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On a fixed point theorem in Banach algebras with applications

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Abstract

In this article it is shown that some of the hypotheses of a fixed point theorem of the present author [B.C. Dhage, On some variants of Schauder's fixed point principle and applications to nonlinear integral equations, J. Math. Phys. Sci. 25 (1988) 603–611] involving two operators in a Banach algebra are redundant. Our claim is also illustrated with the applications to some nonlinear functional integral equations for proving the existence results.

MSC: 47H10

Keywords: Banach algebra; Fixed point theorem; Integral equations

1. Introduction

It is known that the first important hybrid fixed point theorem due to Krasnoselskii [1] which combines the metric fixed point theorem of Banach with the topological fixed point theorem of Schauder in a Banach space has several applications to nonlinear integral equations that arise in the inversion of the perturbed differential equations. Many attempts have been made to improve and weaken the hypotheses of Krasnoselskii's fixed point theorem. See [2] and the references therein. The case with the Krasnoselskii type fixed point theorem of the present author [3] in Banach algebras is similar. The study of the nonlinear integral equations in Banach algebras was initiated by Dhage [3] via fixed point theorems and includes the following second important hybrid fixed point theorem in Banach algebras. See [3,4].

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Theorem 1.1 ([3]). *Let S be a closed, convex and bounded subset of a Banach algebra X and let $A, B : S \to S$ be two operators such that*

(a) *A is Lipschitzian with a Lipschitz constant α,*
(b) *$\left(\frac{I}{A}\right)(x)^{-1}$ exists on $B(S)$, where I is an identity operator and the operator $\frac{I}{A} : X \to X$ is defined by $\left(\frac{I}{A}\right) = \frac{x}{Ax}$,*
(c) *B is completely continuous, and*
(d) *$AxBy \in S, \quad \forall\, x, y \in S$.*

Then the operator equation

$$AxBx = x \tag{1}$$

has a solution, whenever $\alpha M < 1$, where $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$.

Remark 1.1. Note that $\left(\frac{I}{A}\right)^{-1}$ exists if $\left(\frac{I}{A}\right)$ is well defined and one-to-one on X. Further, $\left(\frac{I}{A}\right)$ is well defined if A is regular, i.e. A maps X into the set of all invertible elements of X.

We mention that Theorem 1.1 is useful in the study of nonlinear integral equations of mixed type in a Banach algebra. The above result was further improved in the due course of time by the present author under weaker versions of the hypotheses (a)–(d) thereof. The following re-formulation of Theorem 1.2 is noteworthy.

Theorem 1.2 ([4]). *Let S be a closed, convex and bounded subset of a Banach algebra X and let $A, B : S \to X$ be two operators such that*

(a) *A is Lipschitzian with a Lipschitz constant α,*
(b) *B is completely continuous, and*
(c) *$AxBx \in S$ for all $x \in S$.*

Then the operator equation (1) has a solution, whenever $\alpha M < 1$, where $M = \|B(S)\|$.

The proof of Theorem 1.2 involves the use of the advanced notions of the nonlinear functional analysis such as measures of noncompactness and condensing mappings etc., and so for beginners it is very difficult to grasp the underlying ideas and the strength of the applicability of Theorem 1.2. Therefore it is of interest to prove the improved version of Theorem 1.1 using the ideas of the elementary functional analysis. Most recently the present author has focused his attention on hypothesis (d) of Theorem 1.1 and proved the following improved version under weaker conditions. Before stating the fixed point theorem along these lines, we give a useful definition.

Definition 1.1. A mapping $T : X \to X$ is called $\mathcal{D}$-**Lipschitzian** if there exists a continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|Tx - Ty\| \le \phi(\|x - y\|) \tag{2}$$

for all $x, y \in X$, where $\phi(0) = 0$.

Sometimes we call the function ϕ a $\mathcal{D}$-**function** of T on X. Obviously every Lipschitzian mapping is $\mathcal{D}$-Lipschitzian, but the converse may not be true. If ϕ is not necessarily nondecreasing and satisfies $\phi(r) < r$, for $r > 0$, the mapping T is called a **nonlinear contraction** with a contraction function ϕ. See [6].

Let X be a Banach space and let $T : X \to X$. Then T is called a **compact** operator if $\overline{T(X)}$ is a compact subset of X. Again T is called **totally bounded** if for any bounded subset S of X, $T(S)$ is a totally bounded set of X. Further, T is called **completely continuous** if it is continuous and totally bounded. Note that every compact operator is totally bounded, but the converse may not be true; however, the two notions are equivalent on a bounded subset of X.

Theorem 1.3 ([5]). *Let S be a closed, convex and bounded subset of a Banach algebra X and let $A : X \to X$, $B : S \to X$ be two operators such that*

(a) *A is $\mathcal{D}$-Lipschitzian with a D-function ϕ,*
(b) *$\left(\frac{I}{A}\right)^{-1}$ exists on $B(S)$, where I is an identity operator and the operator $\frac{I}{A} : X \to X$ is defined by $\left(\frac{I}{A}\right)(x) = \frac{x}{Ax}$,*
(c) *B is completely continuous, and*
(d) *$x = AxBy \Rightarrow x \in S$ for all $y \in S$.*

Then the operator equation (1) has a solution, whenever $M\phi(r) < r$, $r > 0$, where $M = \|B(S)\|$.

We note that the hypothesis (b) puts a severe restriction on the class of mappings A and thereby limits the scope of applications of Theorem 1.3 to nonlinear problems of differential and integral equations. In this work we further improve the above Theorem 1.3 by relaxing the hypothesis (b) and increase its utility from the point of view of applications.

2. Fixed point theorem

Theorem 2.1. *Let S be a closed, convex and bounded subset of a Banach algebra X and let $A : X \to X$, $B : S \to X$ be two operators such that*

(a) *A is $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function ϕ,*
(b) *B is completely continuous, and*
(c) *$x = AxBy \Rightarrow x \in S$, for all $y \in S$.*

Then the operator equation (1) has a solution, whenever $M\phi(r) < r$, $r > 0$, where $M = \|B(S)\|$.

Proof. Let $y \in S$ and define a mapping $A_y : X \to X$ by

$$A_y(x) = AxBy, \ \ x \in X.$$

Notice that A_y is a nonlinear contraction on X with a contraction function ψ given by $\psi(r) = M\phi(r)$, $r \in \mathbb{R}^+$, since we have that

$$\begin{aligned}\|A_y x_1 - A_y x_2\| &\le \|Ax_1 - Ax_2\| \|By\| \\ &\le M\phi(\|x_1 - x_2\|)\end{aligned}$$

whenever $x_1, x_2 \in X$. Now an application of a fixed point theorem of Boyd and Wong [6] yields that there is a unique point $x^* \in X$ such that

$$A_y(x^*) = x^*$$

or, equivalently,

$$x^* = Ax^*By.$$

Since hypothesis (c) holds, we have that $x^* \in S$. Define a mapping $N : S \to X$ by

$$Ny = z, \tag{3}$$

where $z \in X$ is the unique solution of the equation

$$z = AzBy, y \in S.$$

We show that N is continuous. Let $\{y_n\}$ be a sequence in S converging to a point y. Since S is closed, $y \in S$. Now

$$\begin{aligned}\|Ny_n - Ny\| &= \|ANy_n By_n - ANyBy\| \\ &\le \|ANy_n By_n - ANyBy_n\| + \|ANyBy_n - ANyBy\| \\ &\le \|ANy_n - AN(y)\|\|By_n\| + \|AN(y)\|\|y_n - y\| \\ &\le M\phi(\|Ny_n - Ny\|) + \|ANy\|\|y_n - y\|\end{aligned}$$

and hence

$$\limsup_n \|Ny_n - Ny\| \le M\phi(\limsup_n \|Ny_n - Ny\|) + \|ANy\|(\limsup_n \|y_n - y\|).$$

This shows that $\lim_n \|Ny_n - Ny\| = 0$ and consequently N is continuous on S. Next we show that N is a compact operator on S. Now for any $z \in S$ we have

$$\begin{aligned}\|Az\| &\le \|Aa\| + \|Az - Aa\| \\ &\le \|Aa\| + \alpha\|z - a\| \\ &\le c\end{aligned}$$

where $c = \|Aa\| + \operatorname{diam}(S)$ for some fixed $a \in S$.

Let $\epsilon > 0$ be given. Since B is completely continuous, $B(S)$ is totally bounded. Then there is a set $Y = \{y_1, \dots, y_n\}$ in S such that

$$B(S) \subset \bigcup_{i=1}^{n} \mathcal{B}_\delta(w_i),$$

where $w_i = B(y_i)$, $\delta = \left(\frac{1-\alpha M}{c}\right)\epsilon$ and $\mathcal{B}_\delta(w_i)$ is an open ball in X centered at w_i of radius δ. Therefore for any $y \in S$ we have a $y_k \in Y$ such that

$$\|By - By_k\| < \left(\frac{1-\alpha M}{c}\right)\epsilon.$$

Also we have

$$\begin{aligned}\|Ny - Ny_k\| &\le \|AzBy - Az_k By_k\| \\ &\le \|AzBy - Az_k By\| + \|Az_k By - Az_k Bz_k\| \\ &\le \|Az - Az_k\|\|By\| + \|Az_k\|\|By_k - By\| \\ &\le (\alpha M)\|z - z_k\| + \|Az_k\|\|By_k - By\| \\ &\le \frac{c}{1-\alpha M}\|By - By_k\| \\ &< \epsilon.\end{aligned}$$

This is true for every $y \in S$ and hence

$$N(S) \subset \bigcup_{i=1}^{n} \mathcal{B}_{\epsilon}(z_i),$$

where $z_i = N(y_i)$. As a result $N(S)$ is totally bounded. Since N is continuous, it is a compact operator on S. Now an application of Schauder's fixed point yields that N has a fixed point in S. Then by the definition of N

$$x = Nx = A(Nx)Bx = AxBx,$$

and so the operator equation $x = AxBx$ has a solution in S. □

Since every Lipschitzian mapping is $\mathcal{D}$-Lipschitzian, we obtain the following interesting corollary to Theorem 2.1 in the applicable form to nonlinear differential and integral equations.

Corollary 2.1. *Let S be a closed, convex and bounded subset of a Banach algebra X and let $A : X \to X$, $B : S \to X$ be two operators such that*

(a) *A is Lipschitzian with a Lipschitz constant α,*
(b) *B is completely continuous, and*
(c) *$x = AxBy \Rightarrow x \in S$, for all $y \in S$.*

Then the operator equation (1) *has a solution, whenever $\alpha M < 1$, where $M = \|B(S)\|$.*

The following sufficient condition which guarantees the hypothesis (c) of Theorem 2.1 appears in [5].

Proposition 2.1. *Let S be a closed, convex and bounded subset of a Banach algebra X such that $S = \{y \in X \mid \|y\| \leq r\}$ for some real number $r > 0$. Let $A : X \to X - \{0\}$, $B : S \to X$ be two operators satisfying hypotheses* (a)–(b) *of* Theorem 2.1. *Further, if*

$$\|x\| \leq \left\| \left(\frac{I}{A} \right) x \right\| \tag{4}$$

for all $x \in X$, then $x \in S$.

3. Functional integral equations

The geometrical and topological fixed point theorems have some nice applications to nonlinear differential and integral equations. See [7] and the references therein. Similarly the hybrid fixed point theorems in Banach algebras are also useful for proving the existence theorems to certain nonlinear differential and integral equations. Here in this section we illustrate the applicability of our Theorem 2.1 and Proposition 2.1 by considering the following examples of nonlinear functional integral equations.

Example 3.1. Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, the set of all real numbers, consider the nonlinear functional integral equation (in short FIE)

$$x(t) = \left[\frac{1}{1 + |x(\theta(t))|} \right] \left(q(t) + \int_0^{\sigma(t)} g(s, x(\eta(s)))\, \mathrm{d}s \right), \tag{5}$$

for all $t \in J$, where $\theta, \eta : J \to J$, $q : J \to \mathbb{R}$, and $g : J \times \mathbb{R} \to \mathbb{R}$ are continuous.

By a solution of the FIE (5) we mean a continuous function $x : J \to \mathbb{R}$ that satisfies FIE (5) on J.

Let $X = C(J, \mathbb{R})$ be a Banach algebra of all continuous real-valued functions on J with the norm

$$\|x\| = \sup_{t \in J} |x(t)|. \tag{6}$$

We shall obtain the solution of FIE (5) under some suitable conditions on the functions involved in (5). Suppose that the function σ and g satisfy the condition

$$\left.\begin{array}{l} \sigma(t) \le t, \\ |g(t, x)| < 1 - \|q\|, \quad \|q\| < 1 \end{array}\right\} \tag{7}$$

for all $t \in J$ and $x \in \mathbb{R}$.

Define a subset S of X by

$$S = \{y \in X \mid \|y\| \le 1\}. \tag{8}$$

Consider the two mappings $A, B : X \to X$ defined by

$$Ax(t) = \frac{1}{1 + |x(\theta(t))|}, \ t \in J, \tag{9}$$

and

$$Bx(t) = q(t) + \int_0^{\sigma(t)} g(s, x(\eta(s)))\, \mathrm{d}s, \ t \in J. \tag{10}$$

Then the FIE (5) is equivalent to the operator equation

$$Ax(t)Bx(t) = x(t), \ t \in J. \tag{11}$$

We shall show that the operators A and B satisfy all the conditions of Corollary 2.1.

Clearly A defines a mapping $A : X \to X - \{0\}$. First we show that A is Lipschitzian on X. Let $x, y \in X$. Then we have

$$\begin{aligned} |Ax(t) - Ay(t)| &= \left| \frac{1}{1 + |x(\theta(t))|} - \frac{1}{1 + |y(\theta(t))|} \right| \\ &= \frac{|x(t)| - |y(t)|}{(1 + |x(\theta(t))|)(1 + |y(\theta(t))|)} \\ &\le \|x - y\|. \end{aligned}$$

Taking the supremum over t we obtain

$$\|Ax - Ay\| \le \|x - y\|$$

which shows that A is a Lipschitzian with a Lipschitz constant $\alpha = 1$.

It is an easy exercise to prove that B is completely continuous on S. We show that $B : S \to S$. Let $x \in S$. Then by (7) and (10),

$$\begin{aligned} |Bx(t)| &\le |q(t)| + \int_0^{\sigma(t)} |g(t, x(\eta(s)))|\, \mathrm{d}s \\ &< |q(t)| + \int_0^{t} (1 - \|q\|) \mathrm{d}s. \end{aligned}$$

Since $Bx \in C(J, \mathbb{R})$, there is a point $t^* \in J$ such that

$$\|Bx\| = |Bx(t^*)| = \max_{t \in J} |Bx(t)|.$$

Therefore we have

$$\begin{aligned}\|Bx\| &= |Bx(t^*)| \\ &< |q(t^*)| + \int_0^{t^*} (1 - \|q\|)\mathrm{d}s \\ &\le \|q\| + \int_0^1 (1 - \|q\|)\mathrm{d}s \\ &= 1,\end{aligned}$$

i.e. $\|Bx\| < 1$. As a result $B : S \to S$. Finally we show that condition (4) of Proposition 2.1 holds. Now for any $x \in X$,

$$\begin{aligned}\left\| \left(\frac{I}{A}\right)(x) \right\| &= \sup_{t \in J} \left| \frac{x(t)}{Ax(t)} \right| \\ &= \sup_{t \in J} \{|x(t)|[1 + |x(\theta(t))|]\} \\ &\ge \|x\|,\end{aligned}$$

and so by Proposition 2.1, condition (c) of Theorem 2.1 is satisfied. Thus the operators A and B satisfy all the conditions of Theorem 2.1 and hence an application of it yields that the operator equation (11) and consequently the FIE (5) has a solution on J.

Example 3.2. Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the nonlinear integral equation (in short IE)

$$x(t) = [1 + \lambda|x(\theta(t))|] \left(q(t) + \int_0^{\sigma(t)} g(s, x(\eta(s)))\, \mathrm{d}s \right), \tag{12}$$

for all $t \in J$, and $0 < \lambda < 1$, where $\theta, \sigma, \eta : J \to J$, $q : J \to \mathbb{R}$, and $g : J \times \mathbb{R} \to \mathbb{R}$ are continuous.

By a solution of the FIE (12) we mean a continuous function $x : J \to \mathbb{R}$ that satisfies FIE (12) on J. Let $X = C(J, \mathbb{R})$ be a Banach algebra of all continuous real valued functions on J with the norm given in (6).

We shall obtain the solution of FIE (12) under some suitable conditions. We assume that the functions involved in (12) satisfy the condition (7). Define a subset S of X by

$$S = \left\{ y \in X \mid \|y\| \le \frac{1}{1 - \lambda} \right\}. \tag{13}$$

Consider the two mappings $A, B : X \to X$ defined by

$$Ax(t) = 1 + \lambda|x(\theta(t))|, \quad t \in J, \tag{14}$$

and

$$Bx(t) = q(t) + \int_0^{\sigma(t)} g(s, x(\eta(s)))\mathrm{d}s, \quad t \in J. \tag{15}$$

We shall show that the operators A and B satisfy all the conditions of Corollary 2.1.

First we show that A is Lipschitzian on X. Let $x, y \in X$. Then

$$\begin{aligned}|Ax(t) - Ay(t)| &= |[1 + \lambda|x(\theta(t))|] - [1 + \lambda|x(\theta(t))|]| \\ &= \lambda \mid |x(\theta(t))| - |y(\theta(t))| \mid \\ &\leq \lambda \ \|x - y\|,\end{aligned}$$

or $\|Ax - Ay\| \leq \lambda \ \|x - y\|$, which shows that A is a Lipschitzian with a Lipschitz constant λ.

It is proved as in Example 3.1 that the operator B is completely continuous on S and that $B : S \to S$. Also we have $\|Bx\| < 1$ for all $x \in S$. Let $x \in S$ be arbitrary with $x = AxBy$ for some $y \in S$. Then we have

$$\begin{aligned}|x(t)| &= |Ax(t)|\,|By(t)| \\ &\leq \|Ax\|\,\|By\| \\ &\leq 1 + \lambda\|x\| \\ &\leq \frac{1}{1-\lambda}\end{aligned}$$

and so, $\|x\| \leq \frac{1}{1-\lambda}$. As a result $x \in S$. Thus hypothesis (c) of Theorem 2.1 is satisfied. Thus if $\lambda < 1$, then an application of Theorem 2.1 yields that the FIE (12) has a solution on J. $\square$

Remark 3.1. We note that the operator A in Example 3.2 does not satisfy condition (4) of Proposition 2.1.

Remark 3.2. It is worthwhile mentioning that in both Examples 3.1 and 3.2, the operator $\left(\frac{I}{A}\right)$ is not one-to-one on X. This proves the advantage of Theorem 2.1 over those of Theorems 1.1 and 1.3.

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