee X \perp Y))\},$$
$$A^- = \{Y \mid Y \text{ is nice } \wedge \exists X \in A\ (X\, E\, Y \wedge (Y < X \vee X \perp Y))\}.$$

Since $A \subseteq \tilde{A}$ and $\forall X \in \tilde{A}\ \exists Y$ nice $(X\, E\, Y \wedge X \not\equiv Y)$, we have either $A^+ \neq \emptyset$ or $A^- \neq \emptyset$: suppose $A^+ \neq \emptyset$ (the other case is analogous).

Let $A_0 = \{Y \mid Y \text{ is nice } \wedge \exists X \in A\ (X\, E\, Y \wedge Y \leqslant X)\}$. $A_0 \neq \emptyset$ because $A \subseteq A_0$ and A_0 is downward closed in each E-class-on-nice. We claim that $A_0 \cap A^+ \neq \emptyset$. If this were not the case then by Lemma 4.4, $\forall X \in A_0 \forall Y \in A^+ \neg X E Y$. This implies, by definition of A^+, $A^+ = \emptyset$. Hence the claim is proved: let $Y \in A_0 \cap A^+$.

Since $Y \in A^+$ there exists $X \in A$ such that $X E Y \wedge (X < Y \vee X \perp Y)$. Since $Y \in A_0$ there exists $X' \in A$ such that $X' E Y \wedge Y \preccurlyeq X'$. Then $X E X'$ and $X \not\approx X'$, as desired. □

Lemma 4.7. *If* $(A, A') \in \mathbf{P} \times_E \mathbf{P}$ *and* $A, A' \preccurlyeq \tilde{A}$ *then*

$$\mathfrak{M} \models \exists X \in A\, \exists Y \in A' (X E Y \wedge X \not\approx Y).$$

Proof. We reason in $\mathfrak{M}$ and define $A_0 = \{X \in A \mid \exists Y \in A' (X E Y \wedge X \approx Y)\}$. If $A_0 = \emptyset$ we are done, otherwise by Lemma 4.6, $\exists X, X' \in A_0\, (X E X' \wedge X \not\approx X')$. Fix such X and X' and let $Y \in A'$ be such that $X' E Y$. Then $X \in A$, $Y \in A'$, $X E Y$ and—since $X \not\approx X' \approx Y$—$X \not\approx Y$. □

Lemma 4.8. *If* $(X, Y) \in \tilde{A} \times \tilde{A}$ *is* $\mathbf{P} \times_E \mathbf{P}$*-generic then* $X \perp Y$.

Proof. Let G be the $\mathbf{P} \times_E \mathbf{P}$-generic filter over $\mathfrak{N}$ such that $\{(X, Y)\} = \bigcap G$, i.e., $G = \{(A, A') \in \mathbf{P} \times_E \mathbf{P} \mid X \in A \wedge Y \in A'\}$. By Lemma 2.6, $\mathfrak{N}[G]$ is an ω-model of $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$ and by Lemma 2.8 for any $X, Y \in \mathfrak{N}[G]$ there exists an evaluation map for $\preccurlyeq$ at (X, Y) and $\mathfrak{N}[G] \models X \preccurlyeq Y$ if and only if $X \preccurlyeq Y$. We claim that $\mathfrak{N}[G] \models X \perp Y$.

First suppose $\mathfrak{N}[G] \models X < Y$. By Lemma 2.5 there exists $(A, A') \in \mathbf{P} \times_E \mathbf{P}$ such that $(A, A') \leqslant (\tilde{A}, \tilde{A})$ and $(A, A') \Vdash_{\mathbf{P} \times_E \mathbf{P}} \dot{X} < \dot{Y}$. By Lemma 4.5, $\mathfrak{M} \models \exists X \in A$ $\exists Y \in A' (X E Y \wedge Y \preccurlyeq X)$. Therefore

$$D = \{(X, Y) \mid X \in A \wedge Y \in A' \wedge X E Y \wedge Y \preccurlyeq X\} \in \mathbf{P}^2_E.$$

By Lemma 2.2 let $((X_0, Y_0), (X_1, Y_1)) \in D \times D$ be $\mathbf{P}^2_E \times_E \mathbf{P}^2_E$-generic. By Lemma 3.4 (X_0, Y_1) and (X_1, Y_0) are $\mathbf{P} \times_E \mathbf{P}$-generic. Since $(X_0, Y_1), (X_1, Y_0) \in A \times A'$ we have $X_0 < Y_1$ and $X_1 < Y_0$. Since $(X_0, Y_0), (X_1, Y_1) \in D$ we have $Y_0 \preccurlyeq X_0$ and $Y_1 \preccurlyeq X_1$. Hence $X_0 < Y_1 \preccurlyeq X_1 < Y_0 \preccurlyeq X_0$, a contradiction.

Similarly one rules out the possibility that $\mathfrak{N}[G] \models Y < X$.

Now suppose $\mathfrak{N}[G] \models X \approx Y$. By Lemma 2.5 there exists $(A, A') \in \mathbf{P} \times_E \mathbf{P}$ such that $(A, A') \leqslant (\tilde{A}, \tilde{A})$ and $(A, A') \Vdash_{\mathbf{P} \times_E \mathbf{P}} \dot{X} \approx \dot{Y}$. By Lemma 4.7, $\mathfrak{M} \models \exists X \in A$ $\exists Y \in A' (X E Y \wedge X \not\approx Y)$. Therefore

$$D = \{(X, Y) \mid X \in A \wedge Y \in A' \wedge X E Y \wedge X \not\approx Y\} \in \mathbf{P}^2_E.$$

Let (X_0, Y_0), (X_1, Y_1) and (X_2, Y_2) be three pairs such that for all $i < 3$, $(X_i, Y_i) \in D$ and for all $i < j < 3$, $((X_i, Y_i), (X_j, Y_j))$ is $\mathbf{P}^2_E \times_E \mathbf{P}^2_E$-generic (these can be obtained by Lemma 3.5). By Lemma 3.4, (X_0, Y_1), (X_2, Y_1) and (X_2, Y_0) are $\mathbf{P} \times_E \mathbf{P}$-generic. Since $(X_0, Y_1), (X_2, Y_1), (X_2, Y_0) \in A \times A'$ we have $X_0 \approx Y_1 \approx X_2 \approx Y_0$. This contradicts $(X_0, Y_0) \in D$.

Thus $\mathfrak{N}[G] \models X \perp Y$ and hence $X \perp Y$. □

From Lemma 4.8 we obtain a perfect set of mutually incomparable elements by applying Lemma 3.5 (for any $A \leqslant \tilde{A}$ let $E_A = E$). In this case the map $F \mapsto X_F$ is one-to-one and hence $\{X_F \mid F \in [T]\}$ is an uncountable analytic set. By the perfect set theorem [16, Theorem V.4.3] this set has a perfect subset and (b) is satisfied. □

Definition. ($\mathbf{ACA}_0$) A *Borel linear ordering* of the reals is a Borel quasi-ordering of the reals $(B, \leqslant)$ with the additional property that any two elements of B are comparable according to $\leqslant$.

Corollary 4.9. ($\mathbf{ATR}_0$) *Every Borel linear ordering of the reals is embeddable in a Borel way in a linear ordering of the form 2^W for some well-ordering W, i.e., the linear orderings of the form 2^W are cofinal in the quasi-ordering of the Borel linear orderings of the reals under Borel embeddability.*

Proof. Immediate from the theorem. □

5. Borel chains

The main result of this section is concerned with the possibility of decomposing a Borel quasi-ordering into a union of chains, i.e., linearly ordered subsets of the original quasi-ordering.

Definition. ($\mathbf{ACA}_0$) Let $(B, \leqslant)$ be a Borel quasi-ordering of the reals. $B' \subseteq B$ is a *chain* if $\forall X, Y \in B'\ (X \leqslant Y \vee Y \leqslant X)$ and $\forall X \in B'\ \forall Y$ nice $(X \approx Y \Rightarrow Y \in B')$.

Definition. ($\mathbf{ACA}_0$) Let $(\mathbb{R}, \leqslant)$ be a Δ^1_1 quasi-ordering of the reals. For any Σ^1_1-on-nice set A let

$$\mathcal{H}_A = \{B \mid B \text{ is } \Delta^1_1\text{-on-nice} \wedge \forall X \in A \cap B\ \forall Y \in A \setminus B\ (X < Y \vee Y < X)\},$$

$$X\, E_A\, Y \iff \forall B \in \mathcal{H}_A\ (X \in B \Leftrightarrow Y \in B).$$

$\mathcal{H}_A$ is Π^1_1 in $\mathbf{\Sigma}^1_1\text{-}\mathbf{AC}_0$ because Δ^1_1 is Π^1_1. The elements of $\mathcal{H}_A$ are recursive codes and hence if nice reals exist, they are nice and $\mathcal{H}_A$ is Π^1_1-on-nice. E_A is a Σ^1_1-on-nice equivalence relation. The main feature of E_A is that all the incomparabilities between elements of A are concentrated within E_A-equivalence classes. Moreover, E_A is an approximation on A of the equivalence relation $E_{\leqslant}$ obtained by taking the transitive closure of $\approx \cup \perp$. $E_{\leqslant}$ has the additional property that its equivalence classes are linearly ordered by $\leqslant$ and is a useful tool in the study of quasi-orderings (see for example [7]). We will use E_A in place of $E_{\leqslant}$ because the former is 'represented' by the Π^1_1 set of Δ^1_1 codes $\mathcal{H}_A$.

The following theorem asserts that Theorem 5.1 of [6] is provable in $\mathbf{ATR}_0$.

Theorem 5.1. ($\mathbf{ATR}_0$) *If* $(B, \preccurlyeq)$ *is a Borel quasi-ordering on the reals then one of the following is true*:

(a) *there exists a sequence* $\{B_n\}$ *of Borel chains such that* $\bigcup_n B_n = B$;

(b) *there exists a perfect set* $P \subseteq \mathbb{R}$ *such that* $\forall X, Y \in P\ (X \neq Y \Rightarrow X \perp Y)$.

Proof. We will prove the lightface version of the statement, substituting Δ^1_1 for Borel: the boldface version will follow by relativization. It is clear that without loss of generality we can consider only the case $B = \mathbb{R}$.

Let $U = \{X \mid \exists B\,(B \text{ is a } \Delta^1_1\text{-on-nice chain} \wedge X \in B)\}$. U is a Π^1_1 set and the proof splits into two cases.

Case I: $U = \mathbb{R}$.

Let $\varphi(B) \Leftrightarrow (B \text{ is a } \Delta^1_1 \text{ chain})$. φ is a Π^1_1 property of natural numbers (the codes for the Δ^1_1 sets) and by case hypothesis $\forall X\, \exists B\, (\varphi(B) \wedge X \in B)$. By Π^1_1-uniformization (provable in $\mathbf{ATR}_0$ by the lightface version of Theorem VIII.4.6 of [16]) there exists a Π^1_1 formula $\psi(X, B)$ such that

$$\forall X\, \forall B\, (\psi(X, B) \Rightarrow \varphi(B) \wedge X \in B) \wedge \forall X\, \exists! B\, \psi(X, B).$$

$\psi(X, B)$ is equivalent to $\forall B' \neq B\, \neg\psi(X, B')$ and hence $\psi(X, B)$ is Δ^1_1. Now let $\varphi_0(B) \Leftrightarrow \exists X\, \psi(X, B)$. φ_0 is Σ^1_1 and $\forall B\, (\varphi_0(B) \Rightarrow \varphi(B))$. By Σ^1_1-separation [16, Theorem V.5.1] there exists a set Z such that

$$\forall B\, ((\varphi_0(B) \Rightarrow B \in Z) \wedge (B \in Z \Rightarrow \varphi(B))).$$

Any enumeration of Z shows that case (a) of the statement holds.

Case II: $U \subsetneqq \mathbb{R}$.

In this case we will obtain case (b) of the statement. Let

$$\tilde{A} = \{X \mid X \text{ is nice} \wedge X \notin U\}.$$

$\tilde{A}$ is Σ^1_1-on-nice and by case hypothesis $\tilde{A} \neq \emptyset$.

Let $\mathfrak{M}$ be a countable ω-model of $\mathbf{\Sigma^1_1}\text{-}\mathbf{AC}_0 + \exists X\, (X \in \tilde{A})$. Let $\mathfrak{N}$ be a countable ω-model of $\mathbf{\Sigma^1_1}\text{-}\mathbf{AC}_0$ such that $\mathfrak{M} \in \mathfrak{N}$. The same considerations made about $\mathfrak{M}$ and $\mathfrak{N}$ at the beginning of case II in the proof of Theorem 4.2 can be repeated here. In particular $\mathfrak{M}$ is a model of ATR_0', $\mathfrak{M} \vDash (\preccurlyeq$ is a quasi-ordering on the reals) and for any Σ^1_1-on-nice set A, $\mathfrak{M} \vDash (E_A$ is an equivalence relation).

Define $\mathbf{P}^n$ as usual. For any $A \in \mathbf{P}$ such that $A \leqslant \tilde{A}$ we will consider the product Gandy forcing $\mathbf{P} \times_{E_A} \mathbf{P}$ below A and show that there is a pair of $\mathbf{P} \times_{E_A} \mathbf{P}$-generic reals whose elements are incomparable. Lemma 3.5 will then give the desired perfect set of mutually incomparable reals.

Lemma 5.2. *If* $A \in \mathbf{P}$ *and* $A \leqslant \tilde{A}$ *then* $\mathfrak{M} \vDash \exists X, Y \in A\ X \perp Y$.

Proof. We reason within $\mathfrak{M}$ and suppose that the conclusion does not hold, i.e., $\forall X, Y \in A\, (X \preccurlyeq Y \vee Y \preccurlyeq X)$. Let $A' = \{Y \text{ nice} \mid \exists X \in A\ X \approx Y\}$. A' is Σ^1_1-on-nice and $A \subseteq A'$. Let

$\varphi(C, D) \iff \forall X, Y \notin C\,(X \preccurlyeq Y \vee Y \preccurlyeq X) \wedge \forall X \notin C\, \forall Y$ nice $(Y \approx X \Rightarrow Y \in D)$.

φ is Π^1_1 on Π^1_1-on-nice, extensional, monotonic upward and continuous downward; moreover $\varphi(A'^{\mathrm{c}}, A')$ holds. By the second reflection principle there exists a Δ^1_1-on-nice set $B \supseteq A'$ such that $\varphi(B^{\mathrm{c}}, B)$ holds. Thus B is a Δ^1_1-on-nice chain and hence $B \cap \tilde{A} = \emptyset$. This contradicts $A \subseteq B \cap \tilde{A}$ and proves the lemma. □

Lemma 5.3. *Let $A \leqslant \tilde{A}$ be such that $(A, A) \Vdash_{\mathbf{P} \times_{E_A} \mathbf{P}} \dot{X} \preccurlyeq \dot{Y} \vee \dot{Y} \preccurlyeq \dot{X}$. Suppose $A_0, A_1 \leqslant A$ and $(A_0, A_1) \Vdash_{\mathbf{P} \times_{E_A} \mathbf{P}} \dot{X} \preccurlyeq \dot{Y}$; then*

$$\mathfrak{M} \models \forall X \in A_0\, \forall Y \in A_1\, (X\, E_A\, Y \Rightarrow X \preccurlyeq Y).$$

Proof. Suppose the conclusion does not hold and let

$$D = \{(X, Y) \mid X \in A_0 \wedge Y \in A_1 \wedge X\, E_A\, Y \wedge X \npreccurlyeq Y\} \in \mathbf{P}^2_{E_A},$$
$$A' = \{Y \mid \exists X\, (X, Y) \in D\} \leqslant A_1.$$

By the hypothesis on A there exists $(A_0', A_1') \in \mathbf{P} \times_{E_A} \mathbf{P}$ such that $(A_0', A_1') \leqslant (A', A')$ and $(A_0', A_1') \Vdash_{\mathbf{P} \times_{E_A} \mathbf{P}} \dot{X} \preccurlyeq \dot{Y}$ (if $(A_0', A_1') \Vdash_{\mathbf{P} \times_{E_A} \mathbf{P}} \dot{Y} \preccurlyeq \dot{X}$ switch A_0' and A_1'). Let $D' = \{(X, Y) \in D \mid Y \in A_1'\}$.

By Lemma 2.2 let $((X_0, Y), X_1) \in D' \times A_0'$ be $\mathbf{P}^2_{E_A} \times_{E_A} \mathbf{P}$-generic. By Lemma 3.4, (X_0, X_1) and (X_1, Y) are $\mathbf{P} \times_{E_A} \mathbf{P}$-generic. Since $(X_0, X_1) \in (A_0, A_0') \leqslant (A_0, A_1)$ by Lemma 2.5 we have $X_0 \preccurlyeq X_1$. Since $(X_1, Y) \in (A_0', A_1')$ by Lemma 2.5 again we have $X_1 \preccurlyeq Y$. Hence $X_0 \preccurlyeq Y$ contradicting $(X_0, Y) \in D' \subseteq D$. □

Lemma 5.4. *Let A, A_0 and A_1 be as in the previous lemma; then*

$$\mathfrak{M} \models \forall X \in A_0\, \forall Y \in A_1\, (X\, E_A\, Y \Rightarrow X \prec Y).$$

Proof. Suppose the conclusion does not hold: by Lemma 5.3

$$A' = \{X \in A_0 \mid \exists Y \in A_1\, (X\, E_A\, Y \wedge X \approx Y)\} \in \mathbf{P}.$$

We reason within $\mathfrak{M}$: for any $X_0, X_1 \in A'$, if $\neg X_0\, E_A\, X_1$ by definition of E_A we have either $X_0 \prec X_1$ or $X_1 \prec X_0$. If $X_0\, E_A\, X_1$ let $Y \in A_1$ be such that $X_1\, E_A\, Y \wedge X_1 \approx Y$: we have $X_0\, E_A\, Y$ and by Lemma 5.3, $X_0 \preccurlyeq Y \approx X_1$.

We have just shown that $\mathfrak{M} \models \forall X_0, X_1 \in A'\ (X_0 \preccurlyeq X_1 \vee X_1 \preccurlyeq X_0)$, which contradicts Lemma 5.2. □

We are now in a position to prove the main lemma.

Lemma 5.5. *If $A \leqslant \tilde{A}$ then it is not the case that $(A, A) \Vdash_{\mathbf{P} \times_{E_A} \mathbf{P}} \dot{X} \preccurlyeq \dot{Y} \vee \dot{Y} \preccurlyeq \dot{X}$.*

Proof. Suppose the conclusion does not hold. Then there exists $(A_0, A_1) \leqslant (A, A)$ such that $(A_0, A_1) \Vdash_{\mathbf{P} \times_{E_A} \mathbf{P}} \dot{X} \preccurlyeq \dot{Y}$. Let

$$A' = \{X \in A \mid \exists X' \in A_0\, (X\, E_A\, X' \wedge X \preccurlyeq X')\}.$$

We reason in $\mathfrak{M}$: we have that $A_0 \subseteq A'$ and that A' is downward closed in each E_A-class-on-A (i.e., $\forall X \in A' \, \forall Y \in A \, (Y \preccurlyeq X \wedge X \, E_A \, Y \Rightarrow Y \in A')$). Moreover, by Lemma 5.4 we have $A' \cap A_1 = \emptyset$. Let

$$\varphi(C, D) \quad \Leftrightarrow \quad \forall X \notin C \, \forall Y \in A \, (Y \preccurlyeq X \wedge X \, E_A \, Y \Rightarrow Y \in D) \wedge A_1 \subseteq C.$$

φ is Π^1_1 on Π^1_1-on-nice, extensional, monotonic upward and continuous downward; moreover $\varphi(A'^{\mathrm{c}}, A')$ holds. By the second reflection principle there exists a Δ^1_1-on-nice set $B \supseteq A'$ such that $\varphi(B^{\mathrm{c}}, B)$ holds. Thus B is downward closed in each E_A-class-on-A and $A_1 \subseteq A \setminus B$.

We now claim that $\mathfrak{M} \models (B \in \mathscr{H}_A)$: this will complete the proof of the lemma because it contradicts $\mathfrak{M} \models (\exists X \in B \, \exists Y \in A \setminus B \, X \, E_A \, Y)$, which is a consequence of $(B, A \setminus B) \geqslant (A_0, A_1) \in \mathbf{P} \times_{E_A} \mathbf{P}$.

Thus our goal is to show that within $\mathfrak{M}$ we have $\forall X \in A \cap B \, \forall Y \in A \setminus B \, (X < Y \vee Y < X)$. For X and Y such that $\neg X \, E_A \, Y$ this is immediate by definition of E_A. If $X \, E_A \, Y$, by the downward closure of B in each E_A-class-on-A we have $X < Y \vee X \perp Y$.

Suppose $\mathfrak{M} \models (\exists X \in A \cap B \, \exists Y \in A \setminus B \, (X \, E_A \, Y \wedge X \perp Y))$ so that

$$D = \{Y \in A \setminus B \mid \exists X \in A \cap B \, (X \, E_A \, Y \wedge X \perp Y)\} \in \mathbf{P}.$$

There exists $(D_0, D_1) \in \mathbf{P} \times_{E_A} \mathbf{P}$ such that $(D_0, D_1) \leqslant (D, D)$ and $(D_0, D_1) \Vdash_{\mathbf{P} \times_{E_A} \mathbf{P}} \dot{X} \preccurlyeq \dot{Y}$. Let $D' = \{(X, Y) \mid X \in A \cap B \wedge Y \in D_1 \wedge X \, E_A \, Y \wedge X \perp Y\} \in \mathbf{P}^2_{E_A}$ and let $((X_0, Y), X_1) \in D' \times D_0$ be $\mathbf{P}^2_{E_A} \times_{E_A} \mathbf{P}$-generic. By Lemma 3.4, (X_0, X_1) and (X_1, Y) are $\mathbf{P} \times_{E_A} \mathbf{P}$-generic. Since $(X_1, Y) \in (D_0, D_1)$ by Lemma 2.5 we have $X_1 \preccurlyeq Y$. Since $(X_0, X_1) \in (A \cap B, D_0) \leqslant (A, A)$ by Lemma 2.5 again we have $X_0 \preccurlyeq X_1 \vee X_1 \preccurlyeq X_0$: the first possibility implies $X_0 \preccurlyeq Y$, which is in contradiction with $(X_0, Y) \in D'$, and hence $X_1 \preccurlyeq X_0$ holds. Now we can apply Lemma 2.5 in the other direction to obtain $(A_0^*, A_1^*) \in \mathbf{P} \times_{E_A} \mathbf{P}$ such that $X_1 \in A_0^*$, $X_0 \in A_1^*$, $(A_0^*, A_1^*) \leqslant (D_0, A \cap B)$ and $(A_0^*, A_1^*) \Vdash_{\mathbf{P} \times_{E_A} \mathbf{P}} \dot{X} \preccurlyeq \dot{Y}$. By Lemma 5.3, $\mathfrak{M} \models \forall X \in A_0^* \, \forall Y \in A_1^* \, (X \, E_A \, Y \Rightarrow X \preccurlyeq Y)$. Since $(A_0^*, A_1^*) \in \mathbf{P} \times_{E_A} \mathbf{P}$ reasoning in $\mathfrak{M}$ there exist $X \in A_0^* \subseteq D_0 \subseteq A \setminus B$ and $Y \in A_1^* \subseteq A \cap B$ such that $X \, E_A \, Y$: $X \preccurlyeq Y$ then violates the downward closure of B in each E_A-class-on-A, providing the desired contradiction. This proves the claim and hence the lemma. □

By Lemma 2.6, $\mathfrak{N}[G]$ is a model of $\mathbf{\Sigma}^1_1$-$\mathbf{AC}_0$, by Lemma 2.8 there exist evaluation maps for $\preccurlyeq$ at any $(X, Y) \in \mathfrak{N}[G]$ and $\mathfrak{N}[G] \models ((\neg X \preccurlyeq Y \wedge \neg Y \preccurlyeq X) \Leftrightarrow X \perp Y)$. Therefore we can apply Lemma 3.5 to $\varphi(X, Y) \equiv \neg X \preccurlyeq Y \wedge \neg Y \preccurlyeq X$ and obtain a perfect set of mutually incomparable elements exactly as in the proof of Theorem 4.2. □

We now draw some corollaries, which are originally due to Simpson (unpublished, to appear in [16]), showing that $\mathbf{ATR}_0$ proves a weak form of Silver's theorem and $\mathbf{\Pi}^1_1$-$\mathbf{CA}_0$ is equivalent over $\mathbf{ACA}_0$ to a stronger form of the same theorem. Actually Simpson's results are slightly stronger, in that they apply to equivalence relations which are coanalytic whereas from the previous theorem we can draw conclusions only about Borel ones (in the case of coanalytic equivalence relations the second clause of case (a) of Corollary 5.6 reads "each B_n is a subset of an E-equivalence class").

Corollary 5.6. ($\mathbf{ATR}_0$) *If E is a Borel equivalence relation on the reals then one of the following is true*:

(a) *there exists a sequence of Borel sets $\{B_n\}$ such that $\forall X\, \exists n\, X \in B_n$ and each B_n is either empty or an E-equivalence class*:

(b) *there exists a perfect set $P \subseteq \mathbb{R}$ such that $\forall X, Y \in P\, (X \neq Y \Rightarrow \neg X\, E\, Y)$.*

Proof. Given E define a Borel quasi-ordering by $X \preccurlyeq Y \Leftrightarrow X\, E\, Y$. Then (a) and (b) are just the restatements of the corresponding cases of Theorem 5.1. □

Definition. Let $ST(E)$ stand for the statement of Silver's theorem for the equivalence relation E which asserts that one of the following is true:

(a) there exists a sequence of reals $\{X_n\}$ such that $\forall X\, \exists n\, X\, E\, X_n$;

(b) there exists a perfect set $P \subseteq \mathbb{R}$ such that $\forall X, Y \in P\, (X \neq Y \Rightarrow \neg X\, E\, Y)$.

The following corollary is a typical reverse mathematics result, in which mathematical theorems are proved to be equivalent to an axiom over a weaker base theory.

Corollary 5.7. ($\mathbf{ACA}_0$) *The following are equivalent*:

(1) $\boldsymbol{\Pi}^1_1$-$\mathbf{CA}_0$;

(2) *if $(B, \preccurlyeq)$ is a Borel quasi-ordering for which there is no perfect set of pairwise incomparable elements then there exists a sequence of reals $\{X_n\}$ such that $\forall X \in B\, \exists n\, (X \preccurlyeq X_n \vee X_n \preccurlyeq X)$*;

(3) *if E is a Borel equivalence relation then $ST(E)$ holds.*

Proof. To show that (1) implies (2) use Theorem 5.1 to get a sequence $\{B_n\}$ of Borel chains such that $\bigcup_n B_n = B$. In $\boldsymbol{\Pi}^1_1$-$\mathbf{CA}_0$ we can form $Z = \{n \mid \exists X\, X \in B_n\}$ and by $\boldsymbol{\Sigma}^1_1$-$\mathbf{AC}_0$ for any $n \in Z$ we can pick $X_n \in B_n$ obtaining the desired conclusion.

(2) implies (3) is trivial and for (3) implies (1) we use the following construction due to Sami (personal communication to Simpson, June 1981). Let $\exists X\, \theta(n, X)$ with θ arithmetical be a $\boldsymbol{\Sigma}^1_1$ formula: we need to show that the set $Z = \{n \mid \exists X\, \theta(n, X)\}$ exists. Define a Borel equivalence relation by setting $(n, X)\, E\, (m, Y)$ if and only if $n = m \wedge (\theta(n, X) \Leftrightarrow \theta(n, Y))$. Clearly case (b) of $ST(E)$ does not hold and hence there exists a sequence $\{(n_k, X_k)\}$ such that $\forall n\, \forall X\, \exists k\, (n, X)\, E\, (n_k, X_k)$. Then we have $\exists X\, \theta(n, X) \Leftrightarrow \exists k\, (n = n_k \wedge \theta(n, X_k))$ and the set Z exists by arithmetical comprehension. □

Directions for further research

From the viewpoint of reverse mathematics it is natural to ask whether the statements of Theorems 4.2 and 5.1, and of Corollaries 4.9 and 5.6 are equivalent to $\mathbf{ATR}_0$ over $\mathbf{ACA}_0$ or some other base theory: we do not know of any such result.

Another interesting result about Borel quasi-orderings is the following theorem of Kada's [7]: "if $(B, \preccurlyeq)$ is a Borel quasi-ordering such that there are at most n pairwise incomparable reals then B is the union of n Borel chains". We do not know whether this theorem, either in its full generality or for any given n, can be proved in $\mathbf{ATR}_0$ or in any other subsystem of second-order arithmetic.

The statement of Corollary 4.9 is a first result in the study of the quasi-ordering of Borel linear orderings under Borel embeddability. Several other results have been obtained in this field by Marker [6], Louveau [10] and Louveau and Saint-Raymond [11]. It would be interesting to investigate how much of this theory can be carried out in $\mathbf{ATR}_0$ or other subsystems of second-order arithmetic.

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