Global Attractivity in Nonlinear Delay Differential Equations

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We obtain sufficient conditions under which every solution of the nonlinear delay differential equation

\[ x(t) = f(t, x(t - \tau_1(t)), \ldots, x(t - \tau_m(t))), \quad t \geq t_0 \]

tends to its equilibrium. Our results have applications to delay differential equations in mathematical biology.

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1. INTRODUCTION AND PRELIMINARY

Our aim in this paper is to investigate the global attractivity of the equilibrium of the following quite general nonlinear and nonautonomous delay differential equation

\[ \dot{x}(t) = f(t, x(t - \tau_1(t)), \ldots, x(t - \tau_m(t))), \quad t \geq t_0 \]

(1.1)

where the function \( f \) and the delays \( \tau_1, \tau_2, \ldots, \tau_m \) satisfy the following hypotheses:

(H.1) \( f \in C([t_0, \infty) \times (L, \infty)^m, \mathbb{R}) \) for some \( L \geq -\infty \), and for every \( t \geq t_0 \), \( f(t, u_1, \ldots, u_m) \) is nonincreasing in each of its arguments \( u_1, \ldots, u_m \).

(H.2) There exists an \( x^* > L \) such that \( f(t, x^*, \ldots, x^*) \equiv 0 \) for all \( t \geq t_0 \);

(H.3) \( \tau_i \in C([t_0, \infty), [0, \infty]) \) and \( \lim_{t \to \infty} |t - \tau_i(t)| = \infty \) for \( i = 1, 2, \ldots, m \).

From (H.2) we see that \( x^* \) is an equilibrium of Eq. (1.1). In the next section, we will establish sufficient conditions for \( x^* \) to be a global

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attractor of all solutions of Eq. (1.1). Our results have applications to delay differential equations in mathematical biology. For example, they apply to the delay differential equation

$$\dot{x}(t) = r(t) \left(1 - \frac{cx(t)x^p(t - \tau(t))}{1 + x^p(t - \tau(t))}\right), \quad t \geq 0,$$

(1.2)

where

$$\left\{ \begin{array}{l}
c, p \in (0, \infty), r, \tau \in C[[0, \infty), [0, \infty)], \\
\lim_{t \to \infty} [t - \tau(t)] = \infty, \text{ and for some } \tau' > 0, \inf_{t \geq t_0} \tau(t) \geq \tau'. \end{array} \right.$$

(1.3)

When \(r(t) = a, c = b/a, \) and \(\tau(t) = \tau\) for some positive constants \(a, b, \) and \(\tau,\) then Eq. (1.2) reduces to

$$\dot{x}(t) = a - \frac{b}{1 + x^n(t - \tau)}, \quad t \geq 0.$$  

(1.4)

Equation (1.4) was proposed by Mackey and Glass [1977] for studying a “dynamic disease” involving respiratory disorders. A sufficient condition for all positive solutions of Eq. (1.4) to tend to its positive equilibrium has been established in [4]. Our result about Eq. (1.2), when restricted to Eq. (1.4), improves the results in [4].

Our results also apply to the delay differential equation

$$\dot{N}(t) = r(t)N(t) \left(1 - \frac{\sum_{i=1}^{m} a_i N^p(t - \tau_i(t))}{\sum_{i=1}^{m} s_i(t) N^p(t - \tau_i(t))}\right), \quad t \geq 0$$

(1.5)

where

$$\left\{ \begin{array}{l}
K, p \in (0, \infty), a_i \in [0, \infty), \tau_i \in C[[0, \infty), (0, \infty)], \\
s_i \in C[[0, \infty), [0, \infty)], \text{ and } \lim_{t \to \infty} [t - \tau_i(t)] = \infty \quad \text{for } i = 1, 2, \ldots. \end{array} \right.$$

(1.6)

A special case of Eq. (1.5) is the population model

$$\dot{N}(t) = r(t)N(t) \left(1 - \frac{N(t - \tau(t))}{K}\right), \quad t \geq 0.$$  

(1.7)

Equation (1.7) and extensions of it have been studied extensively. See, for example, [2] and the references cited therein. In this paper, we improve and extend several known global attractivity results related to this equation.
For any \( T \geq t_0 \), we define

\[
T_{-1} = \min_{1 \leq i \leq m} \left\{ \inf_{t \geq T} \{ t - \tau_i(t) \} \right\}.
\]

By a solution of Eq. (1.1) we mean a function \( x \in C^1([T, \infty), (T, \infty]) \) which satisfies Eq. (1.1) for \( t \geq T \).

With Eq. (1.1) we associate an “initial condition” of the form

\[
x(t) = \phi(t) \quad \text{for} \quad T_{-1} \leq t \leq T
\]

where \( \phi \in C^1([T_{-1}, T], (L, \infty)) \) is a given “initial function.” We assume that the IVP(1.1) and (1.8) have a unique solution \( x \) valid on \( [T, \infty) \); that is, \( x \) is continuously differentiable on \( [T, \infty) \), \( x \) satisfies (1.8), and \( x \) satisfies Eq. (1.1) for \( t \geq T \). As we will see, many equations satisfy this assumption.

2. GLOBAL ATTRACTIVITY OF EQUATION (1.1)

Throughout this paper, we will assume that (H_1)-(H_3) hold and set

\[
\tau(t) = \max\{\tau_i(t) : 1 \leq i \leq m\}, \quad t \geq 0.
\]

The following theorem is the main result in this section.

**Theorem 1.** Assume that \( x^* \) is the only equilibrium of Eq. (1.1) in \((L, \infty)\) and for any \( \mu \in (L, \infty) \) with \( \mu \neq x^* \),

\[
\left| \int_{t_0}^{\infty} f(s, \mu, \ldots, \mu) \, ds \right| = \infty.
\]

Suppose that there exists a nonincreasing function \( F \in C([L, \infty), \mathbb{R}) \) such that the one-sided limit \( F(L^+) = \lim_{x \to L^+} F(x) \in \mathbb{R} \) exists and satisfies the inequality

\[
x^* + F(x^* + F(L^+)) > L.
\]

Suppose also that for \( t \geq t_0 \)

\[
\int_{t - \tau(t)}^{t} f(s, u, \ldots, u) \, ds \leq F(u) \quad \text{for} \quad L < u < x^*
\]

and

\[
\int_{t - \tau(t)}^{t} f(s, u, \ldots, u) \, ds \geq F(u) \quad \text{for} \quad x^* < u < \infty.
\]
Finally, assume that the solution of the IVP
\[
\begin{align*}
  y_{n+1} &= x^* + F(y_n) \\
  y_0 &= x^* + F(L+)
\end{align*}
\]  
(2.3)
tends to \( x^* \) as \( n \) tends to \( \infty \). Then every solution of Eq. (1.1) tends to \( x^* \) as \( t \) tends to \( \infty \).

Proof. First we show that every solution \( x(t) \) which does not oscillate about \( x^* \) tends to \( x^* \). Assume that eventually \( x(t) > x^* \). The proof when eventually \( x(t) < x^* \) is similar and will be omitted. Then from Eq. (1.1) we see that for \( t \) sufficiently large,
\[
\dot{x}(t) \leq f(t, x^*, \ldots, x^*) = 0
\]
and so
\[
\lim_{t \to \infty} x(t) = l \geq x^* \text{ exists and } x(t) \geq l.
\]
Hence it follows that for \( t \) sufficiently large,
\[
l - x(t) = \int_t^\infty f(s, x(s - \tau_1(s)), \ldots, x(s - \tau_m(s))) \, ds
\]
\[
\leq \int_t^\infty f(s, l, \ldots, l) \, ds
\]
which, in view of (2.1), implies that \( l = x^* \).

Next, assume that \( x(t) \) is a solution which oscillates about \( x^* \). The proof of the theorem will be complete once we prove that for any integer \( q \geq 0 \), there exists \( T(q) \geq 0 \) such that
\[
L < y_{2q+1} \leq x(t) \leq y_{2q} \quad \text{for } t \geq T(q) \quad \text{(2.4)}
\]
Since \( x(t) \) oscillates, there is a sequence
\[
t_0 < t_1 < \cdots < t_n < \cdots
\]
such that
\[
\lim_{n \to \infty} t_n = \infty \text{ and } x(t_n) = x^* \text{ for } n = 0, 1, \ldots
\]  
(2.5)
Let \( s_n \in (t_n, t_{n+1}) \) be a point where \( x(t) \) obtains its maximum or minimum in \((t_n, t_{n+1})\). Hence \( \dot{x}(s_n) = 0 \) and so it follows from Eq. (1.1) that
\[
f(s_n, x(s_n - \tau_1(s_n)), \ldots, x(s_n - \tau_m(s_n))) = 0
\]
which, in view of the nonincreasing property of \( f \), implies that there is a point \( \xi_n \in [s_n - \tau(s_n), s_n] \) such that

\[
x(\xi_n) = x^*, \quad n = 0, 1, \ldots
\]

Hence by integrating both sides of Eq. (1.1) from \( \xi_n \) to \( s_n \) we find that

\[
x(s_n) - x^* = \int_{\xi_n}^{s_n} \left( f(t, x(t - \tau(t)), \ldots, x(t - \tau_n(t))) \right) dt
\]

\[
\leq \int_{s_n - \tau(s_n)}^{s_n} f(t, L^+, \ldots, L^+) dt
\]

\[
\leq F(L^+) \quad \text{for } n = 0, 1, \ldots
\]

and so

\[
x(s_n) \leq x^* + F(L^+) = y_0 \quad \text{for } n = 0, 1, \ldots
\]

Since \( y_0 \) is independent of the choice of \( s_n \), it follows that

\[
x(t) \leq y_0 \quad \text{for } t \geq t_0.
\]

In view of (H_3), we see that there exists \( T_0 > t_0 \) such that \( t - \tau(t) \geq t_0 \) for \( t \geq T_0 - \tau(T_0) \) and so

\[
x(t - \tau(t)) \leq y_0 \quad \text{for } t \geq T_0 - \tau(T_0).
\]

Also, it follows from (2.5) that there exists \( N_0 \geq T_0 \) such that

\[
t_n \geq T_0 - \tau(T_0) \quad \text{for } n \geq N_0.
\]

By using (2.8) in (2.6) and by using the nonincreasing nature of \( f \), we see that

\[
x(s_n) - x^* \geq \int_{\xi_n}^{s_n} f(t, y_0, \ldots, y_0) dt
\]

\[
\geq F(y_0) \quad \text{for } n \geq N_0
\]

and so

\[
x(s_n) \geq x^* + F(y_0) = y_1 \quad \text{for } n \geq N_0.
\]

As \( y_1 \) is independent of the choice of \( s_n \) and in view of (2.2), it follows that

\[
x(t) \geq y_1 > L \quad \text{for } t \geq t_{N_0}.
\]

By combining (2.7) and (2.9), we see that

\[
L < y_1 \leq x(t) \leq y_0 \quad \text{for } t \geq t_{N_0}.
\]
Now assume that there exists a \( t_{N_k} \) such that
\[
L < y_{2k+1} \leq x(t) \leq y_{2k} \quad \text{for } t \geq t_{N_k}.
\] (2.10)
Then there exists \( T'_k \geq t_{N_k} \) such that
\[
t - \tau(t) \geq t_{N_k} \quad \text{for } t \geq T'_k - \tau(T'_k)
\]
and so it follows from (2.10) that
\[
x(t - \tau(t)) \geq y_{2k+1} \quad \text{for } t \geq T'_k - \tau(T'_k). \tag{2.11}
\]
Also, there exists \( N'_k \geq T'_k \) such that
\[
t_n \geq T'_k - \tau(T'_k) \quad \text{for } n \geq N'_k.
\]
Hence by using (2.11) in (2.6), we find
\[
x(s_n) - x^* \leq \int_{s_n - \tau(s_n)}^{s_n} f(t, y_{2k+1}, \ldots, y_{2k+1}) \, dt
\]
\[
\leq F(y_{2k+1}) \quad \text{for } n \geq N'_k
\]
and so
\[
x(s_n) \leq x^* + F(y_{2k+1}) = y_{2(k+1)} \quad \text{for } n \geq N'_k
\]
which implies that
\[
x(t) \leq y_{2(k+1)} \quad \text{for } t \geq t_{N'_k}.
\] (2.12)
Then from (2.6) and by using the fact that there exist \( T_{k+1} \geq t_{N'_k} \) such that
\[
x(t - \tau(t)) \leq y_{2(k+1)} \quad \text{for } t \geq T_{k+1} - \tau(T_{k+1})
\]
and \( N_{k+1} \geq T_{k+1} \) such that
\[
t_n \geq T_{k+1} - \tau(T_{k+1}) \quad \text{for } n \geq N_{k+1},
\]
it follows for \( n \geq N_{k+1} \) that
\[
x(s_n) - x^* \geq \int_{s_n - \tau(s_n)}^{s_n} f(t, y_{2(k+1)}, \ldots, y_{2(k+1)}) \, dt
\]
\[
\geq F(y_{2k+1}) \quad \text{for } n \geq N_{k+1},
\]
and so
\[
x(s_n) \geq x^* + F(y_{2(k+1)}) = y_{2(k+1)+1} \quad \text{for } n \geq N_{k+1}.
\]
Hence,
\[ x(t) \geq y_{2(k+1)+1} \quad \text{for } t \geq t_{N_k+1}, \]  
(2.13)

Now from (2.10) and (2.2) and by using the nonincreasing property of \( F \), we obtain
\[ y_{2(k+1)} = x^* + F(y_{2(k+1)}) \leq x^* + F(L+) \]
and so
\[ y_{2(k+1)+1} = x^* + F(y_{2(k+1)}) \geq x^* + F(x^* + F(L+)) > L. \]  
(2.14)

By combining (2.12), (2.13), and (2.14), we find
\[ L < y_{2(k+1)+1} \leq x(t) \leq y_{2(k+1)} \quad \text{for } t \geq t_{N_k+1}, \]
and so by induction it follows that (2.4) holds. Since \( \{y_n\} \) tends to \( x^* \) as \( n \) tends to \( \infty \), it follows from (2.4) that \( x(t) \) also tends to \( x^* \) as \( t \) tends to \( \infty \). The proof is complete.

Now let us consider some special cases of Eq. (1.1). For the autonomous differential equation
\[ \dot{x}(t) = f(x(t - \tau_1), \ldots, x(t - \tau_m)), \quad t \geq t_0, \]  
(2.15)
where \( \tau_1, \ldots, \tau_m \) are nonnegative constants, the following result is a consequence of Theorem 1.

**Corollary 1.** Assume that \( x^* \) is the only equilibrium of Eq. (2.15) in \( (L, \infty) \). Suppose also that a one-sided limit
\[ f(L+) = \lim_{x \to L^+} f(x) \in \mathbb{R} \]
exists and satisfies the inequality
\[ x^* + \tau f(x^* + \tau f(L+)) > L \]
where \( \tau = \max(\tau_i; 1 \leq i \leq m) \). Finally, assume that the solution of the IVP
\[
\begin{align*}
y_{n+1} &= x^* + \tau f(y_n) \\
y_0 &= x^* + \tau f(L+) \end{align*}
\]  
(2.16)
tends to \( x^* \) as \( n \) tends to \( \infty \). Then every solution of Eq. (2.15) tends to \( x^* \) as \( t \) tends to \( \infty \).

**Proof.** Let \( F(u) = \tau f(u, \ldots, u) \). Then it is easy to see that all the hypotheses of Theorem 1 are satisfied and so the proof is complete.
The following result about differential equation with separable variables is also a direct consequence of Theorem 1.

**Corollary 2.** Consider the differential equation

\[ \dot{x}(t) = \sum_{i=1}^{m} p_i(t) f_i(x(t - \tau_i(t))), \quad t \geq t_0 \quad (2.17) \]

where for \( i = 1, 2, \ldots, m \), \( p_i \in C([0, \infty), [0, \infty]) \); \( f_i \in C([L, \infty), \mathbb{R}] \) for some \( L \geq -\infty \); and \( \tau_i \in C([0, \infty), [0, \infty]) \) with \( \lim_{t \to \infty} |t - \tau_i(t)| = \infty \).

Assume that for \( i = 1, 2, \ldots, m \),

\[ \sum_{i=1}^{m} \int_{0}^{\infty} p_i(t) \, dt = \infty \quad \text{and} \quad P_i = \sup \left\{ f_i^t |_{t - \tau_i(t)} \right\} \, ds : t - \tau(t) \geq 0 < \infty. \]

Suppose that for each \( i = 1, 2, \ldots, m \), \( f_i \) is nonincreasing and the one-sided limit

\[ f_i(L^+) = \lim_{x \to L^+} f_i(x) \]

exists. Suppose also that there exists a \( x^* > L \) such that

\[ x^* + \sum_{i=1}^{m} P_i f_i(x) + \sum_{i=1}^{m} P_i f_i(L^+) > L \]

and for \( i = 1, 2, \ldots, m \),

\[ f_i(x^*) = 0 \quad \text{and} \quad f_i(x) \neq 0 \quad \text{for} \ x \neq x^*. \]

Finally, assume that the solution of the IVP

\[
\begin{cases}
    y_{n+1} = x^* + \sum_{i=1}^{m} P_i f_i(y_n) \\
    y_0 = x^* + \sum_{i=1}^{m} P_i f_i(L^+) 
\end{cases}
\quad (2.18)
\]

tends to \( x^* \) as \( n \) tends to \( \infty \). Then every solution of Eq. (2.15) tends to \( x^* \) as \( t \) tends to \( \infty \).

**Proof.** Clearly, \( x^* \) is the only equilibrium of Eq. (2.17). Let \( f(t, u_1, \ldots, u_m) = \sum_{i=1}^{m} p_i(t) f_i(u_i) \) and \( F(u) = \sum_{i=1}^{m} P_i f_i(u) \). We see that all the hypotheses of Theorem 1 are satisfied and so the proof is complete.

### 3. APPLICATIONS

In this section, we apply our results to obtain sufficient conditions for global attractivity of some delay differential equations.
First we introduce some lemmas. The first lemma provides a sufficient condition for the solution of the form of IVP(2.3) to tend to the equilibrium.

**Lemma 1.** Consider the difference equation

$$y_{n+1} = h(y_n)$$

where

$$h \in C^1([l, \infty), (l, \infty)] \quad \text{with} \quad l \geq -\infty.$$ 

Assume that $$y^*$$ is the unique fixed point of $$h$$ and that $$h$$ is a nonincreasing function. Suppose also that the one-sided limit

$$h(l+) = \lim_{y \to l^+} h(y) \in (l, \infty)$$

exists and that

$$h'(y)h'(h(y)) < 1 \quad \text{either for } l < y < y^* \text{ or for } y^* < y < h(l+).$$

Then the solution $$(y_n)$$ of Eq. (3.1) with $$y_0 = h(l+)$$ tends to $$y^*$$ as $$n$$ tends to $$\infty$$.

**Proof.** We will assume that for $$y^* < y < h(l+)$$,

$$h'(y)h'(h(y)) < 1. \quad (3.2)$$

The proof when (3.2) holds for $$l < y < y^*$$ is similar and will be omitted.

From the nonincreasing nature of $$h$$, it is easy to see that $$y_2 \leq y_0$$ and by induction

$$y_1 \leq y_3 \leq \cdots \leq y^* \leq \cdots \leq y_2 \leq y_0.$$ 

Therefore

$$h(l+) \geq \lim_{n \to \infty} y_{2n} = u \geq y^* \quad \text{and} \quad l < \lim_{n \to \infty} y_{2n+1} = v \leq y^*.$$ 

By taking limits on both sides of Eq. (3.1), we find that

$$u = h(v) \quad \text{and} \quad v = h(u)$$

and so

$$u = h(h(u)) \quad \text{with} \quad u \geq y^*. \quad (3.3)$$

We claim that $$u = y^*$$. To this end, consider the function

$$\lambda(y) = y - h(h(y)) \quad \text{for} \quad y \geq y^*$$
and observe that
\[ \lambda(y^*) = 0 \text{ and } \lambda'(y) = 1 - h'(y)h'(h(y)) > 0 \quad \text{for } y > y^*. \]

Then
\[ y > h(h(y)) \quad \text{for } y > y^* \]

which, in view of (3.3), implies that \( u = y^* \). Then it follows that \( v = y^* \) also. Hence \( \{y_n\} \) tends to \( y^* \) as \( n \) tends to \( \infty \) and the proof is complete.

The next lemma which is extracted from [5] will also be useful.

**Lemma 2.** Consider the difference equation
\[ z_{n+1} = \exp \left( \frac{\alpha}{1 + \beta z_n} \right), \quad n = 0, 1, \ldots, \quad (3.4) \]

where
\[ \alpha \in (0, \infty), \beta \in [0, \infty), \text{ and } z_0 \in [0, \infty). \]

Then the equilibrium \( z^* = 1 \) of Eq. (3.4) is globally asymptotically stable if and only if
\[ \frac{\alpha}{1 + \beta} \leq 1. \]

Now let us look at Eq. (1.2). We will only consider the solutions of Eq. (1.2) with the following initial conditions
\[ \begin{cases} 
    x(t) = \phi(t) & \text{for } -\tau^* \leq t < 0, \text{ where} \\
    \tau^* = \sup_{t \geq 0} \tau(t) \leq \infty, \phi \in C\left([\tau^*, 0], [0, \infty]\right) \text{ and } \phi(0) > 0. \quad (3.5) 
\end{cases} \]

Observe that in the interval \([0, \tau']\), Eq. (1.2) reduces to a linear equation whose solution is given by
\[ x(t) = \phi(0) \exp \left( - \int_0^t r(s) \frac{\varphi^p(s - \tau(s))}{1 + \varphi^p(s - \tau(s))} ds \right) \]
\[ + \int_0^t r(s) \exp \left( -c \int_s^t \varphi^p(u - \tau(u)) \frac{\varphi^p(u - \tau(u))}{1 + \varphi^p(u - \tau(u))} du \right) ds. \]

We find that \( x(t) \) exists and is positive throughout the interval \( 0 \leq t \leq \tau' \).
One can see by induction that \( x(t) \) exists and is positive for all \( t \geq 0 \).
Clearly, Eq. (1.2) has a unique positive equilibrium $K$. Then by employing Theorem 1 and Lemma 1 we obtain the following global attractivity result.

**Theorem 2.** Assume that

$$\int_{0}^{\infty} r(t) \, dt = \infty. \quad (3.6)$$

Suppose also that

$$K + r_0 \left( 1 - c \frac{(r_0 + K)^{p+1}}{1 + (r_0 + K)^p} \right) > 0 \quad (3.7)$$

and

$$(cr_0)^2 \frac{K^p (1 + p + K^p)}{(1 + K^p)^2} \cdot \frac{(p + 1)^2}{4p} \leq 1 \quad (3.8)$$

where

$$r_0 = \sup \left\{ \int_{t-\tau(t)}^{t} r(s) \, ds : t - \tau(t) \geq 0 \right\}.$$

Then every positive solution of Eq. (1.2) tends to $K$ as $t$ tends to $\infty$.

**Proof.** Set

$$f(t, u_1, u_2) = r(t) \left( 1 - \frac{cu_1 u_2}{1 + u_2} \right), \quad F(u) = r_0 \left( 1 - c \frac{u^{p+1}}{1 + u^p} \right),$$

and $L = 0$. Clearly, if we show that the solution of the IVP

$$\begin{align*}
  y_{n+1} &= K + r_0 \left( 1 - c \frac{y_n^{p+1}}{1 + y_n^p} \right) \\
  y_0 &= K + r_0
\end{align*} \quad (3.9)$$

tends to the equilibrium $K$ as $n$ tends to $\infty$, then all the hypotheses of Theorem 1 will be satisfied and the proof will be complete.

To this end, set

$$h(u) = K + r_0 \left( 1 - c \frac{u^{p+1}}{1 + u^p} \right) \quad \text{for} \ u > 0.$$
Then, $h \in C^1((0, \infty), \mathbb{R})$ and
\[ \lim_{u \to 0^+} h(u) = K + r_0. \]
Observe that
\[ h'(u) = -cr_0 \frac{u^p(1 + u^p)}{(1 + u^p)^2} \]
and so
\[ h'(u)h'(h(u)) = (cr_0)^2 \frac{u^p(1 + u^p)F^p(u)(1 + F^p(u))}{(1 + u^p)^2(1 + F^p(u))^2}. \]
(3.10)

We claim that
\[ h'(u)h'(h(u)) < 1 \quad \text{for } 0 < u < K. \]  
(3.11)

Set
\[ g(s) = \frac{s^p(1 + s^p)}{(1 + s^p)^2} \quad \text{for } s \geq 0 \]
and observe that
\[ g'(s) = \frac{ps^{p-1}(1 + (1-p)s^p)}{(1 + s^p)^3} \quad \text{for } s \geq 0. \]

If $p \leq 1$, then $g'(s) > 0$ for $s > 0$. Hence for $0 < u < K$, \[ g(u) < g(K) = \frac{K(2 + K)}{(1 + K)^2} \]
and \[ g(F(u)) \leq g(\infty) = 1. \]
Then it follows from (3.8) and (3.10) that
\[ h'(u)h'(h(u)) < (cr_0)^2 \frac{K(2 + K)}{(1 + K)^2} \leq 1 \quad \text{for } 0 < u < K. \]

Now assume that $p > 1$. Then $g'(s) = 0$ for $s = (p + 1)/(p - 1)^{1/p}$. Observe that
\[ g(0) = 0, \quad g(\infty) = 1, \]
and \( g(s) \) is increasing for \( 0 < s < ((p + 1)/(p - 1))^{1/p} \) and decreasing for \( ((p + 1)/(p - 1))^{1/p} < s < \infty \). Then it follows that
\[
g(s) \leq g\left(\left(\frac{p + 1}{p - 1}\right)^{1/p}\right) = \frac{(p + 1)^2}{4p}.
\]

Also as \( u \leq K \leq F(u) \) and either
\[
K > \left(\frac{p + 1}{p - 1}\right)^{1/p} \quad \text{or} \quad K \leq \left(\frac{p + 1}{p - 1}\right)^{1/p},
\]
we find
\[
g(u)g(F(u)) \leq \frac{K^p(1 + p + K^p)}{(1 + K^p)^2} \frac{(p + 1)^2}{4p}.
\]

Thus it follows from (3.10), (3.12), and (3.8) that (3.11) holds. Hence by Lemma 1, the solution of (3.9) tends to \( K \) as \( n \) tends to \( \infty \). Therefore by Theorem 1, every positive solution of Eq. (1.2) tends to \( K \) as \( t \) tends to \( \infty \). The proof is complete.

Observe that
\[
\frac{K^p(1 + p + K^p)}{(1 + K^p)^2} \leq \frac{(p + 1)^2}{4p}
\]
and so the following result is an immediate consequence of Theorem 2.

**Corollary 3.** Assume that (3.6) and (3.7) hold and that
\[
cr_0 \cdot \frac{(p + 1)^2}{4p} \leq 1.
\]
Then every positive solution \( x(t) \) of Eq. (1.2) tends to \( K \) as \( t \) tends to \( \infty \).

For Eq. (1.4), we have the following result.

**Corollary 4.** Assume that \( K \) is the unique positive equilibrium of Eq. (1.4) and that
\[
K + \left( a - b \frac{(K + a\tau)^{p+1}}{1 + (K + a\tau)^p} \right) \tau > 0
\]
and

\[
(b\tau)^2 \frac{K^p (1 + p + K^p)}{(1 + K^p)^2} \frac{(p + 1)^2}{4p} \leq 1.
\]

Then every positive solution of Eq. (1.4) tends to \( K \) as \( t \) tends to \( \infty \); in particular, if (3.13) holds and

\[
(b\tau) \frac{(p + 1)^2}{4p} \leq 1
\]

then every positive solution of Eq. (1.4) tends to \( K \) as \( t \) tends to \( \infty \).

Next we will consider Eq. (1.5). We will only consider the solutions of Eq. (1.5) with the following initial conditions of the form

\[
\begin{cases}
N(t) = \phi(t) & \text{for } -\tau^* \leq t \leq 0 \\
\phi \in C([-\tau^*, 0], [0, \infty)) & \text{and } \phi(0) > 0.
\end{cases}
\]  

(3.14)

Since \( \tau_i(t) > 0 \) for \( i = 1, 2, \ldots, m \) and \( t \geq 0 \), it is not difficult to show as in [1, pp. 10–11] that the solutions of (1.5) and (3.14) are defined for all \( t \geq 0 \) and that they remain positive for \( t \geq 0 \).

Clearly, \( (K/\sum_{i=1}^m a_i)^{1/p} \) is the unique positive equilibrium of Eq. (1.5). Then by employing Theorem 1 and Lemma 2, we establish the following global attractivity result for Eq. (1.5).

**Theorem 3.** Assume that

\[
\int_0^{\infty} \frac{r(t)}{1 + \sum_{i=1}^m s_i(t)} dt = \infty
\]  

(3.15)

and

\[
\frac{pR_0}{1 + s/a} \leq 1
\]  

(3.16)

where

\[
a = \sum_{i=1}^m a_i, \quad s_0 = \inf \left\{ \sum_{i=1}^m s_i(t) \text{ for } t \geq 0 \right\}
\]

and

\[
R_0 = \sup \left\{ \int_{t-\tau(t)}^t r(s) ds \text{ for } t - \tau(t) \geq 0 \right\}.
\]
Then every positive solution \( N(t) \) of Eq. (1.5) tends to its positive equilibrium \( N^* = (K/\sum_{i=1}^{m} a_i)^{1/p} \) as \( t \) tends to \( \infty \).

**Proof.** Set

\[
N(t) = N^* e^{x(t)} \quad \text{for} \quad t \geq 0.
\]

Then \( x(t) \) satisfies the equation

\[
\dot{x}(t) = a^{-1}r(t) \left\{ \frac{\sum_{i=1}^{m} a_i (1 - e^{p(t - \tau_i(t))})}{1 + a^{-1} \sum_{i=1}^{m} s_i(t) e^{p(t - \tau_i(t))}} \right\}.
\]

It suffices to show that \( x(t) \) tends to zero. To this end, set

\[
f(t, u_1, \ldots, u_m) = a^{-1}r(t) \left\{ \frac{\sum_{i=1}^{m} a_i (1 - e^{p(t - \tau_i(t))})}{1 + a^{-1} \sum_{i=1}^{m} s_i(t) e^{p(t - \tau_i(t))}} \right\}
\]

and

\[
F(u) = R_0 \left\{ \frac{1 - e^{pu}}{1 + a^{-1}s_0 e^{pu}} \right\}.
\]

Then it is easy to see that all the hypotheses of Theorem 1 will be satisfied provided the solution of the IVP

\[
\begin{cases}
y_{n+1} = R_0 \left\{ \frac{1 - e^{p y_n}}{1 + a^{-1}s_0 e^{p y_n}} \right\} \\
y_0 = R_0
\end{cases}
\tag{3.17}
\]

tends to zero as \( n \) tends to \( \infty \). Set \( z_n = e^{p y_n} \). Then (3.17) becomes

\[
\begin{cases}
z_{n+1} = \exp \left\{ p R_0 \left( (1 - z_n)/(1 + a^{-1}s_0 z_n) \right) \right\} \\
z_0 = e^{p R_0}
\end{cases}
\tag{3.18}
\]

By Lemma 2 we see that under the hypothesis (3.16), \( \{z_n\} \) tends to 1 and so \( \{y_n\} \) tends to zero as \( n \) tends to \( \infty \). Therefore \( x(t) \) tends to zero which implies that \( N(t) \) tends to \( N^* \) as \( t \) tends to \( \infty \). The proof is complete.

When \( m = 1, a_i = 1, \) and \( \tau_i(t) \equiv \tau > 0 \) is a constant, Eq. (1.5) reduces to

\[
\dot{x}(t) = a^{-1}r(t)N(t) \left\{ \frac{K - N^p(t - \tau)}{K + s(t) N^p(t - \tau)} \right\}, \quad t \geq 0.
\tag{3.19}
Hence the following result, which has been established in [5], is an immediate consequence of Theorem 3.

**Corollary 5.** Assume that

\[ \int_0^\infty \frac{r(t)}{1 + s(t)} \, dt = \infty \quad \text{and} \quad \frac{pR_0}{1 + s_0} \leq 1 \]

where

\[ R_0 = \sup \left\{ \int_{t-\tau}^{t} r(s) \, ds : t - \tau \geq 0 \right\} \quad \text{and} \quad s_0 = \inf \{ s_1(t) : t \geq 0 \} \]

Then every positive solution \( N(t) \) of Eq. (3.19) tends to the positive equilibrium \( K^{1/p} \) as \( t \) tends to \( \infty \).

**Remark 1.** When \( \tau = 0 \), \( p = 1 \), \( r(t) \equiv r \), and \( s_i(t) \equiv s \) are constants, Eq. (3.19) was proposed by Smith [6] as an alternative to the logistic equation for a “food-limited” population. The time-delayed, food-limited model (3.19) has been studied by Gopalsamy *et al.* [3] and Grove *et al.* [5].

Now consider another special case of Eq. (1.5). When \( p = 1 \) and \( s_i(t) \equiv 0 \) for \( i = 1, 2, \ldots, m \), Eq. (1.5) reduces to

\[ \dot{N}(t) = r(t) N(t) \left( 1 - \frac{\sum_{i=1}^{m} a_i N(t - \tau_i)}{K} \right), \quad t \geq 0, \quad (3.20) \]

and so we have the following consequence of Theorem 3.

**Corollary 6.** Assume that

\[ \int_0^\infty r(t) \, dt = \infty \quad \text{and} \quad R_0 \leq 1 \quad (3.21) \]

where

\[ R_0 = \sup \left\{ \int_{t-\tau}^{t} r(s) \, ds \right\}. \]

Then every positive solution of Eq. (3.20) tends to the positive equilibrium \( k / \sum_{i=1}^{m} a_i \) as \( t \) tends to \( \infty \).

Equation (3.20) is well known as a nonautonomous delay logistic equation. The qualitative behavior of this equation has been studied extensively. It has been shown (see [2, p. 57]) that for the special case of Eq. (3.20),

\[ \dot{N}(t) = rN(t) \left( 1 - \frac{a_1 N(t - \tau_1) + a_2 N(t - \tau_2)}{K} \right), \quad t \geq 0, \quad (3.22) \]
where
\[ r, a_1, a_2, \tau_1, \tau_2 \in (0, \infty) \]

if
\[ r(\tau_1 + \tau_2) \exp[r(\tau_1 + \tau_2)] < 1, \quad (3.23) \]

then every positive solution of Eq. (3.22) tends to the positive equilibrium \( K/(a_1 + a_2) \) as \( t \) tends to \( \infty \). However, in this special case, our condition (3.21) reduces to
\[ r \max\{\tau_1, \tau_2\} \leq 1 \]

which clearly is an improvement over (3.23).

For the special case of Eq. (3.20),
\[ \dot{N}(t) = rN(t)\left(1 - \frac{N(t - \tau(t))}{K}\right), \quad (3.24) \]

it has been shown (see [2, p. 65]) that if
\[ r\tau_0 e^{\tau_0 e^{r\tau_0}} < 1 \quad (3.25) \]

where \( \tau_0 = \sup(\tau(t): \tau \geq 0) \), then every positive solution of Eq. (3.24) tends to \( K \) as \( t \) tends to \( \infty \). However, in this special case, our condition (3.21) reduces to
\[ r\tau_0 \leq 1 \]

which is an improvement of (3.25).

Finally, we will apply our results to the delay differential equation
\[ \dot{y}(t) = -r(t)y(t - \tau(t))\left[1 - y^2(t)\right], \quad t \geq t_0, \quad (3.26) \]

where
\[ r, \tau \in C\left[[t_0, \infty), (0, \infty)\right] \quad \text{and} \quad \lim_{t \to \infty} [t - \tau(t)] = \infty, \]

with an initial condition of the form
\[
\begin{cases}
  y(t) = \phi(t) & \text{for} \ -\tau^* \leq t \leq 0 \text{ where} \\
  \phi \in C\left[[-\tau^*, 0], \mathbb{R}\right] & \text{and} \ |\phi(0)| < 1.
\end{cases} \quad (3.27)
\]
It is not difficult to show that the solution of (3.26) and (3.27) exists for all \( t \geq 0 \) and satisfies the condition \(-1 < y(t) < 1\). Set

\[
y(t) = \frac{e^{2x(t)} - 1}{e^{2x(t)} + 1}, \quad t \geq 0.
\]

Then Eq. (3.26) reduces to

\[
\dot{x}(t) = r(t) \left\{ \frac{1 - e^{2x(t-\tau(t))}}{1 + e^{2x(t-\tau(t))}} \right\}, \quad t \geq 0, \tag{3.28}
\]

and so we have the following result.

**Theorem 4.** Assume that

\[
\int_0^\infty r(t) \, dt = \infty \quad \text{and} \quad R_0 \leq 1 \tag{3.29}
\]

where

\[
R_0 = \sup \left\{ \int_{t-\tau(t)}^t r(s) \, ds : t - \tau(t) \geq 0 \right\}.
\]

Then every solution of (3.26) and (3.27) tends to zero as \( n \) tends to \( \infty \).

**Proof.** Set

\[
f(t, u) = r(t) \left\{ \frac{1 - e^{2u}}{1 + e^{2u}} \right\} \quad \text{and} \quad F(u) = R_0 \left\{ \frac{1 - e^{2u}}{1 + e^{2u}} \right\}
\]

Then by an argument similar to that in Theorem 3, we see that under the hypothesis (3.29) the solution of the IVP

\[
\begin{align*}
y_{n+1} &= R_0 \left\{ \frac{1 - e^{2y_n}}{1 + e^{2y_n}} \right\} \\
y_0 &= R_0
\end{align*}
\]

(3.30)

tends to zero. Hence it is easy to see that all the hypotheses of Theorem 1 are satisfied and so every solution of (3.28) tends to zero. Therefore, every solution of (3.26) and (3.27) tends to zero as \( t \) tends to \( \infty \). The proof is complete.
REFERENCES


