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Finiteness of graded local cohomology modules

Reza Sazeedeh*

Department of Mathematics, Urmia University, Urmia, Iran
Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, Iran

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Abstract

Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$ and irrelevant ideal R_+, let M be a finitely generated graded R-module. In this paper we show that $H^1_{\mathfrak{m}_0R}(H^1_{R_+}(M))$ is Artinian and $H^i_{\mathfrak{m}_0R}(H^1_{R_+}(M))$ is Artinian for each i in the case where R_+ is principal. Moreover, for the case where $\operatorname{ara}(R_+) = 2$, we prove that, for each $i \in \mathbb{N}_0$, $H^i_{\mathfrak{m}_0R}(H^2_{R_+}(M))$ is Artinian if and only if $H^{i+2}_{\mathfrak{m}_0R}(H^1_{R_+}(M))$ is Artinian. We also prove that $H^d_{\mathfrak{m}_0}(H^c_{R_+}(M))$ is Artinian, where $d = \dim(R_0)$ and c is the cohomological dimension of M with respect to R_+. Finally we present some examples which show that $H^2_{\mathfrak{m}_0R}(H^1_{R_+}(M))$ and $H^3_{\mathfrak{m}_0R}(H^1_{R_+}(M))$ need not be Artinian.

MSC: Primary: 13D45; 13E10

1. Introduction

Throughout this paper, let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$. So R_0 is a Noetherian ring and there are finitely many elements $l_1, \ldots, l_r \in R_1$ such that $R = R_0[l_1, \ldots, l_r]$. Let $R_+ := \bigoplus_{n\in\mathbb{N}} R_n$ denote the irrelevant ideal of R and let $\mathfrak{m} := \mathfrak{m}_0 \oplus R_+$ denote the graded maximal ideal of R. Moreover, let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R-module.

The finiteness of the graded local cohomology module has an important role in studying the asymptotic behaviour of the n-th graded component $H^i_{R_+}(M)_n$ of $H^i_{R_+}(M)$ for $n \to -\infty$. One of the parts of finiteness of graded local cohomology is Artinianess. As $H^i_{R_+}(M)_n$ is a finitely generated R_0-module for each $n \in \mathbb{Z}$, the Artinianess of the graded local cohomology allows us to draw conclusions on the multiplicity $e_{\mathfrak{q}_0}(H^i_{R_+}(M)_n)$ of $H^i_{R_+}(M)$, where $\mathfrak{q}_0$ is an $\mathfrak{m}_0$-primary ideal of R_0.

Brodmann, Fumasoli and Tajarod in [2] proved that for each $i \in \mathbb{N}_0$, the graded modules $\Gamma_{\mathfrak{m}_0R}(H^i_{R_+}(M))$ and $H^1_{\mathfrak{m}_0R}(H^i_{R_+}(M))$ are Artinian when R_0 is of dimension 1. Moreover Brodmann, Rohrer and Sazeedeh in [3] showed

* Corresponding address: Department of Mathematics, Urmia University, Urmia, Iran.
E-mail address: rsazeedeh@ipm.ir.

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that $H^1_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ is Artinian for each $i \in \mathbb{N}_0$ even if $\dim(R_0) = 2$. The authors in [2,3] have presented several examples which showed that these results need not hold if we remove the condition on the dimension of R_0. Also, the authors in [3,7] studied the Artinianess of $\Gamma_{\mathfrak{m}_0 R}(H^f_{R_+}(M))$ where f was the least non-negative integer i such that $H^i_{R_+}(M)$ is not finitely generated or is not R_+-cofinite.

In this paper we show that, without any condition on R_0, the module $H^1_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is Artinian (cf. Theorem 2.2). In addition, if R_+ is principal, then $H^i_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is Artinian for each i (cf. Proposition 2.6). When $\operatorname{ara}(R_+) = 2$, we also show that, for each $i \in \mathbb{N}_0$, $H^{i+2}_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is Artinian if and only if $H^i_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ is Artinian (cf. Theorem 2.3). This result allows us to draw a conclusion on $\Gamma_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$. In fact we deduce that $\Gamma_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ is Artinian if and only if $H^2_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is Artinian. Finally we shall obtain a result on top local cohomology. We show that if $\dim(R_0) = d$ and c is the cohomological dimension of M with respect to R_+, then $H^d_{\mathfrak{m}_0 R}(H^c_{R_+}(M))$ is Artinian (cf. Proposition 2.8). In the last part we present two examples which show that $H^2_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ and $H^3_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ need not be Artinian even if $\operatorname{ara}(R_+) = 2$ (cf. Examples 2.9 and 2.10).

2. The results

Definition 2.1. Let $\mathfrak{a}$ be an ideal of R. Following [1], the *arithmetic rank of* $\mathfrak{a}$, denoted by $\operatorname{ara}(\mathfrak{a})$, is the least number of elements of R required to generate an ideal which has the same radical as $\mathfrak{a}$. Thus

$$\operatorname{ara}(\mathfrak{a}) = \min\{n \in \mathbb{N}_0 : \exists\, b_1, \ldots, b_n \in R \text{ with } \sqrt{(b_1, \ldots, b_n)R} = \sqrt{\mathfrak{a}}\}.$$

Note that $\operatorname{ara}(0R) = 0$.

Theorem 2.2. *The graded module $H^1_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is Artinian.*

Proof. Let $\operatorname{ara}(R_+) = n$. Then there exist some elements $y_1, \ldots, y_n \in R$ such that $\sqrt{(y_1, \ldots, y_n)R} = \sqrt{R_+}$. As there exists an isomorphism $H^i_{R_+}(M) \cong H^i_{(y_1,\ldots,y_n)R}(M)$ for each $i \in \mathbb{N}_0$, we may assume that $R_+ = (y_1, \ldots, y_n)R$. Moreover, for each $i \in \mathbb{N}_0$, there is an exact sequence of graded R-modules

$$H^{i-1}_{(y_1,\ldots,y_{n-1})R}(M) \xrightarrow{\eta^{i-1}_{y_n}} H^{i-1}_{(y_1,\ldots,y_{n-1})R}(M)_{y_n} \longrightarrow H^i_{R_+}(M) \longrightarrow H^i_{(y_1,\ldots,y_{n-1})R}(M) \xrightarrow{\eta^i_{y_n}} H^i_{(y_1,\ldots,y_{n-1})R}(M)_{y_n}$$

in which $\eta^{i-1}_{y_n}$ and $\eta^i_{y_n}$ are the natural homomorphisms (cf. [1, Exercise 13.1.12] or [2, Theorem 2.5]). Now it follows from [1, Corollary 2.2.18] that $\operatorname{Coker}(\eta^{i-1}_{y_n}) \cong H^1_{y_n R}(H^{i-1}_{(y_1,\ldots,y_{n-1})R}(M))$ and $\operatorname{Ker}(\eta^i_{y_n}) \cong \Gamma_{y_n R}(H^i_{(y_1,\ldots,y_{n-1})R}(M))$ and so we obtain an exact sequence

$$0 \to H^1_{y_n R}(H^{i-1}_{(y_1,\ldots,y_{n-1})R}(M)) \to H^i_{R_+}(M) \to \Gamma_{y_n R}(H^i_{(y_1,\ldots,y_{n-1})R}(M)) \to 0.$$

In particular, if we set $i = 1$, we will have the following exact sequence of graded R-modules:

$$0 \to H^1_{y_n R}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M)) \to H^1_{R_+}(M) \to \Gamma_{y_n R}(H^1_{(y_1,\ldots,y_{n-1})R}(M)) \to 0.$$

Application of the functor $H^1_{\mathfrak{m}_0 R}(-)$ to this exact sequence gives the following exact sequence of R-modules:

$$H^1_{\mathfrak{m}_0 R}(H^1_{y_n R}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M))) \to H^1_{\mathfrak{m}_0 R}(H^1_{R_+}(M)) \to H^1_{\mathfrak{m}_0 R}(\Gamma_{y_n R}(H^1_{(y_1,\ldots,y_{n-1})R}(M))).$$

So, in order to get our assertion, it is enough to show that $H^1_{\mathfrak{m}_0 R}(H^1_{y_n R}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M)))$ and $H^1_{\mathfrak{m}_0 R}(\Gamma_{y_n R}(H^1_{(y_1,\ldots,y_{n-1})R}(M)))$ are Artinian. We now prove Artinianess of the first term. By Grothendieck's spectral sequence, for each $p, q \in \mathbb{N}_0$, we have

$$E_2^{p,q} := H^p_{\mathfrak{m}_0 R}(H^q_{y_n R}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M))) \underset{p}{\Rightarrow} H^{p+q}_{(\mathfrak{m}_0, y_n)R}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M)).$$

Since $E_2^{p,q} = 0$, for all $q \neq 0, 1$, the above spectral sequence gives the following long exact sequence of R-modules:

$$\cdots \to H^2_{(\mathfrak{m}_0, y_n)R}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M)) \to E_2^{1,1} \to E_2^{3,0} \to H^3_{(\mathfrak{m}_0, y_n)R}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M)) \to \cdots.$$

We note that there are the following isomorphisms $H^2_{(\mathfrak{m}_0, y_n)R}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M)) \cong H^2_{\mathfrak{m}}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M))$ and

$E_2^{3,0} = H^3_{\mathfrak{m}_0 R}(\Gamma_{y_n R}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M))) \cong H^3_{\mathfrak{m}}(\Gamma_{R_+}(M))$ and so $H^2_{(\mathfrak{m}_0, y_n)R}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M))$ and $E_2^{3,0}$ are Artinian. Therefore, the previous exact sequence implies that $E_2^{1,1} = H^1_{\mathfrak{m}_0 R}(H^1_{y_n R}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M)))$ is Artinian. It remains to prove the Artinianess of $H^1_{\mathfrak{m}_0 R}(\Gamma_{y_n R}(H^1_{(y_1,\ldots,y_{n-1})R}(M)))$. According to [5, Corollary 11.44], there exists an exact sequence of R-modules

$$0 \to H^1_{\mathfrak{m}_0 R}(\Gamma_{y_n R}(H^1_{(y_1,\ldots,y_{n-1})R}(M))) \to H^1_{(\mathfrak{m}_0, y_n)R}(H^1_{(y_1,\ldots,y_{n-1})R}(M)) \to \cdots.$$

Thus it is enough to show that $H^1_{(\mathfrak{m}_0, y_n)R}(H^1_{(y_1,\ldots,y_{n-1})R}(M))$ is Artinian. Consider the following exact sequence of R-modules:

$$0 \to H^1_{y_{n-1}R}(\Gamma_{(y_1,\ldots,y_{n-2})R}(M)) \to H^1_{(y_1,\ldots,y_{n-1})R}(M) \to \Gamma_{y_{n-1}R}(H^1_{(y_1,\ldots,y_{n-2})R}(M)) \to 0.$$

Application of the functor $H^1_{(\mathfrak{m}_0, y_n)R}(-)$ to this exact sequence induces the following exact sequence:

$$H^1_{(\mathfrak{m}_0, y_n)R}(H^1_{y_{n-1}R}(\Gamma_{(y_1,\ldots,y_{n-2})R}(M))) \to H^1_{(\mathfrak{m}_0, y_n)R}(H^1_{(y_1,\ldots,y_{n-1})R}(M)) \\ \to H^1_{(\mathfrak{m}_0, y_n)R}(\Gamma_{y_{n-1}R}(H^1_{(y_1,\ldots,y_{n-2})R}(M))).$$

Now, in view of the above exact sequence, it suffices to prove that the graded modules $H^1_{(\mathfrak{m}_0, y_n)R}(H^1_{y_{n-1}R}(\Gamma_{(y_1,\ldots,y_{n-2})R}(M)))$ and $H^1_{(\mathfrak{m}_0, y_n)R}(\Gamma_{y_{n-1}R}(H^1_{(y_1,\ldots,y_{n-2})R}(M)))$ are Artinian. By a proof similar to that mentioned for $H^1_{\mathfrak{m}_0 R}(H^1_{y_n R}(\Gamma_{(y_1,\ldots,y_{n-1})R}(M)))$, we can conclude that $H^1_{(\mathfrak{m}_0, y_n)R}(H^1_{y_{n-1}R}(\Gamma_{(y_1,\ldots,y_{n-2})R}(M)))$ is Artinian. Now, we prove Artinianess of the second term. Using again [5, Corollary 11.44], there exists an exact sequence of R-modules

$$0 \to H^1_{(\mathfrak{m}_0, y_n)R}(\Gamma_{y_{n-1}R}(H^1_{(y_1,\ldots,y_{n-2})R}(M))) \to H^1_{(\mathfrak{m}_0, y_n, y_{n-1})R}(H^1_{(y_1,\ldots,y_{n-2})R}(M)) \to \cdots.$$

So it is enough to show that $H^1_{(\mathfrak{m}_0, y_n, y_{n-1})R}(H^1_{(y_1,\ldots,y_{n-2})R}(M))$ is Artinian. If we apply the above argument which was mentioned for $H^1_{(\mathfrak{m}_0, y_n)R}(H^1_{(y_1,\ldots,y_{n-1})R}(M))$, we shall deduce that $H^1_{(\mathfrak{m}_0, y_n, y_{n-1})R}(H^1_{(y_1,\ldots,y_{n-2})R}(M))$ is Artinian if $H^1_{(\mathfrak{m}_0, y_n, y_{n-1}, y_{n-2})R}(H^1_{(y_1,\ldots,y_{n-3})R}(M))$ is Artinian. Now, by repeating this argument, we finally conclude that the graded R-module $H^1_{(\mathfrak{m}_0, y_n, y_{n-1})R}(H^1_{(y_1,\ldots,y_{n-2})R}(M))$ is Artinian if $H^1_{(\mathfrak{m}_0, y_n, y_{n-1},\ldots,y_2)R}(H^1_{y_1 R}(M))$ is Artinian. Therefore it suffices to prove that $H^1_{(\mathfrak{m}_0, y_n, y_{n-1},\ldots,y_2)R}(H^1_{y_1 R}(M))$ is Artinian. By Grothendieck's spectral sequence, for each $p, q \in \mathbb{N}_0$, we have

$$E_2^{p,q} := H^p_{(\mathfrak{m}_0, y_n, y_{n-1},\ldots,y_2)R}(H^q_{y_1 R}(M)) \underset{p}{\Rightarrow} H^{p+q}_{\mathfrak{m}}(M).$$

Since $E_2^{p,q} = 0$ for all $q \neq 0, 1$, the above spectral sequence gives the following long exact sequence of R-modules:

$$\cdots \to H^2_{\mathfrak{m}}(M) \to E_2^{1,1} \to E_2^{3,0} \to \cdots.$$

We note that $H^2_{\mathfrak{m}}(M)$ and $E_2^{3,0} = H^3_{(\mathfrak{m}_0, y_n, y_{n-1},\ldots,y_2)R}(\Gamma_{y_1 R}(M)) \cong H^3_{\mathfrak{m}}(\Gamma_{y_1 R}(M))$ are Artinian; and hence the previous long exact sequence implies that the graded R-module $E_2^{1,1} = H^1_{(\mathfrak{m}_0, y_n, y_{n-1},\ldots,y_2)R}(H^1_{y_1 R}(M))$ is Artinian. □

Theorem 2.3. *Let* $\operatorname{ara}(R_+) = 2$ *and* $i \in \mathbb{N}_0$. *Then* $H^i_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ *is Artinian if and only if* $H^{i+2}_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ *is Artinian.*

Proof. As $\operatorname{ara}(R_+) = 2$, there exist $x, y \in R$ such that $\sqrt{R_+} = \sqrt{(x, y)R}$. So we have the following exact sequence:

$$0 \to H^1_{yR}(H^1_{xR}(M)) \to H^2_{R_+}(M) \to \Gamma_{yR}(H^2_{xR}(M)) \to 0.$$

We note that $H^2_{xR}(M) = 0$ and so the exact sequence above implies the isomorphism $H^2_{R_+}(M) \cong H^1_{yR}(H^1_{xR}(M))$. At first assume that $H^i_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ is Artinian for some $i \in \mathbb{N}_0$ and so we shall prove that $H^{i+2}_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is Artinian. Consider the following exact sequence:

$$0 \to H^1_{yR}(\Gamma_{xR}(M)) \to H^1_{R_+}(M) \to \Gamma_{yR}(H^1_{xR}(M)) \to 0.$$

Application of the functor $H^{i+2}_{\mathfrak{m}_0 R}(-)$ to the above exact sequence induces the following exact sequence:

$$H^{i+2}_{\mathfrak{m}_0 R}(H^1_{yR}(\Gamma_{xR}(M))) \to H^{i+2}_{\mathfrak{m}_0 R}(H^1_{R_+}(M)) \to H^{i+2}_{\mathfrak{m}_0 R}(\Gamma_{yR}(H^1_{xR}(M))) \to H^{i+3}_{\mathfrak{m}_0 R}(H^1_{yR}(\Gamma_{xR}(M))). \qquad (\star)$$

Thus, it is enough to show that $H^{i+2}_{\mathfrak{m}_0 R}(H^1_{yR}(\Gamma_{xR}(M)))$ and $H^{i+2}_{\mathfrak{m}_0 R}(\Gamma_{yR}(H^1_{xR}(M)))$ are Artinian. We first prove that $H^{i+2}_{\mathfrak{m}_0 R}(H^1_{yR}(\Gamma_{xR}(M)))$ is Artinian. By Grothendieck's spectral sequence we have

$$E_2^{p,q} := H^p_{\mathfrak{m}_0 R}(H^q_{yR}(\Gamma_{xR}(M))) \underset{p}{\Rightarrow} H^{p+q}_{(\mathfrak{m}_0,y)R}(\Gamma_{xR}(M)).$$

Since $E_2^{p,q} = 0$ for all $q \neq 0, 1$, this spectral sequence gives the following long exact sequence:

$$\cdots \to H^{i+3}_{(\mathfrak{m}_0,y)R}(\Gamma_{xR}(M)) \to E_2^{i+2,1} \to E_2^{i+4,0} \to H^{i+4}_{(\mathfrak{m}_0,y)R}(\Gamma_{xR}(M)).$$

We note that $H^{i+3}_{(\mathfrak{m}_0,y)R}(\Gamma_{xR}(M)) \cong H^{i+3}_{\mathfrak{m}}(\Gamma_{xR}(M))$ and $E_2^{i+4,0} = H^{i+4}_{\mathfrak{m}_0 R}(\Gamma_{yR}(\Gamma_{xR}(M))) \cong H^{i+4}_{\mathfrak{m}_0 R}(\Gamma_{R_+}(M)) \cong H^{i+4}_{\mathfrak{m}}(\Gamma_{R_+}(M))$ are Artinian and so is $E_2^{i+2,1} = H^{i+2}_{\mathfrak{m}_0 R}(H^1_{yR}(\Gamma_{xR}(M)))$. Now, we prove that $H^{i+2}_{\mathfrak{m}_0 R}(\Gamma_{yR}(H^1_{xR}(M)))$ is Artinian. Using again Grothendieck's spectral sequence, there is a spectral sequence

$$E_2^{p,q} := H^p_{\mathfrak{m}_0 R}(H^q_{yR}(H^1_{xR}(M))) \underset{p}{\Rightarrow} H^{p+q}_{(\mathfrak{m}_0,y)R}(H^1_{xR}(M)).$$

Since $E_2^{p,q} = 0$ for all $q \neq 0, 1$, this spectral sequence gives the following long exact sequence:

$$\cdots \to H^{i+1}_{(\mathfrak{m}_0,y)R}(H^1_{xR}(M)) \to H^i_{\mathfrak{m}_0 R}(H^1_{yR}(H^1_{xR}(M)))$$
$$\to H^{i+2}_{\mathfrak{m}_0 R}(\Gamma_{yR}(H^1_{xR}(M))) \to H^{i+2}_{(\mathfrak{m}_0,y)R}(H^1_{xR}(M)) \to \cdots. \qquad (\star\star)$$

By the hypotheses and the first argument in the our proof, the graded R-module $H^i_{\mathfrak{m}_0 R}(H^1_{yR}(H^1_{xR}(M))) \cong H^i_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ is Artinian. On the other hand, consider Grothendieck's spectral sequence

$$E_2^{p,q} := H^p_{(\mathfrak{m}_0,y)}(H^q_{xR}(M)) \underset{p}{\Rightarrow} H^{p+q}_{\mathfrak{m}}(M).$$

Since $E_2^{p,q} = 0$ for all $q \neq 0, 1$, this spectral sequence gives the long exact sequence of R-modules

$$\cdots \to H^{i+3}_{\mathfrak{m}}(M) \to E_2^{i+2,1} \to E_2^{i+4,0} \to \cdots.$$

We note that $H^{i+3}_{\mathfrak{m}}(M)$ and $E_2^{i+4,0} = H^{i+4}_{(\mathfrak{m}_0,y)}(\Gamma_{xR}(M)) = H^{i+4}_{\mathfrak{m}}(\Gamma_{xR}(M))$ are Artinian and so by the exact sequence above $E_2^{i+2,1} = H^{i+2}_{(\mathfrak{m}_0,y)R}(H^1_{xR}(M))$ is Artinian. Therefore, in view of $(\star\star)$, the module $H^{i+2}_{\mathfrak{m}_0 R}(\Gamma_{yR}(H^1_{xR}(M)))$ is Artinian. Conversely, suppose that $H^{i+2}_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is Artinian and so we shall prove that $H^i_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ is Artinian. By $(\star\star)$ and the first argument mentioned in this proof, it is enough to show that $H^{i+1}_{(\mathfrak{m}_0,y)R}(H^1_{xR}(M))$ and $H^{i+2}_{\mathfrak{m}_0 R}(\Gamma_{yR}(H^1_{xR}(M)))$ is Artinian. A proof similar to that stated for the Artinianess of $H^{i+2}_{(\mathfrak{m}_0,y)R}(H^1_{xR}(M))$ implies that the first term is Artinian. For the second term, by $(\star)$ and the hypotheses, it suffices to show that $H^{i+3}_{\mathfrak{m}_0 R}(H^1_{yR}(\Gamma_{xR}(M)))$ is Artinian. Now, by using again a proof similar to that stated for Artinianess of $H^{i+2}_{(\mathfrak{m}_0,y)R}(H^1_{xR}(M))$, we can show that $H^{i+3}_{\mathfrak{m}_0 R}(H^1_{yR}(\Gamma_{xR}(M)))$ is Artinian too. □

Corollary 2.4. *Let* $\operatorname{ara}(R_+) = 2$. *Then* $\Gamma_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ *is Artinian if and only if* $H^2_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ *is Artinian.*

Proof. The result follows immediately by the previous theorem if we consider $i = 0$. □

Remark 2.5. Any local flat morphism of local Noetherian rings is faithfully flat. So, if R'_0 is flat over R_0 and $\mathfrak{m}_0 R'_0 \subseteq \mathfrak{m}'_0$, then R'_0 is faithfully flat over R_0. Moreover, it follows from [4, Theorem 1] that if $(R'_0, \mathfrak{m}'_0)$ is a faithfully flat local R_0-algebra, then A is a graded Artinian R-module if and only if $A' := R'_0 \otimes_{R_0} A$ is a graded Artinian module over $R' := R'_0 \otimes_{R_0} R$.

Proposition 2.6. *Let* R_+ *be a principal graded ideal of* R. *Then* $H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M))$ *is Artinian for all* $i, j \in \mathbb{N}_0$.

Proof. As R_+ is principal, we have $H^j_{R_+}(M) = 0$ for each $j > 1$. So the assertion is obvious for each $j > 1$. If $j = 0$, then $\Gamma_{R_+}(M)$ is a finitely generated graded R_+-torsion R-module, and hence for each i, there is an isomorphism $H^i_{\mathfrak{m}_0 R}(\Gamma_{R_+}(M)) \cong H^i_{\mathfrak{m}}(\Gamma_{R_+}(M))$. We note that the last term is Artinian and so the result is clear in this case. Now, consider $j = 1$. Let $\mathbf{x}$ be an indeterminate and let $R'_0 := R_0[\mathbf{x}]_{\mathfrak{m}_0 R_0[\mathbf{x}]}$, $\mathfrak{m}'_0 := \mathfrak{m}_0 R'_0$, $R' = R'_0 \otimes_{R_0} R$ and $M' := R'_0 \otimes_{R_0} M$. Then by the flat base change property of local cohomology, for each $i \in \mathbb{N}_0$ we have $R'_0 \otimes_{R_0} H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M)) \cong H^i_{\mathfrak{m}'_0 R}(H^j_{R'_+}(M'))$. So, in view of Remark 2.5, we can assume that the residue field $R_0/\mathfrak{m}_0$ is infinite. Now, consider $j = 1$. Since there is an isomorphism $H^1_{R_+}(M) \cong H^1_{R_+}(M/\Gamma_{R_+}(M))$, we may assume that $\Gamma_{R_+}(M) = 0$. Thus there exists an element $x \in R_1$ which is a non-zero-divisor with respect to M and so there is the following exact sequence $0 \to M(-1) \xrightarrow{x.} M \to M/xM \to 0$. Application of the functor $H^j_{R_+}(-)$ to this exact sequence induces the following exact sequence:

$$0 \to \Gamma_{R_+}(M/xM) \to H^1_{R_+}(M)(-1) \xrightarrow{x.} H^1_{R_+}(M) \to 0.$$

Now, if we apply the functor $H^i_{\mathfrak{m}_0 R}(-)$ to this exact sequence, we get the following exact sequence:

$$H^i_{\mathfrak{m}_0 R}(\Gamma_{R_+}(M/xM)) \to H^i_{\mathfrak{m}_0 R}(H^1_{R_+}(M))(-1) \xrightarrow{x.} H^i_{\mathfrak{m}_0 R}(H^1_{R_+}(M)).$$

It should be noted that $H^i_{\mathfrak{m}_0 R}(\Gamma_{R_+}(M/xM)) \cong H^i_{\mathfrak{m}}(\Gamma_{R_+}(M/xM))$ is Artinian. So this fact implies that $(0 :_{H^i_{\mathfrak{m}_0 R}(H^1_{R_+}(M))(-1)} x)$ is Artinian. Now, since $H^i_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is x-torsion, using Melkersson's Lemma this module is Artinian. □

Remark 2.7. (i) We denote by $c := c_{R_+}(M)$ the *cohomological dimension of M with respect to R_+* which is

$$c := c_{R_+}(M) = \sup\{i \in \mathbb{N}_0 | H^i_{R_+}(M) \neq 0\}.$$

One can easily show that $c_{R_+}(M) = \dim_R(M/\mathfrak{m}_0 M)$ (see [6, 1.2]).
(ii) If $c_{R_+}(M) > 0$ and $\Gamma_{R_+}(M) = 0$, following [6, 1.3.7], there exists a homogeneous element $x \in R_+$ which is a non-zero-divisor with respect to M and $c_{R_+}(M/xM) = c_{R_+}(M) - 1$. In fact this follows by choosing x to avoid all the minimal primes of $(\mathfrak{m}_0 M :_R M)$.

Proposition 2.8. *Let* $\dim(R_0) = d$ *and let* $c = c_{R_+}(M)$ *be the cohomological dimension of M with respect to R_+. Then* $H^d_{\mathfrak{m}_0 R}(H^c_{R_+}(M))$ *is Artinian.*

Proof. We proceed the assertion by induction on c. If $c = 0$, then $H^d_{\mathfrak{m}_0 R}(\Gamma_{R_+}(M)) \cong H^d_{\mathfrak{m}}(\Gamma_{R_+}(M))$ and the last module is Artinian. Now, suppose inductively that the result has been proved for all values smaller than c and so we prove it for c. As usual we may assume that the residue field $R_0/\mathfrak{m}_0$ is infinite. Since there is an isomorphism $H^i_{R_+}(M) \cong H^i_{R_+}(M/\Gamma_{R_+}(M))$ for each i, we may assume that $\Gamma_{R_+}(M) = 0$. In view of Remark 2.7, there exists a homogeneous element $x \in R_+$ with $\deg(x) = a$ which is a non-zero-divisor with respect to M and $c_{R_+}(M/xM) = c_{R_+}(M) - 1 = c - 1$. On the other hand the usual exact sequence $0 \to M(-a) \xrightarrow{x.} M \to M/xM \to 0$ gives the following exact sequence:

$$H^{c-1}_{R_+}(M/xM) \to H^c_{R_+}(M)(-a) \xrightarrow{x.} H^c_{R_+}(M) \to 0.$$

Application of the right exact functor $H^d_{\mathfrak{m}_0 R}(-)$ to the above exact sequence induces the following exact sequence:

$$H^d_{\mathfrak{m}_0 R}(H^{c-1}_{R_+}(M/xM)) \to H^d_{\mathfrak{m}_0 R}(H^c_{R_+}(M))(-a) \xrightarrow{x.} H^d_{\mathfrak{m}_0 R}(H^c_{R_+}(M)) \to 0.$$

Using induction hypothesis, the module $H^d_{\mathfrak{m}_0 R}(H^{c-1}_{R_+}(M/xM))$ is Artinian and so is $(0 :_{H^d_{\mathfrak{m}_0 R}(H^c_{R_+}(M))(-a)} x)$. Now, since $H^d_{\mathfrak{m}_0 R}(H^c_{R_+}(M))$ is x-torsion, by using Melkersson's Lemma, it is Artinian. □

In the rest of this section we present some examples which show that $H^2_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ and $H^3_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ need not be Artinian.

In the following example which has already been presented in [2] we show that $H^2_{\mathfrak{m}_0 R}(H^1_{R_+}(R))$ is not Artinian even if R_0 is a regular local ring of dimension 2 and $\mathrm{ara}(R_+) = 2$.

Example 2.9. Let K be a field, let x, y, t be indeterminates, let $R_0 := K[x, y]_{(x,y)}$ and $\mathfrak{m}_0 := (x, y)R_0$. Moreover let $R := R_0[\mathfrak{m}_0 t]$ be the (truncated) Rees ring of $\mathfrak{m}_0$. One can easily see that $R_+ = (xt, yt)R$; and hence $\mathrm{ara}(R_+) = 2$. On the other hand, it follows from [2, Example 4.2] that $\Gamma_{\mathfrak{m}_0 R}(H^2_{R_+}(R))$ is not Artinian. Now, using Corollary 2.4, we deduce that $H^2_{\mathfrak{m}_0 R}(H^1_{R_+}(R))$ is not Artinian.

The following example which has already been presented in [3] shows that $H^3_{\mathfrak{m}_0 R}(H^1_{R_+}(R))$ is not Artinian even if $\mathrm{ara}(R_+) = 2$.

Example 2.10. Let K be a field, let $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{t}$ be indeterminates and let $R_0 := K[\mathbf{x}, \mathbf{y}, \mathbf{z}]_{(\mathbf{x},\mathbf{y},\mathbf{z})}$ and $\mathfrak{m}_0 := (\mathbf{x}, \mathbf{y}, \mathbf{z})R_0$. Furnish the polynomial ring $S := R_0[\mathbf{u}, \mathbf{v}]$ with its standard grading and consider the Noetherian homogeneous R_0-algebra $R := S/(\mathbf{xv}-\mathbf{yu})S$, which is canonically isomorphic to the Rees ring $R_0[(\mathbf{x}, \mathbf{y})\mathbf{t}]$ of R_0 with respect to the ideal $(\mathbf{x}, \mathbf{y}) \subseteq R_0$. One can easily see that R_+ is generated by two elements; and hence $\mathrm{ara}(R_+) = 2$. On the other hand, it follows from [3, Example 5.11] that $H^1_{\mathfrak{m}_0 R}(H^2_{R_+}(R))$ is not Artinian. Now, using Theorem 2.3, we deduce that $H^3_{\mathfrak{m}_0 R}(H^1_{R_+}(R))$ is not Artinian.

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