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# Note

# On Cayley's Formula for Counting Forests

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In 1889, A. Cayley stated that the number of forests with $n$ labeled vertices that consist of $s$ distinct trees such that $s$ specified vertices belong to distinct trees is $sn^{n-s-1}$ for $1 \leqslant s \leqslant n$. In this paper Cayley's formula is proved in a very simple way. © 1990 Academic Press, Inc.

## 1. INTRODUCTION

A forest is a simple graph that has no cycles. In other words, a forest is a simple graph, all of whose components are trees.

Denote by $F(n, s)$, $1 \leqslant s \leqslant n$, the number of forests having vertex set $\{1, 2, ..., n\}$ and $s$ components which are trees such that vertices $1, 2, ..., s$ all belong to different trees. We have

$$F(n, s) = s\, n^{n-s-1} \tag{1}$$

for $1 \leqslant s \leqslant n$. In particular,

$$F(n, 1) = n^{n-2} \tag{2}$$

is the number of distinct trees with $n$ labeled vertices.

In 1889, A. Cayley [2] discovered formula (2). Since then various proofs have been given for (2) by O. Dziobek [4], H. Prüfer [10], G. Bol [1], L. E. Clarke [3], J. W. Moon [6, 7], A. Rényi [11, 12], and others.

A. Cayley [2] also stated formula (1), but he did not indicate how to prove it. In 1959 A. Rényi [11] gave an analytic proof for (1). For other proofs of (1) we refer to J. W. Moon [8, 9], J. Riordan [13], V. F. Kolchin [5], and V. N. Sachkov [14]. All these proofs are analytic in nature and presume that the particular case (2) has already been proved. In what follows we shall give an elementary proof of (1) which does not presume (2).



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## 2. PROOF OF CAYLEY'S FORMULA

The proof of (1) is based on the following recurrence formula. If $n > 1$ and $1 \leqslant s \leqslant n$, then

$$F(n, s) = \sum_{j=0}^{n-s} \binom{n-s}{j} F(n-1, s+j-1), \tag{3}$$

where $F(1, 1) = 1$ and $F(n, 0) = 0$ for $n \geqslant 1$. To prove (3) let us consider a forest having vertex set $\{1, 2, ..., n\}$ and $s$ components which are trees such that vertices $1, 2, ..., s$ all belong to different trees. In this forest vertex 1 may have degree $j = 0, 1, ..., n-s$; that is, vertex 1 may be connected to $j$ vertices in the set of vertices $\{s+1, ..., n\}$. Let us call these $j$ vertices secondary vertices. We can choose $j$ secondary vertices among the $n-s$ vertices in $\binom{n-s}{j}$ ways. Now let us remove vertex 1 and all the $j$ edges emanating from vertex 1. Then the remaining graph becomes a forest having vertex set $\{2, 3, ..., n\}$ and consisting of $s+j-1$ components which are trees such that vertices $2, 3, ..., s$ and the $j$ secondary vertices all belong to different trees. The number of such forests is $F(n-1, s+j-1)$. If we add $\binom{n-s}{j} F(n-1, s+j-1)$ for all possible values of $j$ we obtain $F(n, s)$. This proves (3).

By using (3) we can prove (1) by mathematical induction. If $n = 1$, then (1) is evidently true. Let us suppose that $F(n-1, i) = i(n-1)^{n-i-2}$ for $1 \leqslant i \leqslant n-1$ and $n > 1$. Then by (3),

$$F(n, s) = \sum_{j=0}^{n-s} \binom{n-s}{j} (s+j-1)(n-1)^{n-s-j-1} = s\, n^{n-s-1} \tag{4}$$

for $1 \leqslant s \leqslant n$ and $n > 1$. In (4) we write $j\binom{n-s}{j} = (n-s)\binom{n-s-1}{j-1}$ for $j \geqslant 1$ and apply the binomial theorem. This completes the proof of (1).

## REFERENCES

1. G. BOL, Über eine kombinatorische Frage, *Abh. Math. Sem. Hansischen Univ.* **12** (1938), 242–245.
2. A. CAYLEY, A theorem on trees, *Quart. J. Pure Appl. Math.* **23** (1889), 376–378. (The Collected Mathematical Papers of Arthur Cayley, Vol. XIII, pp. 26–28, Cambridge Univ. Press, Cambridge, 1897.
3. L. E. CLARKE, On Cayley's formula for counting trees, *J. London Math. Soc.* **33** (1958), 471–474.
4. O. DZIOBEK, Eine Formel der Substitutionstheorie, *Sitzungsber. Berlin. Math. Gesell.* **17** (1917), 64–67.
5. V. F. KOLCHIN, "Random Mappings," pp. 150–151, Optimization Software, Inc., New York, 1986.

6. J. W. MOON, Various proofs of Cayley's formula for counting trees, *in* "A Seminar on Graph Theory" (F. Harary and L. Beineke, Eds.), pp. 70–78, Holt, Rinehart & Winston, New York, 1967.
7. J. W. MOON, Another proof of Cayley's formula for counting trees, *Amer. Math. Monthly* **70** (1968), 846–847.
8. J. W. MOON, A tree counting problem, *Ann. Math. Statist.* **39** (1968), 242–245.
9. J. W. MOON, "Counting Labelled Trees," pp. 26–27, Canad. Math. Congr. Ottawa, 1970.
10. H. PRÜFER, Neuer Beweis eines Satzes über Permutationen, *Arch. Math. Phys.* **27** (1918), 142–144.
11. A. RÉNYI, Some remarks on the theory of trees, *Publ. Math. Inst. Hungar. Acad. Sci.* **4** (1959), 73–85.
12. A. RÉNYI, On the enumeration of trees, *in* "Combinatorial Structures and Their Applications" (R. Guy, H. Hanani, N. Sauer, and J. Schonheim, Eds.), pp. 355–360, Gordon & Breach, New York, 1970.
13. J. RIORDAN, Forests of labeled trees, *J. Combin. Theory* **5** (1968), 90–103.
14. V. N. SACHKOV, "Combinatorial Methods of Discrete Mathematics," pp. 181–183, "Nauka," Moscow, 1977. [Russian]