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Point sets that minimize $(\leq k)$-edges, 3-decomposable drawings, and the rectilinear crossing number of K_{30}

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ABSTRACT

There are two properties shared by all known crossing-minimizing geometric drawings of K_n, for n a multiple of 3. First, the underlying n-point set of these drawings minimizes the number of $(\leq k)$-edges, that means, has exactly $3\binom{k+2}{2}$ $(\leq k)$-edges, for all $0 \leq k < n/3$. Second, all such drawings have the n points divided into three groups of equal size; this last property is captured under the concept of 3-decomposability. In this paper we show that these properties are closely related: every n-point set with exactly $3\binom{k+2}{2}$ $(\leq k)$-edges for all $0 \leq k < n/3$, is 3-decomposable. The converse, however, is easy to see that it is false. As an application, we prove that the rectilinear crossing number of K_{30} is 9726.

1. Introduction

The *rectilinear crossing number* $\overline{\text{cr}}(G)$ of a graph G, is the minimum number of edge crossings in a *geometric drawing* of G in the plane, that is, a drawing of G in the plane where the vertices are points in general position and the edges are straight segments. Determining $\overline{\text{cr}}(K_n)$, where K_n is the complete graph with n vertices, is a well-know open problem in combinatorial geometry initiated by Guy [11].

The rectilinear crossing number problem is related with the concept of k-edges. A *k-edge* of an n-point set P, with $0 \leq k \leq n/2-1$, is a line through two points of P leaving exactly k points on one side. A $(\leq k)$-edge is an i-edge with $0 \leq i \leq k$. Let $E_k(P)$ denote the number of k-edges of P and $E_{\leq k}(P)$ denote the number of $(\leq k)$-edges, that is, $E_{\leq k}(P) = \sum_{j=0}^{k} E_j(P)$. Finally, $E_{\leq k}(n)$ denotes the minimum of $E_{\leq k}(P)$ taken over all n-point sets P.

The exact determination of $E_{\leq k}(n)$ is another notable open problem in combinatorial geometry. In 2005 [6], Aichholzer et al. gave the following lower bound for $E_{\leq k}(n)$:

$$E_{\leq k}(n) \geq 3\binom{k+2}{2} + 3\binom{k+2-\lfloor n/3 \rfloor}{2} - \max\{0, (k+1-\lfloor n/3 \rfloor)(n-3\lfloor n/3 \rfloor)\}, \tag{1}$$

later, in 2007 [7], Aichholzer et al. proved that this lower bound is tight for $k \leq \lfloor 5n/12 \rfloor - 1$.

The number of crossings in a geometric drawing of K_n and the number of k- and $(\leq k)$-edges in the underlying n-point set P are closely related by the following equality, independently proved by Lóvasz et al. [12] and Ábrego and Fernández-

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Merchant [3]. For any set P of n points

$$\overline{\mathrm{cr}}(P) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2\rfloor -1} k\,(n-k-2)\,E_k(P), \quad \text{or equivalently,}$$

$$\overline{\mathrm{cr}}(P) = \left(\sum_{k=0}^{\lfloor n/2\rfloor -1} (n-2k-3)\,E_{\le k}(P)\right) - \frac{3}{4}\binom{n}{3} + \left(1+(-1)^{n+1}\right)\frac{1}{8}\binom{n}{2}. \tag{2}$$

Another concept that plays a central role in this paper is 3-decomposability, which is a property shared by all known crossing-minimizing geometric drawings of K_n, for n a multiple of 3. Formally, we say that a finite point set P is 3-*decomposable* if it can be partitioned into three equal sized sets A, B and C such that there exists a triangle T enclosing the point set P and the orthogonal projection of P onto the three sides of T show A between B and C on one side, B between C and A on the second side, and C between A and B on the third side. We say that a geometric drawing of K_n is 3-decomposable if its underlying point set is 3-decomposable.

In the following result we establish the relationship between 3-decomposability and the number of $(\le k)$-edges.

Theorem 1 (*Main Theorem*). *Let P be an n-point set, for n a multiple of* 3*, with exactly* $3\binom{k+2}{2}$ *$(\le k)$-edges for all* $0 \le k < n/3$*, then P is* 3*-decomposable.*

In fact, in [2] Ábrego et al. conjectured that for each positive integer n multiple of 3, all crossing-minimal geometric drawings of K_n are 3-decomposable.

As an application of the Main Theorem we prove that a 30-point set that minimizes the crossing number is 3-decomposable. Aichholzer established 9726 as the upper bound for $\overline{\mathrm{cr}}(K_{30})$ [5], moreover we have the following theorem.

Theorem 2 (*The Rectilinear Crossing Number of K_{30}*). *$\overline{\mathrm{cr}}(K_{30})$ is* 9726*.*

All the results of this paper are proved in a more general context of generalized configuration of points. In this scope we define by analogy the *pseudolinear crossing number* $\widetilde{\mathrm{cr}}(K_n)$.

Our main tools are the allowable sequences which will be formally defined in Section 2, and we mention some preliminary results due to Lóvasz et al. in [12]. In Section 3 we prove the Main Theorem. In Section 4 we use the Main Theorem to establish that a configuration with 30 points that minimize the crossing number is 3-decomposable and we give some implications of 3-decomposability. Finally, in Section 5 is the formal proof of Theorem 2.

2. Allowable sequences

An *allowable sequence* $\mathbf{\Pi}$ is a doubly infinite sequence $\ldots, \pi_{-1}, \pi_0, \pi_1, \ldots$ of permutations of n elements, where consecutive permutations differ by a transposition of neighboring elements, and π_i is the reverse permutation of $\pi_{i+\binom{n}{2}}$. Thus $\mathbf{\Pi}$ has period $2\binom{n}{2}$, and the hole information of $\mathbf{\Pi}$ is contained in any of its n-half-periods, which we call n-*half-periods*. We usually denote by Π an n-half-period of $\mathbf{\Pi}$.

It is know that if P is a set of n points in the plane in general position, then all the combinatorial information of P can be encoded by an allowable sequence $\mathbf{\Pi_P}$ on the set P, called *circular sequence* associated with P [10]. It is important to note that most allowable sequences are not circular sequences, however there is a one-to-one correspondence between allowable sequences and generalized configurations of points [10].

We have the following definitions and notations for allowable sequences. A transposition that occurs between elements in sites i and $i+1$ is an i-*transposition*, and we say that it moves through the *ith gate*. In this new setting an i-transposition, or $(n-i)$-transposition corresponds to an $(i-1)$-edge. For $i \le n/2$, an i-*critical transposition* is either an i-transposition or an $(n-i)$-transposition, and a $(\le k)$-*critical transposition* is a transposition that is i-critical for some $1 \le i \le k$. If Π is an n-half-period, then $N_k(\Pi)$ and $N_{\le k}(\Pi)$ denote the number of k-critical transpositions and $(\le k)$-critical transpositions in Π, respectively. Therefore $N_k(\Pi) = E_{k-1}(\Pi)$, $N_{\le k}(\Pi) = E_{\le k-1}(\Pi)$. When n is even an $n/2$-transposition is also called *halving* and $h(\Pi)$ denotes the number of halvings, and thus $h(\Pi) = E_{n/2-1}(\Pi)$.

Identity (2) relating k-edges to crossing number was originally proved for allowable sequences. All these definitions and functions coincide with their original counterparts for P when Π is the circular sequence of P. However, when $\overline{\mathrm{cr}}(n)$, and $E_{\le k}(n)$ are minimized over all allowable sequences on n points rather than over all sets of n points, the corresponding quantities may change so we define the notation $\widetilde{\mathrm{cr}}(n)$ and $\widetilde{E}_{\le k}(n)$. But it is clear that $\widetilde{\mathrm{cr}}(n) \le \overline{\mathrm{cr}}(n)$ and $\widetilde{E}_{\le k}(n) \le E_{\le k}(n)$. Ábrego et al. [1] proved that the lower bound (1) on $E_{\le k}(n)$ is also a lower bound on $\widetilde{E}_{\le k}(n)$ and use it to extend the lower bound on $\overline{\mathrm{cr}}(n)$ to $\widetilde{\mathrm{cr}}(n)$.

Let $\Pi = (\pi_0, \pi_1, \ldots, \pi_{\binom{n}{2}})$ be an n-half-period. For each $k < n/2$, define $m = m(k, n) := n - 2k$. In order to keep track of $(\le k)$-critical transpositions in Π, it is convenient to label the points so that the starting permutation is

$$\pi_0 = (a_k, a_{k-1}, \ldots, a_1, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_k).$$

Sometimes it will be necessary to say when an element is moving, so we will say that an element x *exits* (respectively, *enters*) *through the ith A-gate* if it moves from the position $k-i+1$ to the position $k-i+2$ (respectively, from the position

$k-i+2$ to the position $k-i+1$) during a transposition with another element. Similarly, x *exits* (respectively, *enters*) *through the ith C-gate* if it moves from the position $m+k+i$ to the position $m+k+i-1$ (respectively, from $m+k+i-1$ to $m+k+i$) during a transposition.

An $a \in \{a_1,\ldots,a_k\}$ (respectively, $c \in \{c_1,\ldots,c_k\}$) is *confined* until the first time it exits through the first A-gate (respectively, C-gate); then it becomes *free*. A transposition is *confined* if both elements involved are confined.

The following results, from Proposition 3 to Proposition 7, are due to Lovász et al. in paper [12]:

Proposition 3. *Let Π_0 be an n-half-period, and let $k < n/2$. Then there is an n-half-period Π, with the same number of $(\leq k)$-critical transpositions as Π_0, and with no confined transpositions.*

In view of this statement, for the rest of this section we assume that the n-half-period Π under consideration has no confined transpositions.

The *liberation sequence* $\sigma(\Pi)$ (or simply σ if no confusion arises) of Π contains all the a's and all the c's, in the order in which they become free in Π. Since Π has no confined transpositions, the a's appear in increasing order, as do the c's. We let $T(a_i)$ (respectively $T(c_i)$) denote the set of all those c's (respectively a's) that appear after a_i (respectively c_i) in σ.

A transposition that swaps elements in the positions i and $i+1$ occurs in the *A-Zone* (respectively, *C-Zone*) if $i \leq k$ (respectively, $i \geq k+m$). Such transpositions are of obvious relevance: a transposition is $(\leq k)$-critical if and only if it occurs either in the A-Zone or in the C-Zone.

For $1 \leq i \leq j \leq k$, the ith A-gate is a *compulsory exit-gate* for a_j, and the ith C-gate is a *compulsory entry-gate* for a_j: that is, a_j has to exit through the ith A-gate at least once, and enter the ith C-gate at least once. Analogous definitions and observations hold for c_j: the ith A-gate is a *compulsory entry-gate* for c_j, and the ith C-gate is a *compulsory exit-gate* for c_j. A transposition in which an element enters (respectively, exits) one of its compulsory entry (respectively, exit) gate for the first time is a *discovery* transposition *for* the element. A transposition is a *discovery* transposition if it is a discovery transposition for at least one of the elements involved. If it is a discovery transposition for both elements, then it is a *double-discovery* transposition (for the reader familiar with [12], what we call double-discovery transpositions are the transpositions represented by a directed edge in the *savings digraph* of [12]).

Discovery and double-discovery transpositions play a central role in [12]. The key results are the following, which hold for any n-half-period with no confined transpositions (the first statement is a straightforward counting, whereas the second definitely requires a proof).

Observation 4. *There are (exactly) $2\binom{k+1}{2}$ transpositions that are discovery transpositions for some a, and (exactly) $2\binom{k+1}{2}$ transpositions that are discovery transpositions for some c.*

Proposition 5. *There are at most $\binom{k+1}{2}$ double-discovery transpositions.*

Since each discovery transposition is $(\leq k)$-critical, these statements immediately imply the following.

Proposition 6. *There are at least $3\binom{k+1}{2}$ $(\leq k)$-critical transpositions.*

An n-half-period Π with no confined transpositions is *perfect* if the following hold:

(a) Each transposition in Π that occurs in the A-Zone or in the C-Zone is a discovery transposition.
(b) a_i is involved in (exactly) $\min\{i, |T(a_i)|\}$ double-discovery transpositions in the C-Zone.
(c) Each c_i is involved in (exactly) $\min\{i, |T(c_i)|\}$ double-discovery transpositions in the A-Zone.

The following result is implicit in the proof of Theorem 10 in [12].

Proposition 7. *If Π is perfect, then it has exactly $3\binom{k+1}{2}$ $(\leq k)$-critical transpositions for all $k \leq m$. Conversely, if Π has no confined transpositions, and has exactly $3\binom{k+1}{2}$ $(\leq k)$-critical transpositions for all $k \leq m$, then it is perfect.*

3. Proof of Main Theorem

The concept of 3-decomposability for n-point sets is also generalized in the setting of allowable sequences. An n-half-period Π of an allowable sequence $\mathbf{\Pi}$ is 3-*decomposable* if the elements in Π can be labeled $A=\{a_{n/3},a_{n/3-1},\ldots,a_1\}$, $B=\{b_1,b_2,\ldots,b_{n/3}\}$, $C=\{c_1,c_2,\ldots,c_{n/3}\}$ and if $\pi_0=(a_{n/3},a_{n/3-1},\ldots,a_1,\ b_1,b_2,\ldots,b_{n/3},\ c_1,c_2,\ldots,c_{n/3})$ is the first permutation of Π, thus, all transpositions between an element of A and an element of B occur before the transpositions between C and $A\cup B$, after occur all the transpositions between A and C prior that the transposition between B and C and later occur all the transpositions between C and B. In particular, there are some indices $0<s<t<\binom{n}{2}$, such that π_{s+1} shows all the b-elements followed by all the a-elements followed by all the c-elements, and π_{t+1} shows all the b-elements followed by all the c-elements followed by all the a-elements. An allowable sequence is 3-*decomposable* if it contains an n-half-period 3-decomposable.

Before proving the Main Theorem, we must first state two propositions:

Proposition 8. *Suppose that Π is perfect. Then, in the liberation sequence σ of Π, either all the a's occur consecutively or all the c's occur consecutively.*

Proof. The last entry in σ is either a_k or c_k, and by symmetry we may assume without any loss of generality that it is a_k. Our strategy is to suppose that $a_{t-1}c_\ell c_{\ell+1}\cdots c_k a_t \cdots a_k$ is a suffix of σ, where $\ell > 1$ and $2 \le t \le k$, and derive a contradiction.

We claim that a_{t-1} swaps with c_k in the C-Zone. We start by noting that since Π is perfect, and $|T(a_{t-1})| = k-\ell+1 \ge 1$, it follows that a_{t-1} is involved in a double-discovery transposition in the C-Zone with at least one c. If this transposition involves (a_{t-1} and) c_k, then our claim obviously holds. Thus suppose that it involves (a_{t-1} and) c_i for some $i < k$. Then, right after a_{t-1} and c_i swap, c_k is to the right of a_{t-1}, since no confined transpositions occur in Π. Note that all transpositions that swap a_{t-1} to the left involve an a_j with $j > t-1$. On the other hand, since a_t (moreover, every a_j with $j \ge t$) gets freed after c_k, it follows that before any transposition can move a_{t-1} left, c_k must be freed (and before that it must transpose with a_{t-1}). This shows that the transposition μ that swaps a_{t-1} with c_k occurs in the C-Zone.

Thus, right after μ occurs, a_{t-1} is at position r, where $r \ge k+m+1$. We claim that $\max\{r, k+m+t-1\} < 2k+m$. Since $t-1 < k$, then $k+m+t-1 < 2k+m$, and so it suffices to show that if $r > k+m+t-1$, then $r < 2k+m$. So suppose that $r > k+m+t-1$. Note that the final position in Π (that is, the position in $\pi_{\binom{n}{2}}$) of a_{t-1} is $k+m+t-1$, and so by the time μ occurs there has been a transposition τ that moves a_{t-1} to the right of its final position (we remark that possibly $\tau = \mu$). Since τ occurs in the C-Zone and clearly is not a discovery step for a_{t-1}, and Π is perfect, it follows that τ is a discovery step for a c_i. Moreover, $|T(a_{t-1})| = k-\ell+1$ is greater than $t-1$, as otherwise (by the perfectness of Π) the transposition between a_{t-1} and c_i would have to be a double-discovery step. Thus $|T(a_{t-1})| > t-1$, and again invoking the perfectness of Π we get that a_{t-1} is involved with (exactly) $t-1$ double-discovery steps in the C-Zone, each with an element in $\{c_\ell, \ldots, c_k\}$. Therefore the number of possible transpositions that move a_{t-1} to the right of its final position $k+m+t-1$ is at most $k-\ell+1-(t-1)$. Thus the rightmost position of a_{t-1} throughout Π (and consequently r) is at most $k+m+t-1+k-\ell+1-(t-1) = 2k+m+1-\ell < 2k+m$.

Let R be the set of the points that occupy the positions $r+1, r+2, \ldots, 2k+m$ immediately after μ occurs. Since at this time every a_j with $j > t-1$ is confined, it follows that each point in R is either a b, a free c (this follows easily since there are no confined transpositions, and a_{t-1} reached the position r by transposing with c_k), or an a_j with $j < t-1$. In particular, each element in R still has to transpose with a_{t-1}.

We claim that a_{t-1} must move back to the B-Zone (after μ occurs). Seeking a contradiction, suppose that a_{t-1} does not go back to the B-Zone. We then claim that there is a transposition ρ of a_{t-1} with an element in R that is not a discovery transposition. Then the key observation is that at most $k+m+t-1-r$ transpositions of a_{t-1} with elements of R can be discovery transpositions. In order to prove this assertion, first we note that no transposition of a_{t-1} with an element in R can be discovery transposition for the element in R (recall that each element in R is either a b, a free c, or an a_j with $j < t-1$), so if such a transposition is a discovery one, it is so for a_{t-1} (recall that we assume that a_{t-1} does not go back to the B-Zone). But once a_{t-1} has reached r, it has at most $k+m+t-1-r$ discovery transpositions to do (since the rightmost compulsory entry-gate for a_{t-1} is the $(t-1)$st C-gate). Now since R has $2k+m-r$ elements, and $2k+m-r > k+m+t-1-r$, it follows that there is at least one transposition ρ of a_{t-1} with an element of R that is not a discovery transposition, as claimed. But the perfectness of Π implies that such a transposition must occur in the B-Zone, contradicting (precisely) our assumption that a_{t-1} did not move back to the B-Zone.

Thus, after μ occurs, a_{t-1} eventually re-enters the B-Zone, and since its final position is $k+m+t-1$, afterward it has to re-enter the C-Zone via a transposition λ that moves a_{t-1} to the right and an element $x \in R$ to the left. Since λ occurs in the C-Zone, and Π is perfect, then λ must be a discovery transposition. We complete the proof by arriving at a contradiction: λ *cannot* be a discovery transposition. Indeed, λ cannot be discovery for a_{t-1} (since it had already been in the C-Zone), so it must be a discovery step for x. On the other hand, since each $x \in R$ is either a b, a free c, or an a_j with $j < t-1$, λ it follows that λ cannot be a discovery transposition for x either. □

Our next statement shows that we can actually go a bit further: there is a perfect n-half-period Π' whose liberation sequence has all a's followed by all c's or vice versa.

Proposition 9. *Suppose that Π is a perfect n-half-period of an allowable sequence $\mathbf{\Pi}$. Then $\mathbf{\Pi}$ contains a perfect n-half-period Π', with initial permutation $a'_k a'_{k-1} \ldots a'_1\ b'_1 \ldots b'_m\ c'_1 c'_2 \ldots c'_k$, and whose liberation sequence is either $a'_1 a'_2 \ldots a'_k c'_1 c'_2 \ldots c'_k$ or $c'_1 c'_2 \ldots c'_k a'_1 a'_2 \ldots a'_k$.*

Proof. Let $\Pi = (\pi_0, \pi_1, \ldots, \pi_{\binom{n}{2}})$ be any perfect n-half-period, with initial permutation $\pi_0 = (a_k a_{k-1} \ldots a_1 b_1 \ldots b_m c_1 c_2 \ldots c_k)$, and let σ be the liberation sequence associated with Π. Thus the last entry of σ is either a_k or c_k, and a straightforward symmetry argument shows that we may assume without loss of generality that the last entry in σ is a_k. If σ is $c_1 c_2 \ldots c_k a_1 a_2 \ldots a_k$, then we are done. Thus we may assume that there is a $t, 2 \le t \le k$, such that $a_{t-1}, c_1, c_2, \ldots, c_k, a_t, a_{t+1}, \ldots, a_k$ is a suffix of σ.

In order to define the n-half-period Π' claimed by the proposition, we establish some facts regarding Π.

(A) *Let π_{i+1} be the permutation where c_1 becomes free. Then π_i is of the form $(a_k, a_{k-1}, \ldots, a_t, d_1, d_2, \ldots, d_p c_1, c_2, \ldots, c_k)$ where $p = t-1+m$ and each d_j is either a b or a free a.*

Proof of (A). The perfectness of Π readily implies that every transposition in the A-Zone that involves an element in $L := \{a_t, a_{t+1}, \ldots, a_k\}$ is a double-discovery transposition. In particular, the first element that moves an element in L must involve a c. Therefore, as long as no c becomes free, all the elements in L must stay in their original position. Finally, we observe that when c_1 becomes free, $a_1, a_2, \ldots, a_{t-1}$ are already free, so each d_j is either a b or a free a, as claimed.

(B) *No element in* $\{a_k a_{k-1} \ldots a_t d_1, \ldots, d_{t-1}\}$ (*these are the elements that are in the A-Zone, in the given order, in* π_i) *leaves the A-Zone before* c_k *becomes free*.

Proof of (B). Seeking a contradiction, let e be the first element in $\{a_k a_{k-1} \ldots a_t d_1, \ldots, d_{t-1}\}$ that moves out of the A-Zone before c_k becomes free. The perfectness of Π readily implies that the element that takes e out of the A-Zone is some c_j (where by assumption $j \neq k$). Now right after c_j swaps with e, c_j and c_k are in the A- and C-Zones, respectively. In particular, at this point c_j and c_k have not swapped. Now as we observed above, every transposition in the A-Zone involving an element in L is double-discovery, and so it follows that c_j never gets beyond (to the left of) the position $k - j + 1$. No matter where the $(c_j, c_k) \mapsto (c_k, c_j)$ transposition occurs, this implies that c_j must at some point be in a position r, with $k-j+1 \le r \le k$, and then move (right) to position $r + 1$. Now in order to reach its final position, c_j must eventually move back to the position r, via some transposition $\varepsilon = (x, c_j) \mapsto (c_j, x)$. Since Π is perfect, and ε occurs in the A-Zone, ε is a discovery transposition. But it clearly cannot be discovery for c_j, since c_j is re-visiting the position r. Now $x \in \{a_k, a_{k-1}, \ldots, a_t, d_1, \ldots, d_{t-1}\}$, since these were the elements to the left of c_j when it first entered the A-Zone. Clearly x cannot be a d, since each d is either a b or a free a, and ε must be discovery for x. Thus x must be in $L = \{a_k, a_{k-1}, \ldots, a_t\}$. But this is also impossible, since (see Proof of (A)) every transposition that involves an element in L must be a double-discovery transposition.

(C) *Suppose that two elements that are in the A-Zone* (*respectively*, *C-Zone*) *in* π_i *transpose with each other in the A-Zone* (*respectively*, *C-Zone*) *after* π_i. *Then at least one of these elements leaves the A-Zone* (*respectively*, *C-Zone*) *after* π_i *and before this transposition occurs*.

Proof of (C). First we note that the elements that are in the C-Zone in π_i are $c_1, c_2, \ldots, c_k$, in this order, and that if two of them transpose before at least one of them leaves the C-Zone, this transposition would be confined, contradicting the assumption that Π is perfect. That takes care of the C-Zone part of (C).

Now we recall that the elements that are in the A-Zone in π_i are $a_k, a_{k-1}, \ldots, a_t, d_1, d_2, \ldots, d_{t-1}$, in this order. Suppose that two such elements transpose in the A-Zone after π_i, and that between π_i and this transposition (call it λ) none of them leaves the A-Zone. It follows from the perfectness of Π that, for each a_j, every move of a_j until it leaves the A-Zone must involve some c_ℓ. Thus none of the elements involved in λ can be an a_j, that is, both must be d_j's. But such a transposition would clearly not be discovery (recall that each d is a free a or a b), contradicting the perfectness of Π. This completes the proof of (C).

(D) *After* π_i, *the elements in the A-Zone leave it in the order* $d_{t-1}, d_{t-2}, \ldots, d_1, a_t, \ldots, a_{k-1}, a_k$, *and the elements in the C-Zone leave it in the order* $c_1, c_2, \ldots, c_k$.

Proof of (D). This is an immediate corollary of (C).

Now define $\Pi' := (\pi_i, \pi_{i+1}, \ldots, \pi_{\binom{n}{2}}) = (\pi_0^{-1}, \pi_1^{-1}, \ldots, \pi_{i-1}^{-1}, \pi_i^{-1})$. It is straightforward to check that Π' is an n-half-period. Define the relabeling $a_i \mapsto a'_i$ for $i = t, t+1, \ldots, k$; $d_s \mapsto a'_{t-s}$ for $s = 1, \ldots, t-1$; $d_s \mapsto b'_{s-t+1}$ for $s = t, t+1, \ldots, p$; and $c_i \mapsto c'_i$ for $i = 1, \ldots, k$, so that the initial permutation of Π' (namely $\pi_i = (a_k a_{k-1} \ldots a_t d_1 d_2, \ldots, d_p c_1 c_2 \ldots c_k)$) is $(a'_k a'_{k-1} \ldots a'_1 b'_1 b'_2 \ldots b'_m c'_1 c'_2 \ldots c'_k)$.

To complete the proof, we check that (i) the liberation sequence of Π' is $c'_1 c'_2 \ldots c'_k a'_1 a'_2 \ldots a'_k$; and that (ii) Π' is perfect. We note that (i) follows immediately from (B) and (D). Now in view of Proposition 7, in order to prove that Π' is perfect it suffices to show that it has no confined transpositions, and that it has exactly $3\binom{k+1}{2}$ $(\le k)$-critical transpositions. From (C) it follows that Π' has no confined transpositions. On the other hand, an application of Proposition 7 to Π (which is perfect) yields that Π has $3\binom{k+1}{2}$ $(\le k)$-critical transpositions. The construction of Π' clearly reveals that Π and Π' have the same number of $(\le k)$-critical transpositions, and so Π' has $3\binom{k+1}{2}$ $(\le k)$-critical transpositions, as required. □

Proof of Theorem 1. Let Π be an n-half-period of $\mathbf{\Pi}_P$, for n a multiple of 3. By the hypothesis of the Main Theorem and the fact that $E_{\le k-1}(P) = N_{\le k}(\Pi)$, we have $N_{\le k}(\Pi) = 3\binom{k+1}{2}$ for each $1 \le k \le n/3$. This equality and Proposition 3 guarantee that $\mathbf{\Pi}_P$ contains an n-half-period, say Π_P, that satisfies the hypothesis of Proposition 7. Thus Π_P is perfect, and using Proposition 9 we get an n-half-period which behaves as we need for $\mathbf{\Pi}_P$ to be 3-decomposable. □

4. On allowable sequences that minimize the crossing number of K_{30}

This section is devoted to the study of allowable sequences which come from configurations of 30 points that minimize the crossing number. In particular, each result presented in this section is focused on establishing features of such sequences. Later, in Section 5, each of these properties will be used in the proof of Theorem 2.

We begin by proving, with the help of Theorem 1, that all optimal sequence of K_{30} are 3-decomposable.

We have the following bounds given by Ábrego et al. [4] for any n-half-period $\mathbf{\Pi}$ of an allowable sequence.

$$N_{\lfloor n/2\rfloor}(\Pi) \leq \begin{cases} \left\lfloor \frac{1}{2}\binom{n}{2} - \frac{1}{2}N_{\leq \lfloor n/2\rfloor -2}(\Pi)\right\rfloor, & \text{if } n \text{ is even,} \\ \left\lfloor \frac{2}{3}\binom{n}{2} - \frac{2}{3}N_{\leq \lfloor n/2\rfloor -2}(\Pi) + \frac{1}{3}\right\rfloor, & \text{if } n \text{ is odd.} \end{cases} \quad (3)$$

and

$$N_{\leq \lfloor n/2\rfloor -1}(\Pi) \geq \begin{cases} \binom{n}{2} - \left\lfloor \frac{1}{24}n(n+30) - 3\right\rfloor, & \text{if } n \text{ is even,} \\ \binom{n}{2} - \left\lfloor \frac{1}{18}(n-3)(n+45) + \frac{1}{9}\right\rfloor, & \text{if } n \text{ is odd.} \end{cases} \quad (4)$$

Now, if Π is a 30-half-period associated with a generalized configuration P of 30 points, then from (3) we know that $N_{15}(\Pi) \leq 72$ and if we combine (1) and (4) we get that $N_{14}(\Pi) \geq 72$. With these bounds in (2) we have 9723 as a lower bound for $\widetilde{\text{cr}}(K_{30})$. Moreover, if for some $k = 0, \ldots, 12$, (1) is not tight, then a simple calculation in (2) shows that $\widetilde{\text{cr}}(P) \geq 9727$ and therefore P will be worse than the best known configuration given implicitly by Aichholzer and Krasser in [8], which establishes 9726 as an upper bound. Besides $72 \leq N_{14}(\Pi) \leq 75$ or $\widetilde{\text{cr}}(P) \geq 9727$. So, in an optimal configuration with 30 points, (1) must be tight for each $k = 0, \ldots, 12$ and so, by the Main Theorem, P is 3-decomposable.

For the remainder of this subsection, let us assume that $\Pi = (\pi_0, \pi_1, \ldots, \pi_{\binom{30}{2}})$ is a 3-decomposable 30-half-period, with initial permutation $\pi_0 = (a_{10}, a_9, \ldots, a_1, b_1, b_2, \ldots, b_{10}, c_1, c_2, \ldots, c_{10})$ and $A = \{a_{10}, a_9, \ldots, a_1\}$, $B = \{b_1, b_2, \ldots, b_{10}\}$ and $C = \{c_1, c_2, \ldots, c_{10}\}$.

In order to count the number of $(\leq k)$-critical transposition in Π, we define two types of transpositions. A transposition is *monochromatic* if it occurs between two elements of the same set A, B or C, otherwise it is called *bichromatic*. We denote the number of monochromatic (respectively, bichromatic) $(\leq k)$-critical transpositions in Π by $N^{\text{mono}}_{\leq k}(\Pi)$ (respectively, $N^{bi}_{\leq k}(\Pi)$). Note that $N_{\leq k}(\Pi) = N^{\text{mono}}_{\leq k}(\Pi) + N^{bi}_{\leq k}(\Pi)$.

From [2] we get the next account for bichromatic transpositions on a 3-decomposable n-half-period Π':

$$N^{bi}_{\leq k}(\Pi') = \begin{cases} 3\binom{k+1}{2} & \text{if } k \leq n/3, \\ 3\binom{n/3+1}{2} + (k-n/3)n & \text{if } n/3 < k < n/2. \end{cases} \quad (5)$$

As a consequence of (5) we have the next two corollaries:

Corollary 10. $N^{bi}_{k}(\Pi) = 3k$ *for* $k = 1, 2, \ldots, 10$.

Corollary 11. $N^{bi}_{k}(\Pi) = 30$ *for* $k = 11, 12, 13, 14$.

Lemma 12. $N^{bi}_{15}(\Pi) = 15$.

Proof. The number of bichromatic transpositions between A and B is 100 because there is, exactly, one bichromatic transposition for each element of $A \times B$. For the same reason there are 100 bichromatic transpositions between A and C and 100 between B and C. So $N^{bi}_{\leq 15}(\Pi) = 300$. The desired result follows from Corollaries 10 and 11 and the fact that $N^{bi}_{15}(\Pi) = 300 - \sum_{k=1}^{14} N^{bi}_{k}(\Pi)$. □

From the above discussion, Corollary 10 and Theorem 1 it follows that all monochromatic transpositions occur in the *middle third*. Where the middle third is the space from the 11th position to 20th position.

4.1. Digraphs

Let Π be a 3-decomposable n-half-period of an allowable sequence $\mathbf{\Pi}$. A transposition between elements in the positions i and $i+1$ with $k < i < n-k$ is called a $(>k)$-*transposition*. All these transpositions are said to occur in the k-*center*. Let us denote the number of monochromatic transpositions that occur in the k-center and are of the kinds aa, bb, and cc by $N^{aa}_{>k}(\Pi)$, $N^{bb}_{>k}(\Pi)$, and $N^{cc}_{>k}(\Pi)$, respectively. Since each monochromatic transposition is an aa- or bb- or cc-transposition, then $N^{aa}_{>k}(\Pi) + N^{bb}_{>k}(\Pi) + N^{cc}_{>k}(\Pi)$ is the total number of monochromatic transpositions that occur in the k-center.

Let D_k be the digraph with vertex set $\{n/3, n/3-1, \ldots, 1\}$, and such that there is a directed edge from i to j if and only if $i > j$ and the transposition $a_i a_j$ occurs in the k-center. Note that the number of edges of D_k is exactly $N^{aa}_{>k}(\Pi)$.

In order to count the edges in D_k, let $\mathcal{D}_{v,m}$ be the class of all digraphs on v vertices, say $v, v-1, \ldots, 1$, satisfying that $[i]^+ \leq m + [i]^-$ for all $v \geq i \geq 1$, where $[i]^+$ and $[i]^-$ denote the outdegree and the indegree of the vertex i, respectively, and if we have an edge from i to j, $i \to j$, then $i > j$. Let $D_0(v, m)$ be the graph in $\mathcal{D}_{v,m}$ with vertices $v, v-1, \ldots, 1$ recursively defined by

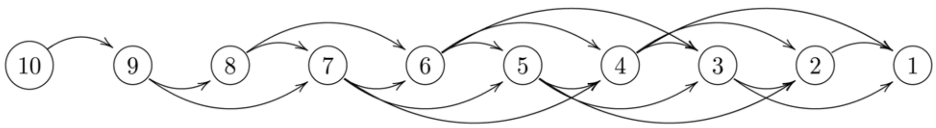

Fig. 1. Digraphs $D_0(10, 1)$.

- $[v]^- = 0$,
- $[i]^+ = \min\{[i]^- + m, i - 1\}$ for each $v \geq i \geq 1$, and
- for all $v \geq i > j \geq 1$, $i \to j$ if and only if $i - 1 \geq j \geq i - 1 - [i]^-$.

Balogh and Salazar prove in [9] that the maximum number of edges of a digraph in $\mathcal{D}_{v,m}$ is attained by $D_0(v, m)$. We note that D_k is in $\mathcal{D}_{n/3,n-2k-1}$, and hence the number of edges in D_k is bounded above by the number of edges in $D_0(n/3, n - 2k - 1)$.

From the preceding information, we can deduce that the number of edges in D_{14} is at most 20 (Fig. 1). This means that $N_{15}^{aa}(\Pi) \leq 20$, $N_{15}^{bb}(\Pi) \leq 20$, and $N_{15}^{cc}(\Pi) \leq 20$. Similarly, the number of edges in D_{13} is at most 33 and we know that $\binom{30}{2} - N_{\leq 13}(\Pi) = 144$ because all the bounds for $(\leq k)$-sets, for $k = 1, \ldots, 13$, are tight. Thus $N_{14}(\Pi) + h(\Pi) = 144$, besides from Corollary 11 and Lemma 12 we get that $N_{14}^{bi}(\Pi) + N_{15}^{bi}(\Pi) = 45$. This implies that $N_{>13}^{\text{mono}}(\Pi) = 99$ and therefore there are exactly 33-monochromatic transpositions in the 13-center per each set A, B and C.

Lemma 13. *If D is a digraph in $\mathcal{D}_{10,3}$ with 33 edges, then for $i, j = 10, 9, 8, 7$ and $i > j$ there is an edge from i to j.*

Proof. Clearly, the number of edges with tail in $\{10, 9, 8, 7\}$ and head in $\{6, 5, \ldots, 1\}$ is at most 12 and the number of edges in the vertex set $\{6, 5, \ldots, 1\}$ is at most 15 (this is attained by $D_0(6, 3)$). Then we need the 6 edges between the elements in $\{10, 9, 8, 7\}$ in order to get the 33 edges in D. $\square$

4.2. Restrictions in the monochromatic transpositions

From now on, we shall use $\Pi = (\pi_0, \pi_1, \ldots, \pi_{\binom{30}{2}})$ to denote a 3-decomposable 30-half-period of an optimal configuration for K_{30} and $\pi_0 = (a_{10}, a_9, \ldots, a_1, b_{l_1}, \ldots, b_{l_{10}}, c_1, c_2, \ldots, c_{10})$ to denote its first permutation. Also we assume that $A := \{a_1, a_2, \ldots, a_{10}\}$, $B := \{b_{l_1}, b_{l_2}, \ldots, b_{l_{10}}\}$ and $C := \{c_1, c_2 \ldots, c_{10}\}$.

As Π is 3-decomposable and all monochromatic transpositions occur in the middle third, it follows that there is a unique element of B that reaches the position 1 (or 30). We shall denote by b_{10} such element of B. For the same reasons, for $i = 2, 3, \ldots, 10$, there is a unique element of B, which we denote by b_{10-i+1}, that reaches the position i (or $30 - i + 1$) but not the position $i - 1$ (or $30 - i + 2$). Clearly, $B = \{b_1, b_2, \ldots, b_{10}\}$.

In this subsection we use that in Π the lower bound given in (1) is tight for $k = 0, \ldots, 12$ in order to deduce some restrictions about the monochromatic transpositions.

Remark 14. Because Π is 3-decomposable (A can interchange the role with B or C), everything that we say for A is also valid for B or C.

Lemma 15. *Each transposition of Π that contributes to $N_{11}^{\text{mono}}(\Pi) + N_{12}^{\text{mono}}(\Pi) + N_{13}^{\text{mono}}(\Pi)$ involves some of $a_{10}, a_9, a_8, b_{10}, b_9, b_8, c_{10}, c_9$ or c_8.*

Proof. Since we have exactly 33 monochromatic transpositions in the 13-center, then, by Lemma 13, the mandatory transpositions between elements of $\{a_{10}, a_9, a_8, a_7\}$ occur in the 13-center.

From Eq. (5) and the fact that (1) is tight for $k = 0, \ldots, 12$, we get that $N_{11}^{\text{mono}}(\Pi) = 6$, $N_{12}^{\text{mono}}(\Pi) = 12$ and $N_{13}^{\text{mono}}(\Pi) = 18$. Because no other a is behind a_{10}, it is not possible to have more than one monochromatic transposition per gate involving a_{10}. Furthermore, a_{10} should change with $a_9, a_8, \ldots, a_1$ in the 10-center (middle third). Thus a_{10} has one monochromatic transposition in each gate of the middle third. By Remark 14 the same happens with b_{10} and c_{10}. Thus, the $2 \cdot 3$ monochromatic transpositions due to a_{10}, b_{10} and c_{10} are all the monochromatic transpositions associated with $N_{11}^{\text{mono}}(\Pi)$.

For the preceding, every monochromatic transposition involving a_9 occurs in 11-center. Since the swap between a_{10} and a_9 occurs in the 13-center, a_9 contributes 2 to $N_{12}^{\text{mono}}(\Pi)$. So we have 4 different monochromatic transpositions due to a_{10} and a_9. By Remark 14, we get $2 \cdot 3 + 2 \cdot 3$ monochromatic transpositions due to a_{10}, a_9, b_{10}, b_9 and c_{10}, c_9 and they are all the monochromatic transpositions associated with $N_{12}^{\text{mono}}(\Pi)$.

So each monochromatic transposition involving a_8 occurs in the 12-center. Thus a_8 contributes 2 to $N_{13}^{\text{mono}}(\Pi)$. a_{10} and a_9 also have other 2 monochromatic transpositions there, and all the transpositions are different because a_{10}, a_9 and a_8 change in the 13-center. Hence the $2 \cdot 3 + 2 \cdot 3 + 2 \cdot 3$ monochromatic transposition associated with $N_{13}^{\text{mono}}(\Pi)$ are generated by $a_{10}, a_9, a_8, b_{10}, b_9, b_8, c_{10}, c_9$ and c_8. $\square$

Let $k \in \{10, 11, \ldots, 14\}$. Note that every element in a 3-decomposable 30-half-period Π' occupies each position of the 10-center at least once. From now on, if τ is the first (respectively, last) transposition in which $x \in A \cup C$ enters (respectively, *leaves*) the k-center, then we say that τ is the swap in which x *enters* (respectively, *leaves*) the k-center of Π'.

Lemma 16. *For $x \in \{a, c\}$, the elements $x_1, x_2, \ldots, x_{10}$ enter (respectively, leave) the 13-center of Π in ascending (respectively, descending) order. Moreover, for $i = 1, 2, \ldots, 7$;*

(1) *the swap between a_i and b_{7-i+1} occurs in the 13th gate and it is precisely the swap in which a_i enters (and b_{7-i+1} leaves) the 13-center of Π,*
(2) *the swap between a_{7-i+1} and c_i occurs in the 17th gate and it is precisely the swap in which a_{7-i+1} leaves (and c_i enters) the 13-center of Π and,*
(3) *the swap between b_i and c_{7-i+1} occurs in the 13th gate and it is precisely the swap in which c_{7-i+1} leaves (and b_i enters) the 13-center of Π.*

It follows from (3) (respectively, (1)) that $b_1, b_2, \ldots, b_7$ also enter (respectively, leave) the 13-center of Π in ascending (respectively, descending) order.

Proof. By Lemma 13 and the fact that there are exactly 33 monochromatic transpositions in the 13-center of Π, each transposition between elements of $\{x_{10}, x_9, x_8, x_7\}$ occurs in the 13-center. Also, by Lemma 15, each transposition between elements of $\{x_7, x_6, \ldots, x_1\}$ occurs in the 13-center. Together, these two conclusions, imply that the elements of $\{x_{10}, x_9, \ldots, x_1\}$ enter (respectively, leave) the 13-center of Π in ascending (respectively, descending) order.

We only show (1). The parts (2) and (3) are analogous.

Let $w \in \{a, b\}$. Because all monochromatic transpositions of Π occur in the 10-center, the elements of $\{w_{10}, w_9, \ldots, w_1\}$ enter (respectively, leave) the 10-center of Π in ascending (respectively, descending) order.

For $t = 1, 2, 3$ we know (Lemma 15) that every monochromatic transposition involving b_{10-t+1} occurs in the $(10+t-1)$-center. This and the fact that the b's leave the 10-center in descending order imply that the swap between a_1 and b_{10-t+1} occurs in the $(10 + t - 1)$th gate.

Since (Lemma 15) each transposition between elements of $\{b_7, b_6, \ldots, b_1\}$ occurs in the 13-center and they leave the 10-center in descending order, then the swap where a_j enters the 13-center must be with b_{7-j+1}, where $j = 1, 2, \ldots, 7$. $\square$

Lemma 17. *Let $\pi_{a_{10}}$ be the permutation of Π where a_{10} enters in the 13-center. Then $\pi_{a_{10}}$ looks like*

$$(B, a_{\le 4}, a_{\le 5}, a_{\le 6}, a_{10}, a_i, a_j, a_k, a_{\le 6}, a_{\le 5}, a_{\le 4}, C)$$

where $a_{\le p}$ is an a_u with $1 \le u \le p$, further $\{i, j, k\} = \{7, 8, 9\}$.

Proof. For $j = 7, 6, \ldots, 1$ let τ_j be the transposition in which a_j enters the 13-center. So, when τ_5 occurs there is at least one $r \in \{1, 2, 3, 4\}$ such that a_r is to the right hand side of the 13-center (without loss of generality, we assume that a_r is the rightmost a element). By Lemma 15, all the monochromatic transpositions between elements of $\{a_7, a_6, \ldots, a_1\}$ or between elements of $\{a_{10}, a_9, a_8, a_7\}$ occur in the 13-center. Thus a_r does not move to the left until after a_{10} exits of the 13-center. On the other hand, since all monochromatic transpositions occur in the middle third, when a_{10} enters the 13-center a_r must be at position 20. Using similar arguments with τ_6 and τ_7 we get the restriction on the right hand side.

Let a_{l_j} be the a that swaps with a_{10} in the $(14-j)$th gate (where $j = 3, 2, 1$). Since each aa transposition that contributes to $N_{11}(\Pi) + N_{12}(\Pi) + N_{13}(\Pi)$ involves a_{10}, a_9 or a_8 and the transpositions between elements of $\{a_{10}, a_9, a_8, a_7\}$ occur in the 13-center, then $l_j \le 6$. Thus a_{l_j} needs j transpositions of kind $a_{l_j}c$ in order to move to 13-center. Hence a_{l_j} will remain to the left hand side of the 13-center until after c_j enters the 13-center. But, by Lemma 16, when c_j enters the 13-center all a_n's with $n \ge 8 - j$ have left there. Hence $l_j \le 7 - j$. $\square$

Let $\mathrm{hal}(a_j)$ denote the number of a_i elements, $i < j$, such that a_j changes with a_i in the 15th gate. This means, the outdegree of the vertex a_j in the digraph D_{14} associated with $N^{aa}_{>14}(\Pi)$.

Some facts are easier to see in Π^*, the *reverse half-period of* Π. We define the reverse half-period of Π as $\Pi^* = (\pi^*_0, \pi^*_1, \ldots, \pi^*_l, \ldots, \pi^*_{\binom{30}{2}}) := \left(\pi^{-1}_{\binom{30}{2}}, \pi^{-1}_{\binom{30}{2}-1}, \ldots, \pi^{-1}_{\binom{30}{2}-l}, \ldots, \pi^{-1}_0\right)$. It is clear that Π and Π^* have the same combinatorial properties.

Lemma 18. *Let $\pi_{a_{10}}$ be the permutation of Π where a_{10} enters the 13-center. If a_i, $1 \le i \le 5$, is at position $10 + l$ or at position $20 - l + 1$, $1 \le l \le 3$, then $\mathrm{hal}(a_i) \le l$.*

Proof. We just prove the case when a_i is at position $10 + l$, otherwise we look at Π^*. Let $B(a_i)$ be the set of $l - 1$ a's that are behind a_i in $\pi_{a_{10}}$. Let j be the number of elements in $B(a_i)$ with index smaller than i. This means that in $\pi_{a_{10}}$, a_i has already changed with each element of $B(a_i)$ with index smaller than i. Note that these transpositions contribute at most j to $\mathrm{hal}(a_i)$. On the other hand, each element of $B(a_i)$ with index greater than i moves a_i to the left one time, then a_i could make at most $((l - 1) - j) + 1$ transpositions in the 15th gate which involve an a with index smaller than i. Thus $\mathrm{hal}(a_i) \le j + (((l - 1) - j) + 1) = l$. $\square$

Corollary 19. *$N^{aa}_{15}(\Pi) \le 19$, $N^{bb}_{15}(\Pi) \le 19$ and $N^{cc}_{15}(\Pi) \le 19$.*

Proof. What we say for A also apply for B and C. By Lemmas 17 and 18, $\mathrm{hal}(a_4) + \mathrm{hal}(a_5) \le 5$ and hence the digraph D_{14} associated with $N^{aa}_{>14}(\Pi)$ has at most 19 edges: at most 5 edges with tail in $\{a_{10}, a_9, a_8, a_7, a_6\}$ and head in $\{a_5, a_4, a_3, a_2, a_1\}$, at most 6 edges between the elements of $\{a_{10}, a_9, a_8, a_7, a_6\}$, at most 5 edges with tail in $\{a_5, a_4\}$, and at most 3 edges between the elements of $\{a_3, a_2, a_1\}$. $\square$

Remark 20. In fact, if we want to have 19 halvings, $\mathrm{hal}(a_{10}) + \mathrm{hal}(a_9) + \cdots + \mathrm{hal}(a_6)$ must be 11, $\mathrm{hal}(a_5) + \mathrm{hal}(a_4)$ must be 5 and $\mathrm{hal}(a_3) + \mathrm{hal}(a_2) + \mathrm{hal}(a_1)$ must be 3. The latter means that a_3, a_2, a_1 have to change in the 15th gate.

Corollary 21. *If $N^{aa}_{15}(\Pi) = 19$, then in the permutation $\pi_{a_{10}}$ of Π in which a_{10} enters the 13-center, a_1 and a_2 are at positions 11 and 20, respectively, or vice versa.*

Proof. From Lemma 18 and Remark 20 it follows that a_4 is not at position 11 or 20 in $\pi_{a_{10}}$. On the other hand, by Lemma 17 we know that a_6 is at position 13 (position 18), then a_4, a_5 occupy the positions 18 and 19 (positions 12 and 13) or they occupy the positions 12 and 18 (positions 13 and 19), not necessarily in that order. Because $\mathrm{hal}(a_3)$ must be 2, then, by Lemma 18 and with the prior discussion, a_3 must be at position 12 or 19. So we get that a_1, a_2 are at positions 11 and 20, not necessarily in that order. $\square$

Before proceeding with the proof of Theorem 2, we need to establish two more lemmas.

Lemma 22. *Let $\pi_{a_{10}}, \pi_{c_{10}}$ and $\pi_{b_{10}}$ be the permutations of Π where a_{10}, c_{10} and b_{10} enter the 13-center, respectively. If a_5 is at position 12 or 19 in $\pi_{a_{10}}$, then $N^{aa}_{15}(\Pi) < 19$, $N^{bb}_{15}(\Pi) < 19$ and $N^{cc}_{15}(\Pi) < 19$.*

Proof. Suppose that a_5 is at position 12 in $\pi_{a_{10}}$ (the case when a_5 is at position 19 is the same if we see Π^*). So $\pi_{a_{10}}$ looks like

$$\pi_{a_{10}} = (B, a_{i_1}, a_5, a_{i_2} | a_{10} - - - | a_{i_3}, a_{i_4}, a_{i_5}, C). \tag{6}$$

Since there are no aa-transpositions after $\pi_{a_{10}}$ on the left hand side of the 13-center, a_5 moves to the 13-center by means of two ac-transpositions. By Lemma 16, the swap between a_5 and c_3 occurs in the 17th gate, and hence, a_5 is moved from the positions 12–13-center by c_1 and c_2. On the other hand, because all the transpositions between elements of $\{c_1, c_2, \ldots, c_7\}$ or between elements of $\{c_7, c_8, c_9, c_{10}\}$ occur in the 13-center, when c_{10} enters the 13-center, c_1 and c_2 are at positions 11 and 12, not necessarily in that order. So $\pi_{c_{10}}$ looks like

$$\pi_{c_{10}} = (B, c_{1 \text{ or } 2}, c_{2 \text{ or } 1}, c_{j_1} | - - - c_{10} | c_{j_2}, c_{j_3}, c_{j_4}, A), \tag{7}$$

and by Lemma 17, $j_4 \in \{3, 4\}$.

Now we deduce some restrictions on $\pi_{b_{10}}$. As before, since there are no cc-transpositions after $\pi_{c_{10}}$ on the right hand side of the 13-center, c_{j_4} moves to the 13-center by means of three bc-transpositions. By Lemma 16, the swap between c_{j_4} and b_{7-j_4+1} occurs in the 13th gate, and hence, c_{j_4} is moved from the positions 20–13-center by three b's, say b_{k_1}, b_{k_2}, and b_{k_3}, such that $k_1, k_2, k_3 < 7 - j_4 + 1 \le 5$. Thus, when $\pi_{b_{10}}$ occurs, b_{k_1}, b_{k_2} and b_{k_3} are at positions 18, 19 and 20. So $\pi_{b_{10}}$ looks like $(C, b_{k_6}, b_{k_5}, b_{k_4} | b_{10} - - - | b_{k_3}, b_{k_2}, b_{k_1}, A)$. Thus, by Lemma 17, $k_4 = 6$ and $k_5 = 5$ and $\pi_{b_{10}}$ looks like

$$\pi_{b_{10}} = (C, b_{k_6}, b_5, b_6 | b_{10} - - - | b_{k_3}, b_{k_2}, b_{k_1}, A). \tag{8}$$

In a similar way as (7) was obtained from (6), it is possible to obtain (9) (respectively, (11)) from (8) (respectively, (10)); (10) can be obtained from (9) as (8) was obtained from (7).

$$\pi_{a_{10}+\binom{30}{2}} = (C, a_{1 \text{ or } 2}, a_{2 \text{ or } 1}, a_{i_3} | - - - a_{10} | a_{i_2}, a_5, a_{i_1}, B). \tag{9}$$

$$\pi_{c_{10}+\binom{30}{2}} = (A, c_{j_4}, c_5, c_6 | c_{10} - - - | c_{j_1}, c_{2 \text{ or } 1}, c_{1 \text{ or } 2}, B). \tag{10}$$

$$\pi_{b_{10}+\binom{30}{2}} = (A, b_{1 \text{ or } 2}, b_{2 \text{ or } 1}, b_p | - - - b_{10} | b_6, b_5, b_{k_6}, C). \tag{11}$$

The desired result is immediate from (9)–(11) and Corollary 21. $\square$

Lemma 23. *Let $\pi_{a_{10}}, \pi_{c_{10}}$ and $\pi_{b_{10}}$ as in Lemma 22. If $N^{aa}_{15}(\Pi) = 19$ and for $x = a, b, c$; x_j occupies the 11th or 20th position in $\pi_{x_{10}}$, then $j \in \{1, 2\}$.*

Proof. We only prove the case $x = c$ (the cases $x = a$ and $x = b$ are analogous). Suppose that c_j occupies the 11th or 20th position in $\pi_{c_{10}}$.

Case 1. c_j occupies the 11th position in $\pi_{c_{10}}$. Suppose that a_t occupies the 13th position in $\pi_{a_{10}}$. By Lemma 22 we know that $t \in \{5, 6\}$.

By Lemma 16, the swap between a_t and c_{7-t+1} occurs in the 17th gate, and hence, a_t is moved from the positions 13–13-center by a c_r such that $r \leq 7-t \leq 2$. On the other hand, by Lemma 15 we know that c_r does not have monochromatic transpositions on the left hand side of the 13-center until after $\pi_{c_{10}}$ occurs. Thus $c_r = c_j$.
CASE 2. c_j occupies the 20th position in $\pi_{c_{10}}$. Seeking a contradiction, suppose that $j \notin \{1, 2\}$. So by Lemma 17, $j \in \{3, 4\}$. Again, by Lemma 16, the swap between c_j and b_{7-j+1} occurs in the 13th gate, and hence, c_j is moved from the position 20–13-center by three b's, say b_{j_1}, b_{j_2}, and b_{j_3}, such that $j_1, j_2, j_3 < 7 - j + 1 \leq 5$. It follows from Lemma 15 that none of b_{j_1}, b_{j_2}, and b_{j_3} moves until after $\pi_{b_{10}}$ occurs. This implies that b_{j_1}, b_{j_2}, and b_{j_3} occupy the positions 18, 19 and 20 in $\pi_{b_{10}}$. By Lemma 17, b_5 is in the 12th position and by Remark 14 and Lemma 22, $N_{15}^{aa}(\Pi) < 19$. $\square$

5. The rectilinear crossing number of K_{30}: proof of Theorem 2

Let $\pi_{a_{10}}$, $\pi_{b_{10}}$ and $\pi_{c_{10}}$ as in Lemma 22. By Lemmas 22 and 23, if $N_{15}^{aa}(\Pi) = 19$ then, without loss of generality, $\pi_{a_{10}}$ looks like

$$\pi_{a_{10}} = (B, a_{i_1}, a_{i_2}, a_6 | a_{10} - - - | a_5, a_{i_3}, a_{i_4}, C), \tag{12}$$

with $\{i_1, i_4\} = \{1, 2\}$, otherwise we look for Π^*, besides in the 13-center are a_9, a_8, a_7 in some order.

By Lemma 16, a_6 leaves the 13-center with c_2, so a_6 re-enters the 13-center with the transposition with c_1. Thus c_1 occupies the 11th position of $\pi_{c_{10}}$. So by Lemma 23, $\pi_{c_{10}}$ looks like

$$\pi_{c_{10}} = (B, c_1, c_{j_1}, c_{j_2} | - - - c_{10} | c_{j_3}, c_{j_4}, c_2, A). \tag{13}$$

Again, since b_6 enters the 13-center with the swap with c_2, $\pi_{b_{10}}$ looks like $(C, b_{k_1}, b_{k_2}, b_{k_3} | b_{10} - - - | b_{k_4}, b_{k_5}, b_{k_6}, A)$ with $k_4, k_5, k_6 \leq 5$. Thus, by Lemmas 17 and 22, $\pi_{b_{10}}$ looks like

$$\pi_{b_{10}} = (C, b_{k_1}, b_{k_2}, b_6 | b_{10} - - - | b_5, b_{k_5}, b_{k_6}, A). \tag{14}$$

In a similar way as (13) was obtained from (12), it is possible to obtain (15) (respectively, (17)) from (14) (respectively, (16)); (16) can be obtained from (15) as (14) was obtained from (13).

$$\pi_{a_{10}+\binom{30}{2}} = (C, a_1, a_{i_3}, a_5 | - - - a_{10} | a_6, a_{i_2}, a_2, B). \tag{15}$$

$$\pi_{c_{10}+\binom{30}{2}} = (A, c_2, c_{j_4}, c_6 | c_{10} - - - | c_5, c_{j_1}, c_1, B). \tag{16}$$

$$\pi_{b_{10}+\binom{30}{2}} = (A, b_1, b_{k_5}, b_5 | - - - b_{10} | b_6, b_{k_2}, b_2, C). \tag{17}$$

So we have only two cases, when i_2 equals to 3 or 4.
CASE $i_2 = 4$. The permutation $\pi_{a_{10}}$ is $(B, a_2, a_4, a_6 | a_{10} - - - | a_5, a_3, a_1, C)$. By Lemma 16, a_4 leaves the 13-center with c_4, then a_4 must re-enter the 13-center with c_3 and therefore $\pi_{c_{10}}$ is $(B, c_1, c_3, c_5 | - - - c_{10} | c_6, c_4, c_2, A)$, and for similar reasons, the permutation $\pi_{b_{10}}$ is $(C, b_2, b_4, b_6 | b_{10} - - - | b_5, b_3, b_1, A)$.

Claim 24. *If* $\mathrm{hal}(a_3) + \mathrm{hal}(a_2) + \mathrm{hal}(a_1) = 3$, *then* $\mathrm{hal}(c_5) \leq 2$. *Hence* $N_{15}^{cc}(\Pi) \leq 18$.

Proof of Claim 24. Since $N_{15}^{aa}(\Pi) = 19$, by Remark 20, $\mathrm{hal}(a_3)+\mathrm{hal}(a_2)+\mathrm{hal}(a_1) = 3$. By Lemma 16, a_3 leaves the 13-center swapping with c_5, and the permutation is

$$(B, c_1, \{c_3, a_2\} | \{c_2, c_4, a_1\} c_5 | a_3, \ldots),$$

where the notation { } means that c_2, c_4, a_1 occupy those positions, but not necessarily in that order, similarly for a_2 and c_3. Because a_2 must change with a_1 in the 15th gate, this is only possible if c_5 changes with a_1 in the 15th gate, but then c_5 does not change with neither c_2 or c_4 in the 15th gate, and therefore $\mathrm{hal}(c_5) \leq 2$. $N_{15}^{cc}(\Pi) \leq 18$ is a consequence of Remark 20. This completes the proof of Claim 24. $\square$

If $N_{15}^{cc}(\Pi) = 18$ and with the fact that $\mathrm{hal}(c_5) \leq 2$, by Remark 20, we conclude that $\mathrm{hal}(c_3) + \mathrm{hal}(c_2) + \mathrm{hal}(c_1) = 3$. Since $\pi_{c_{10}}$ has the same configuration as $\pi_{a_{10}}$, named $(B, c_1, c_3, c_5 | - - - c_{10} | c_6, c_4, c_2, A)$ and also satisfies the hypotheses of Claim 24, we conclude that $N_{15}^{bb}(\Pi) \leq 18$. Now if $N_{15}^{bb}(\Pi) = 18$, B satisfies Claim 24 too and implies that $N_{15}^{aa}(\Pi) \leq 18$, which is a contradiction. Then $N_{15}^{aa}(\Pi) = 19$, $N_{15}^{cc}(\Pi) = 18$ and $N_{15}^{bb}(\Pi) \leq 17$.

So we suppose that $N_{15}^{cc}(\Pi) \leq 17$. The only case we have to worry about is when $N_{15}^{bb}(\Pi) = 19$, but recall that when b_{10} enters the 13-center, the permutation $\pi_{b_{10}}$ is

$$\pi_{b_{10}} = (C, b_2, b_4, b_6 | b_{10} - - - | b_5, b_3, b_2, A)$$

and B holds the hypotheses of Claim 24, which implies that $N_{15}^{aa}(\Pi) \leq 18$, and this is a contradiction. Thus $N_{15}^{aa}(\Pi) = 19$, $N_{15}^{cc}(\Pi) \leq 17$ and $N_{15}^{bb}(\Pi) \leq 18$.
CASE $i_2 = 3$. So, $\pi_{a_{10}} = (B, a_2, a_3, a_6 | a_{10} - - - | a_5, a_4, a_1, C)$. By Lemma 16, a_3 leaves the 13-center with c_5, then a_3 re-enters 13-center with c_3 or c_4.

Suppose that a_3 re-enter with c_3, then $\pi_{c_{10}}$ looks like

$$\pi_{c_{10}} = (B, c_1, c_3, c_5| - - - c_{10}|c_6, c_4, c_2, A),$$

but c_4 leaves the 13-center with b_4, then c_4 must re-enter with b_3, so we have

$$\pi_{b_{10}} = (C, b_2, b_4, b_6|b_{10} - - - |b_5, b_3, b_1, A),$$

but again, b_4 leaves the 13-center with a_4, so b_4 re-enters with a_3, and then we get

$$\pi_{a_{10}+\binom{30}{2}} = (C, a_1, a_3, a_5| - - - a_{10}|a_6, a_4, a_2, B),$$

which is a contradiction. Thus a_3 re-enters the 13-center with c_4.

Here, just for convenience we work in Π^*. Let $\pi^*_{a_{10}}$ be the permutation of Π^* where a_{10} enters the 13-center. So,

$$\pi^*_{a_{10}} = (C, a_1, a_4, a_5|a_{10} - - - |a_6, a_3, a_2, B).$$

Claim 25. *b_2 does not change with b_1 or, if* $\text{hal}(a_5) = 3$ *then b_3 does not change with b_1 in the* 15*th gate. Moreover, in both cases* $N^{bb}_{15}(\Pi) \le 18$.

Proof of Claim 25. If b_2 does not change with b_1 in the 15th gate, by Remark 20, $N^{bb}_{15}(\Pi) \le 18$.

So we assume that b_2 changes with b_1 in the 15th gate. Like $N^{aa}_{15}(\Pi) = 19$, by Remark 20 and Lemma 18, $\text{hal}(a_5)$ is 3. When a_6 leaves the 13-center, this swap is with b_2, so in that moment we have the following situation

$$(\dots|\{a_2, a_3, b_1\}b_2|a_6, \dots).$$

When b_2 changes with b_1 in the 15th gate, we have the following

$$(\dots|a_{2\text{ or }3}, b_2, b_1, a_{3\text{ or }2}|\dots),$$

a_5 re-enters the 13-center with b_2 and must change with either a_2 or a_3 in the 15th gate to complete 3 halvings because at most a_5 has changed in the 15th gate with a_1 and a_4, this implies that there must be an a in the 16th position and that is only possible if b_1 swaps with the leftmost a of the 13-center, and so when a_5 leaves the 13-center and b_3 enters it, the permutation is

$$(\dots|b_1, \{a_{2\text{ or }3}, a_{3\text{ or }2}\}, b_3|a_5\dots),$$

but a_4 re-enters the 13-center with b_3, and there are no more b's in the 13-center until after a_4 leaves it, thus no one moves b_1 from the 13th position and therefore b_3 does not change with b_1 in the 15th gate. This and Remark 20 imply $N^{bb}_{15}(\Pi) \le 18$. This completes the proof of Claim 25.

If $N^{bb}_{15}(\Pi)$ is 18 and knowing that $\text{hal}(b_3)+\text{hal}(b_2)+\text{hal}(b_1) \le 2$, by Remark 20 we get that $\text{hal}(b_5)$ is 3 and also we have the same configuration $(C, b_2, b_3, b_6|---b_{10}|b_5, b_4, b_1, A)$. Then the hypotheses of Claim 25 are satisfied and consequently $N^{cc}_{15}(\Pi) \le 18$.

But again, if $N^{cc}_{15}(\Pi) = 18$ and $\text{hal}(c_3) + \text{hal}(c_2) + \text{hal}(c_1) \le 2$ then $\text{hal}(c_5)$ is equal to 3 and, by Claim 25, $N^{aa}_{15}(\Pi) \le 18$, and this is a contradiction. So $N^{aa}_{15}(\Pi) = 19$, $N^{bb}_{15}(\Pi) = 18$ and $N^{cc}_{15}(\Pi) \le 17$.

Now we suppose that $N^{bb}_{15}(\Pi) \le 17$. The only case we are concerned about is when $N^{cc}_{15}(\Pi) = 19$. Since C satisfies Claim 25, in the moment that C changes with A we will get $N^{aa}_{15}(\Pi) \le 18$, which is a contradiction. Thus $N^{aa}_{15}(\Pi) = 19$, $N^{bb}_{15}(\Pi) \le 17$ and $N^{cc}_{15}(\Pi) \le 18$.

So, $N_{15}(\Pi) = N^{\text{mono}}_{15}(\Pi) + N^{bi}_{15}(\Pi) = 69$. This implies that $N_{14}(\Pi) = 75$, and by (2) we are done. □

6. Concluding remarks

In this paper we have presented a result that relates the number of $(\le k)$-edges with 3-decomposability. That is, every set of points in the plane which has a certain number of $(\le k)$-edges, can be grouped into three independent equal sized sets. Theorem 1 goes a step forward to the understanding of the structure of sets minimizing the number of $(\le k)$-edges. Aichholzer et al. [6] established that such sets always have a triangular convex hull. Here we show that these sets also are 3-decomposable.

As an application of Theorem 1, we give a free computer-assisted proof that the rectilinear crossing number of K_{30} is 9726. This closes the gap between 9723 and 9726, the best lower and upper bounds previously known.

In view of Theorem 1, we now give a more precise version of Conjecture 1 in [2]:

Conjecture 26. *For each positive integer n multiple of* 3*, all crossing-minimal geometric drawings of K_n have exactly* $3\binom{k+2}{2}$ *$(\le k)$-edges for all* $0 \le k \le n/3$.

We believe that Conjecture 26 is one of the main problems to be solved in order to understand the basic structure of the crossing-minimal geometric drawings of K_n.

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