Optimal energy decay rate for a class of weakly dissipative second-order systems with memory

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In this paper we study the asymptotic behavior of a class of linear dissipative integral differential equations $u_{tt}(t) + Au(t) - \int_{0}^{\infty} g(s) A^\alpha u(t - s) ds = 0$. We show that the solution decays and the decay rate is optimal.

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1. Introduction

Let $(H, \langle \cdot, \cdot \rangle, \| \cdot \|)$ be a Hilbert space. Let $A$ be a positive self-adjoint operator with domain $\mathcal{D}(A) \subset H$ with compact embeddings in $H$. Let $\alpha \in [0, 1)$ and $g$ be a nonnegative absolutely continuous nonincreasing function on $[0, +\infty)$. We introduce a class of second-order abstract models

$$u_{tt}(t) + Au(t) - \int_{0}^{\infty} g(s) A^\alpha u(t - s) ds = 0,$$

satisfying the initial conditions $u(-t) = u_0(t), \ t \geq 0, \ u_t(0) = u_1$, where the initial data $u_0$ and $u_1$ belong to suitable spaces we will define later. Here, the subscript $\cdot_t$ denotes the time derivative. If $\alpha = 0$, this differential problem rules the evolution of the electromagnetic field in the ionosphere, whereas, if $\alpha \in (0, 1)$, it may arise in the case of a generalized Kirchhoff viscoelastic beam, where a bending moment relation with memory is considered (see, for more details, [1.2] if $\alpha = 0$, and [3] if $\alpha \in (0, 1)$). Following the approach of Dafermos [4], we consider $\eta = \eta'(s)$, the relative history of $u$, defined as $\eta'(s) = u(t) - u(t - s)$. Hence, putting $\mu_0 = \int_{0}^{\infty} g(s) ds$, Eq. (1) turns into the system

$$u_{tt}(t) + Au(t) - \mu_0 A^\alpha u(t) + \int_{0}^{\infty} g(s) A^\alpha \eta'(s) ds = 0,$$

$$\eta'_t = -\eta'_s + u_t(t),$$

where the subscript $\cdot_s$ denotes the distributional derivative with respect to the internal variable $s$. Accordingly, the initial conditions become

$$u(0) = u_0, \quad u_t(0) = u_1, \quad \eta_0 = \eta_0, \quad \eta_1'(0) = \lim_{s \to 0} \eta_1'(s) = 0, \quad t \geq 0,$$
having set \( u_0 = u_0(0) \) and \( \eta_0(s) = u_0(0) - u_0(s) \). The energy of the system is given by \( E(t) = \frac{1}{2} \left[ \| A^{1/2} u(t) \|^2 - \mu_0 \| A^{\alpha/2} u(t) \|^2 + \int_0^\infty g(s) \| A^{\alpha/2} \eta(s) \|^2 \, ds + \| v(t) \|^2 \right] \). The loss of energy is due to the memory term solely (see (5)). The decay of the energy \( E(t) \) has been already studied in [1,2] when \( \alpha = 0 \), and in [3] when \( \alpha \in [0, 1) \). There, the authors proved that the system is not exponentially stable but there exists a polynomial decay rate.

The aim of this paper is to complete the analysis studying the optimal energy rate decay for problem (2)-(3) (see Theorem 3.1). The method we use is based on the necessary condition given in [5, Theorem 4.2].

2. Preliminaries, abstract formulation and semigroup of solutions

As a departure for obtaining the proofs of well-posedness and of stability, we will consider the system (2)-(3) as an abstract evolution equation in a certain Hilbert space, for which we introduce the following definitions, notions and assumptions.

For \( r \in \mathbb{R} \), we consider the scale of Hilbert spaces \( \mathcal{D}(A^{r/2}) \), endowed with the usual inner products \( \langle v_1, v_2 \rangle_{\mathcal{D}(A^{r/2})} = \langle A^{r/2} v_1, A^{r/2} v_2 \rangle \). The embeddings \( \mathcal{D}(A^{r/2}) \subseteq \mathcal{D}(A^{s/2}) \) are compact whenever \( r > s \). Then

\[
\| A^{1/2} u \|^2 - \mu_0 \| A^{\alpha/2} u \|^2 > k_1 \| A^{1/2} u \|^2, \quad \forall u \in \mathcal{D}(A^{1/2}),
\]

where \( k_1 \) is a positive constant and the coefficient \( \mu_0 \) will be assumed positive (see (h.1)). We suppose

\[
g(s) + \delta g(s) \leq 0, \quad \forall s \in (0, +\infty), \quad \delta > 0.
\]

This hypothesis allows us to introduce the weighted \( L^2 \)-space with respect to the measure \( g(s) \, ds \), \( \mathcal{M}_1 = L^2(\mathbb{R}^+, \mathcal{D}(A^{r/2})) \), endowed with the usual inner product. To give an accurate formulation of the evolution problem we introduce the product Hilbert spaces \( Z = \mathcal{D}(A^{1/2}) \times H \times \mathcal{M}_1 \), endowed with the following inner product (cf. (4)) \( \langle \phi, \zeta \rangle_Z = \langle \phi_1, \psi_1 \rangle_{\mathcal{D}(A^{1/2})} - \mu_0 \langle \phi_1 \psi_1 \rangle_{\mathcal{D}(A^{r/2})} + \langle \psi_2, \psi_2 \rangle_{\mathcal{M}_1} + \langle \psi_3, \psi_3 \rangle_{\mathcal{M}_1} \), where \( \phi = [\phi_1, \phi_2, \phi_3]^\top \), \( \zeta = [\psi_1, \psi_2, \psi_3]^\top \in Z \).

Moreover, to study the asymptotic behavior of the solution to our evolution problem, we suppose that the kernel \( g \) decays exponentially to zero, as \( t \to +\infty \), namely

\[
g(s) + \delta g(s) \leq 0, \quad \forall s \in (0, +\infty), \quad \delta > 0.
\]

Then, considering the notations and the assumptions introduced above, setting \( u = u_t \) and \( z(t) = [u(t), v(t), \eta(t)]^\top \), \( z_0 = [u_0, u_1, \eta_0]^\top \in Z \), system (2)-(3) can be rewritten as \( \frac{d}{dt} z(t) = \mathcal{L} z(t), t > 0, z(0) = z_0 \), where the linear operator \( \mathcal{L} \) is defined as

\[
\begin{bmatrix}
u \\
u \\
\eta
\end{bmatrix} =
\begin{bmatrix}
-Au + \mu_0 A^\alpha u - \int_0^\infty g(s) A^\alpha \eta(s) \, ds \\
\eta + \nu \\
\eta
\end{bmatrix},
\]

with domain \( \mathcal{D}(\mathcal{L}) = \{ z \in Z : u \in \mathcal{D}(A), v \in \mathcal{D}(A^{1/2}), \eta \in \mathcal{M}_1, \eta_0 \in \mathcal{M}_1, \eta_0 = 0, 0 \int_0^\infty g(s) A^\alpha \eta(s) \, ds \in H \} \), and the embedding \( \mathcal{D}(\mathcal{L}) \subseteq Z \) is compact. By means of

\[
\langle \mathcal{L} z, z \rangle_Z = -\langle \eta, \eta \rangle_{\mathcal{M}_1} + \frac{1}{2} \int_0^\infty g(s) \| A^{\alpha/2} \eta(s) \|^2 \, ds \leq 0, \quad \forall \eta \in \mathcal{D}(T),
\]

it is possible to apply the Lumer–Phillips Theorem (see, e.g., [6]), so to obtain

**Theorem 2.1.** Assume that the memory kernel \( g \) satisfies condition (h.1). The linear operator \( \mathcal{L} \) is the infinitesimal generator of a \( C_0 \)-semigroup \( S(t) = e^{\mathcal{L} t} \) of contractions on \( Z \).

**Remark 2.2.** For any initial data \( z_0 \in Z \), the solution \( z \) is given by \( z(t) = S(t) z_0 \). The energy \( E(t) \) associated with (2)-(3) assumes in this new context the simpler form \( E(t) = \frac{1}{2} \| S(t) z_0 \|_Z^2 \).

3. Polynomial stability and optimal rate decay

In the next theorem, we first show that, when \( \alpha \in [0, 1) \), the associated energy to problem (2)-(3) is polynomially stable, and subsequently we find the optimal rate of decay. For the sake of simplicity, throughout the section, we will employ the same symbols \( C \), depending only on the structural parameters of the system under consideration (unless otherwise specified), for different positive constants, even in the same formula. Also, we shall often make use without explicit mention of the Young and the Hölder inequalities.

**Theorem 3.1.** Let (h.1)-(h.2) hold. If \( z_0 \in \mathcal{D}(\mathcal{L}) \) and \( \alpha \in [0, 1) \), then we have that the solution \( z \) decays polynomially to zero as

\[
\| z(t) \|_Z \leq C \left[ \frac{\ln(t)}{t} \right]^{1/(2-2\alpha)} \ln(t) \| z_0 \|_{\mathcal{D}(\mathcal{L})}.
\]

Moreover, the rate of decay is optimal in the sense that the rate \( 1/t^{1/(2-2\alpha)} \) cannot be improved on \( \mathcal{D}(\mathcal{L}) \).
The first part of the proof is based on [7, Theorem 2.1]. Moreover, in order to show the optimality of the decay rate we need the result given in [5, Theorem 4.2].

**Proof.** We split the proof into two parts.

*Polynomial stability.* Let $F = [F_1, F_2, F_3]^T \in Z$ and consider the following equation $(i \beta I - \mathbb{L})z = F$, which can be read as

\[
\begin{align*}
\begin{cases}
i \beta u - v &= F_1 \\
i \beta v + Au - \mu_0 A^\alpha u + \int_0^\infty g(s) A^\alpha \eta(s) \, ds &= F_2 \\
i \beta \eta - v - T \eta &= F_3.
\end{cases}
\end{align*}
\]  

(6)

Taking inner product with $z$ in $Z$ we get $i \beta \|z\|_Z^2 - \langle \mathbb{L} z, z \rangle_Z = \langle F, z \rangle_Z$. By using relation (5) and considering the real part, it follows

\[
\frac{1}{2} \int_0^\infty g'(s) \|A^{\alpha/2} \eta(s)\|^2 \, ds \leq \text{Re}(\langle z, F \rangle_Z).
\]

(7)

Multiplying Eq. (6)_2 by $\int_0^\infty g(s) A^{\alpha-1} \eta(s) \, ds$, and applying assumptions (h.1)–(h.2),

\[
\|A^{(\alpha-1)/2} v\|^2 \leq C \int_0^\infty -g'(s) \|A^{\alpha/2} \eta(s)\|^2 \, ds + C \|A^{\alpha/2} u\| \int_0^\infty -g'(s) \|A^{\alpha/2} \eta(s)\|^2 \, ds + C \|z\|_Z \|F\|_Z,
\]

which implies that $\|A^{(\alpha-1)/2} v\|^2 \leq C \|z\|_Z \|F\|_Z$. Multiplying Eq. (6)_2 by $A^{\alpha-1} u$ we have

\[
\|A^{\alpha/2} u\|^2 \leq C \|A^{(\alpha-1)/2} v\|^2 + C \int_0^\infty -g'(s) \|A^{\alpha/2} \eta(s)\|^2 \, ds + C \|z\|_Z \|F\|_Z,
\]

and then, we find $\|A^{\alpha/2} u\|^2 \leq C \|z\|_Z \|F\|_Z$. By Eq. (6)_1 we obtain $\|A^{\alpha/2} v\| \leq C |\beta| \|A^{\alpha/2} u\| + \|A^{\alpha/2} F_1\|$, and consequently, we have

\[
\|A^{\alpha/2} v\| \leq C |\beta| \|z\|_Z^{1/2} \|F\|_Z^{1/2} + \|A^{\alpha/2} F_1\|. 
\]

(8)

Using interpolation we can estimate $\|v\| \leq C \|A^{(\alpha-1)/2} v\|^a \|A^{\alpha/2} v\|^{1-a}$, and by previous estimates we find

\[
\|v\| \leq C \left(\|z\|_Z^{1/2} \|F\|_Z^{1/2}\right)^a \left(|\beta| \|z\|_Z^{1/2} \|F\|_Z^{1/2} + \|A^{\alpha/2} F_1\|\right)^{1-a}. 
\]

(9)

On the other hand, multiplying (6)_2 by $u$ and recalling (6)_1, we get

\[
\|A^{1/2} u\|^2 \leq C \|u\|^2 + C \int_0^\infty -g'(s) \|A^{\alpha/2} \eta(s)\|^2 \, ds + C \|z\|_Z \|F\|_Z.
\]

and then, we obtain $\|A^{1/2} u\|^2 \leq C \|v\|^2 + C \|z\|_Z \|F\|_Z$. Finally, from previous inequalities we can estimate

\[
\|z\|^2 \leq C \left(\|z\|_Z^{1/2} \|F\|_Z^{1/2}\right)^{2a} \left(|\beta| \|z\|_Z^{1/2} \|F\|_Z^{1/2} + \|A^{\alpha/2} F_1\|\right)^{2-2a} + C \|z\|_Z \|F\|_Z,
\]

which implies $\|z\|^2 \leq C \beta^{2-2a} \|z\|_Z \|F\|_Z + C \|z\|_Z \|F\|_Z$. It follows that $\|z\|_Z \leq C \beta^{2-2a} \|F\|_Z$, for $\beta$ large enough, and by [7, Theorem 2.1] the proof is complete.

*Optimal decay rate.* To show the optimality we use the necessary condition given by [5, Theorem 4.2]. To do so we take $F_1 = F_2 = 0$, $g(t) = e^{-\gamma t}$, $\gamma > 1$. Let us denote by $\lambda_j$ the eigenvalues of $A$. We look for solutions of the form $u = p \lambda_j$, $v = q \lambda_j$, $\eta = h(s) \lambda_j$. From (6) we get that

\[
\begin{align*}
-\beta^2 + \lambda_j - \mu_0 \lambda_j^\alpha \end{align*} p + \int_0^\infty e^{-\gamma s} h(s) \, ds \lambda_j^\alpha = F_2
\]

(10)

\[
i \beta h(s) - i \beta p + h'(s) = 0.
\]

(11)

From the last equation we get $h(s) = -pe^{-i \beta s} + p$. Substitution of $h$ into Eq. (10) we get $\frac{(\gamma \beta^2 - \lambda_j^\alpha + \gamma \sigma_j^\alpha)}{\beta^2 + \gamma^2} p = F_2$, where $\sigma_j = \lambda_j - \mu_0 \lambda_j^\alpha + \frac{\lambda_j^\alpha}{\gamma}$. Here it is clear that there exists the solution $u = p \lambda_j$, $v = q \lambda_j$ and $\eta = h(s) \lambda_j$. Taking norm we get

\[
\frac{(\gamma \beta^2 - \lambda_j^\alpha + \gamma \sigma_j^\alpha)^2 + \beta^2 (\beta^2 - \sigma_j^2)^2}{\beta^2 + \gamma^2} |p|^2 = |F_2|^2.
\]

Taking $\beta = \beta_j = \sqrt{\sigma_j}$, we get $\frac{\lambda_j^2}{\sigma_j + \gamma} |p|^2 = |F_2|^2$. Since $\sigma_j \approx \lambda_j$, we find
∥z∥ ≤ C∥A^{1/2}u∥^2 = λ_j \frac{σ_j + γ^2}{λ_j^{2α}}|F_2|^2 \approx \frac{1}{λ_j^{2α-2}}|F_2|^2,

for λ_j large. If we can improve the rate of decay given in this theorem to t^{1/(2−2α−ε)} for ε small, then from [5, Theorem 4.2] we get that $1/β_j^{2α−2−γ} ∥z∥$, must be bounded. But this is not the case because $1/β_j^{2α−2−γ} ∥z∥ ≥ \frac{1}{β_j^{2α−2−γ}}|F_2| = \frac{σ_j^{1+α+ε}}{λ_j^{2α−1}}|F_2| \approx λ_j^ε|F_2| → ∞$, as β_j → ∞. By application of [5, Theorem 4.2], the proof is now complete. □

References