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TRIANGULATED EDGE INTERSECTION GRAPHS OF PATHS IN A TREE

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1. Preliminaries

Let $T=(V(T), E(T))$ be a tree and $\mathcal{P}=\{P_i\}$ denote a collection of non-trivial (i.e., of length at least 1) simple paths in T, where a path $P=(v_1, v_2, \ldots, v_d)$ is considered in the sequel as the collection $\{\{v_1\}, \{v_2\}, \ldots, \{v_d\}, \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{d-1}, v_d\}\}$. The intersection graph $\Omega(S, \mathcal{P})$ of $\mathcal{P}$ over set S has vertices which correspond to the paths in $\mathcal{P}$ and two vertices v_k and v_l are adjacent if the corresponding paths P_k and P_l intersect over S, that is, if $P_k \cap P_l \cap S = \emptyset$. Two kinds of intersection graphs of $\mathcal{P}$ have been considered so far, for $S=V(T)$ and $S=E(T)$, where in the former case the vertex set of T is considered as the collection of one-element sets. A graph G is *a vertex intersection graph of paths in a tree* (shortly, *VPT graph*) if $G=\Omega(V(T), \mathcal{P})$ for a certain tree T and a path collection $\mathcal{P}$ in T, and G is called *an edge intersection graph of paths in a tree* (*EPT graph*) if $G=\Omega(E(T), \mathcal{P})$ for some path collection $\mathcal{P}$ in a tree T. Neither of these two classes of graphs is contained in the other. Every cycle $C_k (k \geq 4)$ is an EPT graph but no $C_k (k \geq 4)$ is a VPT graph. Some examples of VPT graphs which are not EPT graphs can be found in [8] and [4]. The VPT graphs were defined and completely characterized in [6] and the algorithm for their recognition appeared in [1]. The EPT graphs have been introduced and partially characterized in [7] and [8]. Recently, it has been shown that the recognition problem for EPT graphs is NP-complete (see [4]), hence it is unlikely that there exists an elegant and complete characterization of these graphs.

The main purpose of this paper is to clarify the status of triangulated EPT graphs, where a graph G is *triangulated* if every cycle of length at least 4 has a chord, that is, an edge joining two nonconsecutive vertices of the cycle. It was shown in [8] that if G is a triangulated EPT graph then for every vertex v of G, the subgraph of G generated by $N(v)$, the neighbor set of v, is an interval graph. Based on this fact, it was conjectured that every triangulated EPT graph is a VPT graph. Here, we prove this conjecture. The graphs which are both VPT and EPT have been characterized in [4] (see Lemma 1); however, the conjecture does not follow from this result.

Lemma 1 ([4]). *The following statements are equivalent for a graph G:*
(i) *G is both a* VPT *and* EPT *graph.*
(ii) *$G=\Omega(V(T),\mathcal{P})$ for a path collection $\mathcal{P}$ on a tree T of maximum degree* 3.
(iii) *$G=\Omega(E(S),\mathcal{Q})$ for a path collection $\mathcal{Q}$ on a tree S of maximum degree* 3.

We conclude this section with a simple observation on the representations of EPT graphs.

Lemma 2. *If G is an* EPT *graph then there exist a tree T and a path collection $\{P_v: v\in V(G)\}$ on T such that*
(i) *no two end-vertices of the paths in $\{P_v\}$ coincide, and*
(ii) *end-vertices of the paths in $\{P_v\}$ are of degree* 2 *in T.*

Note that the condition (i) and (ii) are independent.
The definitions of graph-theoretic terms not given here can be found in [5].

2. Results

We shall prove in this section the following theorem.

Theorem 3. *Every triangulated* EPT *graph is also a* VPT *graph.*

Proof. Let G be a triangulated EPT graph and let G be represented on a tree T by a path collection $\mathcal{P}=\{P_v: v\in V(G)\}$ which satisfies the conditions of Lemma 2. If T has no vertices of degree greater than 3, then, by Lemma 1, G is also a VPT graph. Otherwise, we shall show that $\mathcal{P}$ on T can be transformed into a path collection $\mathcal{P}^*=\{P^*_v: v\in V(G)\}$ on a certain tree T^* such that $G=\Omega(V(T^*),\mathcal{P}^*)$, i.e., G is also a VPT graph. Let $w\in V(T)$, $\deg(w)\geqslant 4$ and T_w denote the subtree of T generated by w and its neighbors. If $\mathcal{P}_w$ denotes the subcollection of all paths of $\mathcal{P}$ which contain vertex w, then it is clear that $G_w=\Omega(E(T_w),\mathcal{P}_w)$ is a line graph as an EPT graph on a star, see also [3]. Moreover, G_w is triangulated as an induced subgraph of the triangulated graph G. We now need a characterization of triangulated line graphs.

Lemma 4. *The line graph $\mathcal{L}(G)$ of a multigraph G is triangulated if and only if G contains no cycle C_l ($l\geqslant 4$) as a subgraph.*

Proof. If G contains a cycle C_l ($l\geqslant 4$) then $\mathcal{L}(G)$ contains the chordless cycle of the same length l generated by the edges of C_l. On the other hand, a chordless cycle $C_l(l\geqslant 4)$ of $\mathcal{L}(G)$ corresponds to the cycle of length l in G. □

Every multigraph with no cycle C_l ($l\geqslant 4$) is triangulated, hence we have:

Corollary 5. *A multigraph G and its line graph $\mathscr{L}(G)$ are both triangulated if and only if G contains no cycle C_l ($l \geq 4$).*

A graph $K_4 - x$, where x is an edge of K_4, is the simplest triangulated graph whose line graph is not triangulated. Based on Lemma 4 we can easily determine all multigraphs whose line graphs are triangulated.

Corollary 6. *The line graph $\mathscr{L}(G)$ of a multigraph G is triangulated if and only if every 2-connected component of G is a triangle, possibly with parallel edges.*

Let us return to a graph G_w for a certain vertex w in T of degree greater than 3. There exists a multigraph F_w such that $G_w = \mathscr{L}(F_w)$. Equivalently, the edges of F_w are in a one-to-one correspondence with the paths of $\mathscr{P}_w$ and two paths in $\mathscr{P}_w$ share an edge in T_w if and only if the corresponding edges of F_w share a vertex. Hence, there exists a one-to-one correspondence between the vertices of F_w and the edges of T_w. Since, by Corollary 6, F_w may contain a triangle (in general, a triangle with parallel edges), we transform each triangle of F_w into a 3-star (see Fig. 1), in which each triangle-edge e is represented by a 2-edge path e'. Let F'_w denote the graph(tree) obtained by applying this transformation to F_w. It is easy to see that the intersection graph $\Omega(V(F'_w), E'(F'_w))$ is isomorphic to G_w, where $E'(F'_w)$ consists of all non-triangle edges of F_w and 2-edge paths of F'_w corresponding to triangle-edges of F_w. Moreover, every vertex v of F'_w, except those inserted by the removal of triangles, is incident with exactly those edges or 2-edge paths of $E'(F'_w)$ whose corresponding paths in $\mathscr{P}_w$ intersect over the edge of T_w corresponding to v.

This transformation of a star of T can be applied to every subtree T_w of T for which $\deg(w) \geq 4$. Let T^* be the tree obtained from T by inserting F'_w in the place of T_w, for every vertex w of degree greater than 3. Let $e = (u, w)$ denote an edge of T. If e belongs to two stars T_u and T_w then the vertices in F'_u and F'_w corresponding to e are merged in T^*. Otherwise, if only one end-vertex of e, say w, is of degree greater than 3 then the vertex of F'_w corresponding to e is merged with u in T^*. It is easy to see that every path $P_v \in \mathscr{P}$ is transformed into a path P^*_v

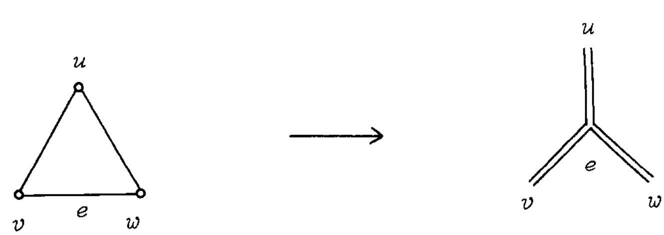

Fig. 1.

in T^*, and $P_u \cap P_w \cap E(T) \neq \emptyset$ if and only if $P_u^* \cap P_w^* \cap V(T^*) \neq \emptyset$ for every u, $w \in V(G)$, $u \neq w$. Therefore, G is a VPT graph and Theorem 3 is proved. □

Hence we can conclude:

Corollary 7. *It is an* NP-*complete problem to test if a triangulated graph is an* EPT *graph.*

Proof. If G is triangulated, then by Theorem 3, G must be a VPT graph if G is also an EPT graph. It is NP-complete however to test if a VPT graph is an EPT graph (see [4]). □

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