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Hilbert–Kunz theory for nodal cubics, via sheaves

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ABSTRACT

Suppose $B = F[x, y, z]/h$ is the homogeneous coordinate ring of a characteristic p degree 3 irreducible plane curve C with a node. Let J be a homogeneous (x, y, z)-primary ideal and $n \to e_n$ be the Hilbert–Kunz function of B with respect to J.

Let $q = p^n$. When $J = (x, y, z)$, it is known that $e_n = \frac{7}{3}q^2 - \frac{1}{3}q - R$ where $R = \frac{5}{3}$ if $q \equiv 2$ (3), and is 1 otherwise. We generalize this, showing that $e_n = \mu q^2 + \alpha q - R$ where R only depends on q mod 3. We describe α and R in terms of classification data for a vector bundle on C.

Introduction

Let h be a form of degree > 0 in $A = F[x, y, z]$ where F is algebraically closed of characteristic $p > 0$. Suppose J is a homogeneous ideal of A. If $q = p^n$, let $J^{[q]}$ be the ideal generated by all u^q, u in J. Let e_n be the F-dimension of $A/(J^{[q]}, h)$.

Problem: If $e_0 < \infty$, how does e_n depend on n?

The problem was treated by elementary methods, when $J = (x, y, z)$ and degree h is small, by several authors. In particular, Pardue in his thesis (see [3] for an exposition) showed that when h is an irreducible nodal cubic then e_n is $\frac{7}{3}q^2 - \frac{1}{3}q - \frac{5}{3}$ if $q \equiv 2$ (3), and is $\frac{7}{3}q^2 - \frac{1}{3}q - 1$ otherwise.

For arbitrary h and J, sheaf-theoretic methods were introduced by Brenner [1] and Trivedi [8]. They calculated $\mu = \lim_{n\to\infty} \frac{e_n}{q^2}$, showing that μ is rational. When h has coefficients in a finite field and defines a smooth plane curve C, Brenner [2] showed further that $\mu q^2 - e_n$ is an eventually periodic function of n. In [7], the author returned to the case $J = (x, y, z)$, and adapted Brenner's

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method to treat all h defining reduced irreducible C. (But now μq^2 must be replaced by something a bit more complicated.)

In the present paper we restrict our attention to nodal cubics but allow J to be arbitrary. Using sheaf-theoretic methods as in [7] we recover Pardue's result when $J = (x, y, z)$. For arbitrary J we get a result nearly as precise. What allows us to get sharp results is the well-developed theory of vector bundles on nodal cubic curves. (See Igor Burban [4] and the references therein.) We are indebted to Burban for pointing us towards this theory, and for the result essential to us that he derives in [4].

1. A little sheaf theory

Definition 1.1. If M is a finitely generated $\mathbb{Z}$-graded $A = F[x, y, z]$ module, $\mathrm{hilb}(M) = \sum \dim(M_d)T^d$ and $\mathrm{poincar\acute{e}}(M) = (1-T)^3\,\mathrm{hilb}(M)$. (Note that $\mathrm{poincar\acute{e}}(M)$ is in $\mathbb{Z}[T, T^{-1}]$.)

Throughout the paper we adopt the notation of the introduction, with $h \in A$ a degree 3 form defining a nodal $C \subset \mathbb{P}^2$, having desingularization $X = \mathbb{P}^1$. Hartshorne [6] is a good reference for what follows.

Even though C is singular there is a good theory of torsion-free sheaves on C. One may define the degree of such a sheaf, all such sheaves are reflexive, and one has Riemann–Roch and Serre duality. In some ways C is like an elliptic curve. For example, if Y is rank 1 torsion-free, $h^0(Y) = \deg Y$ if $\deg Y > 0$, and is 0 if $\deg Y < 0$. When $\deg Y = 0$, $h^0(Y)$ is 1 if Y is isomorphic to O_C and is 0 otherwise.

Definition 1.2. $\mathrm{poincar\acute{e}}(Y) = (1-T)^3 \sum h^0(Y(n))T^n$, where $Y(n)$ is the twist of Y by $O_C(n)$. (Riemann–Roch shows that $(1-T)^{-1}\,\mathrm{poincar\acute{e}}(Y)$ is in $\mathbb{Z}[T, T^{-1}]$.)

Example 1.3.

(a) $\mathrm{poincar\acute{e}}(O_C) = (1-T)^3(1 + 3T + 6T^2 + 9T^3 + \cdots) = 1 - T^3$.
(b) $\mathrm{poincar\acute{e}}(\bigoplus O_C(-d_i)) = (1 - T^3) \cdot \sum T^{d_i}$.
(c) If L has rank 1 and degree $-n$, then:

$$\begin{aligned}(1-T)^{-1}\,\mathrm{poincar\acute{e}}(L) &= T^{\frac{n+2}{3}}(2+T) \quad \text{if } n \equiv 1 \quad (3)\\ &= T^{\frac{n+1}{3}}(1+2T) \quad \text{if } n \equiv 2 \quad (3)\\ &= T^{\frac{n}{3}}\left(1+T+T^2\right) \quad \text{if } L \approx O_C\left(-\frac{n}{3}\right)\\ &= T^{\frac{n}{3}}(3T) \quad \text{otherwise.}\end{aligned}$$

Lemma 1.4. *Suppose L and M are rank 1 torsion-free, that neither is isomorphic to any $O_C(k)$, and that $\deg M \leqslant 1 + \deg L$. Then if $0 \to L \to U \to M \to 0$ is exact,* $\mathrm{poincar\acute{e}}(U) = \mathrm{poincar\acute{e}}(L) + \mathrm{poincar\acute{e}}(M)$.

Proof. Since $\deg M(n) \leqslant 1 + \deg L(n)$ for each n, it's enough to show that $h^0(U) = h^0(L) + h^0(M)$. If $\deg L \geqslant 0$, $\deg L^\vee \leqslant 0$ and $L^\vee$ is not isomorphic to O_C. So $h^1(L) = h^0(L^\vee) = 0$, and we use the exact sequence of cohomology. If $\deg L < 0$, $\deg M \leqslant 0$, and M is not isomorphic to O_C. So $h^0(M) = 0$, and the result follows. $\square$

Now fix a homogeneous ideal J of A with $\dim A/(J, h) < \infty$, and forms $g_1, \ldots, g_s$ generating $(J, h)/h$, with $\deg g_i = d_i$. Then the sheaf map $\bigoplus O_C(-d_i) \to O_C$ defined by the g_i is onto. So if W is the kernel of this map, W is locally free of rank $s - 1$ and degree $-3\sum d_i$.

Lemma 1.5.

(1) poincaré$(A/(J,h)) = (1-T)^3(1-\sum T^{d_i}) +$ poincaré(W).
(2) *More generally, let $q = p^n$ and $W^{[q]}$ be the pull-back of W by Φ^n, where $\Phi : C \to C$ is the Frobenius map. Then*:

$$\text{poincaré}\big(A/\big(J^{[q]}, h\big)\big) = \big(1-T^3\big)\Big(1-\sum T^{qd_i}\Big) + \text{poincaré}\big(W^{[q]}\big).$$

Proof. For each d we have an exact sequence $0 \to W(d) \to \bigoplus O_C(d-d_i) \to O_C(d)$, giving a corresponding exact sequence on global sections. Since $H^0(O_C(d))$ identifies with $(A/h)_d$, the cokernel of the map $H^0(\bigoplus O_C(d-d_i)) \to H^0(O_C(d))$ identifies with $(A/(J,h))_d$. It follows that $\dim(A/(J,h))_d = h^0(O_C(d)) - h^0(\bigoplus O_C(d-d_i)) + h^0(W(d))$. Multiplying by T^d, summing over d, and using (a) and (b) of Example 1.3, we get (1). Furthermore, replacing each g_i by g_i^q replaces J by $J^{[q]}$ and W by $W^{[q]}$. So (2) is a consequence of (1). □

Remark 1.6. Lemma 1.5 allows us to replace the problem of the dependence of poincaré$(A/(J^{[q]},h))$ on q by a more geometric question: if W is a vector bundle on C, how does poincaré$(W^{[q]})$ vary with q? A generalization of Lemma 1.5 is a key to the sheaf-theoretic approach to Hilbert–Kunz theory taken by Brenner and Trivedi.

For the rest of this section we take $J = (x, y, z)$, $g_1 = x$, $g_2 = y$, $g_3 = z$ so that the W of Lemma 1.5 has rank 2 and degree -9. We'll use sheaf theory on C to give another proof of Pardue's results.

Lemma 1.7. *W maps onto a rank 1 degree -4 torsion-free sheaf, M, whose stalk at the node is the maximal ideal m of the local ring $\mathcal{O}$.*

Proof. $W(1)$ identifies with the kernel of the map $O_C \oplus O_C \oplus O_C \to O_C(1)$ given by x, y and z. By Lemma 7.1 of [7], $W(1)$ maps onto a rank 1 degree -1 torsion-free sheaf whose stalk at the node is m, and we twist by $O_C(-1)$. □

Lemma 1.8. *Suppose $q = p^n$. Let M be the sheaf of Lemma 1.7. Pull M back by $\Phi^n : C \to C$ and quotient out the maximal torsion subsheaf to get a rank 1 torsion-free sheaf M_n. Then $\deg M_n = -5q + 1$.*

Proof. Theorem 2.8 of [7] together with Lemma 1.7 above shows that $\deg M_n = \text{constant} \cdot q - \dim(\mathcal{O}/m^{[q]})$. Passing to the completion we find that $\dim(\mathcal{O}/m^{[q]}) = \dim(F[\![x,y]\!]/(xy, x^q, y^q)) = 2q-1$. So $\deg(M_n) = (\text{constant}) \cdot q + 1$. Since $\deg(M) = -4$, the constant is -5. □

Lemma 1.9. *Let L_n be the kernel of the obvious map $W^{[q]} \to M_n$. Then*:

(1) *There is an exact sequence $0 \to L_n \to W^{[q]} \to M_n \to 0$ with $\deg M_n = -5q + 1$, $\deg L_n = -4q - 1$.*
(2) *Neither L_n nor M_n is free at the node.*
(3) poincaré$(W^{[q]}) =$ poincaré$(L_n) +$ poincaré(M_n).

Proof. Since $W^{[q]}$ and M_n have degrees $-9q$ and $-5q+1$ we get (1). If M_n is locally free, the exact sequence (1) shows that L_n is also. Since we have an exact sequence $0 \to M_n^{\vee} \to (W^{[q]})^{\vee} \to L_n^{\vee} \to 0$ we see conversely that if L_n is locally free then so is $M_n^{\vee\vee} = M_n$. Suppose now that L_n and M_n are locally free. Then $q > 1$. Let L_n' and M_n' be the pull-backs of L_n and M_n by Frobenius so that we have an exact sequence $0 \to L_n' \to W^{[pq]} \to M_n' \to 0$. Then $\deg L_{n+1} - \deg M_n' = (-4pq-1) - p(-5q+1) = pq - p - 1 > 0$. So the map $L_{n+1} \to W^{[pq]}/L_n' = M_n'$ is the zero-map, and $L_{n+1} \subset L_n'$. But $\deg L_{n+1} > \deg L_n'$, and this contradiction establishes (2). Finally, $\deg M_n - \deg L_n = 2 - q \leqslant 1$. Combining this with (2) and Lemma 1.4 we get (3). □

Corollary 1.10.

$$\begin{aligned}(1-T)^{-1}\operatorname{poincaré}\bigl(W^{[q]}\bigr) &= T^{\frac{4q+2}{3}}(1+2T)+T^{\frac{5q+1}{3}}(2+T) \quad \textit{if } q\equiv 1 \quad (3)\\ &= T^{\frac{4q+1}{3}}(3T)+T^{\frac{5q+2}{3}}(3) \quad \textit{if } q\equiv 2 \quad (3)\\ &= T^{\frac{4q}{3}}\bigl(2T+T^2\bigr)+T^{\frac{5q}{3}}(1+2T) \quad \textit{if } q\equiv 0 \quad (3).\end{aligned}$$

Proof. Suppose first that $q\equiv 1$ (3). Since $4q+1\equiv 2$ (3), $(1-T)^{-1}\operatorname{poincaré}(L_n)=T^{\frac{4q+2}{3}}(1+2T)$ by Example 1.3(c). Similarly, since $5q-1\equiv 1$ (3), $(1-T)^{-1}\operatorname{poincaré}(M_n)$ is $T^{\frac{5q+1}{3}}(2+T)$. Now use (3) of Lemma 1.9. The cases $q\equiv 2$ (3) and $q\equiv 0$ (3) are handled similarly. (When $q\equiv 2$ (3) we use the fact that neither L_n nor M_n is locally free.)

Now let $e_n=\dim(A/(J^{[q]},h))$. Pardue's formula for e_n is easily derived from Corollary 1.10. Let $u_n=(1-T)^{-1}\operatorname{poincaré}(W^{[q]})$. By Lemma 1.5, $(1-T)^2\operatorname{hilb}A/(J^{[q]},h)=(1+T+T^2)(1-3T^q)+u_n$. Applying $(\frac{\mathrm{d}}{\mathrm{d}T})^2$, dividing by 2, and evaluating at $T=1$ we find that $e_n=\frac{1}{2}(u_n''(1)-(9q^2+9q+4))$. Suppose that $q\equiv 1$ (3). Then Corollary 1.10 shows that $u_n''(1)=(\frac{4q+2}{3})(4q+3)+(\frac{5q+1}{3})(5q)=\frac{41}{3}q^2+\frac{25}{3}q+2$. When $q\equiv 2$ (3), $u_n''(1)=(\frac{4q+1}{3})(4q+4)+(\frac{5q+2}{3})(5q-1)=\frac{41}{3}q^2+\frac{25}{3}q+\frac{2}{3}$. And when $q\equiv 0$ (3), $u_n''(1)=(\frac{4q+3}{3})(4q+2)+(\frac{5q}{3})(5q+1)=\frac{41}{3}q^2+\frac{25}{3}q+2$. So $e_n=\frac{7}{3}q^2-\frac{1}{3}q-\frac{5}{3}$ if $q\equiv 2$ (3), and is $\frac{7}{3}q^2-\frac{1}{3}q-1$ otherwise. □

2. Elements of $\mathbb{Z}[T,T^{-1}]$ attached to cycles

In Corollary 1.10 we calculated all the $(1-T)^{-1}\operatorname{poincaré}(W^{[q]})$ for a certain rank 2 bundle, W. In this section we develop some combinatorial machinery that we'll use later to get similar results for arbitrary W.

Definition 2.1. Suppose $r>0$. A cycle (of length r) is an ordered r-tuple of integers, defined up to cyclic permutation. If a is a cycle, $a(k)$ is the cycle obtained from a by adding $3k$ to each cycle entry.

Definition 2.2.

$\gamma_1(a)$ is the number of entries of a that are $\geqslant 0$.
$\gamma_2(a)=\sum\max(a_i,0)$, where a_i runs over the entries of a.

Note that $\gamma_1(a)+\gamma_2(a)=\sum\max(a_i+1,0)$ where a_i runs over the entries of a. We now compute $(1-T)^2\sum\gamma_2(a(k))T^k$. This is evidently a sum of contributions, one for each entry in a. An entry of 2 gives a contribution of $(1-T)^2(2+5T+8T^2+\cdots)=2+T$; similarly an entry of 1 (resp. 0) gives a contribution of $(1+2T)$ (resp. $3T$). If follows easily that an entry of $-n$ gives a contribution of $T^{n+2}3(2+T)$, $T^{n+1}3(1+2T)$ or $T^n 3(3T)$ according as $n\equiv 1$, 2 or 0 mod 3. We may express this in a slightly different way:

Lemma 2.3. *Suppose the distinct entries in the cycle a are $-n_i$ with $-n_i$ appearing r_i times in the cycle. Then $P_2(a)=(1-T)^2\sum\gamma_2(a(k))T^k$ lies in $\mathbb{Z}[T,T^{-1}]$, and is the sum of contributions, one from each n_i. The contribution from n_i is:*

$$\begin{aligned}T^{\frac{n_i+2}{3}}(2r_i+r_iT) &\quad \textit{if } n_i\equiv 1 \quad (3),\\ T^{\frac{n_i+1}{3}}(r_i+2r_iT) &\quad \textit{if } n_i\equiv 2 \quad (3),\\ T^{\frac{n_i}{3}}(3r_iT) &\quad \textit{if } n_i\equiv 0 \quad (3).\end{aligned}$$

Observe next that the cycle a gives rise to an integer-valued function of period r on $\mathbb{Z}$, defined up to translation. We say that the cycle is "aperiodic" if this function has no period $< r$. For the rest of this section we assume that $r > 1$ and that a is aperiodic.

Definition 2.4. A "bloc", b, of a with entry N consists of consecutive entries of a each of which is N, with both the cycle entry preceding the first bloc entry and the cycle entry following the last bloc entry unequal to N. The length, $l(b)$, of b is the number of entries in b.

Since $r > 1$ and the cycle is aperiodic, there are at least 2 blocs in a. The blocs of a appear in cyclic order and fill out a; their lengths sum to r.

Definition 2.5. Let b be a bloc with entry N.

(1) If the blocs just before and just after b have entries $< N$, b is locally maximal and $\varepsilon(b) = 1$.
(2) If the blocs just before and just after b have entries $> N$, b is locally minimal and $\varepsilon(b) = -1$.
(3) If b is neither locally maximal nor locally minimal, $\varepsilon(b) = 0$.

Remark 2.6. Between any 2 locally maximal blocs there is a locally minimal bloc, and between any 2 locally minimals there is a locally maximal. Since there are at least 2 blocs, $\sum \varepsilon(b) = 0$.

Definition 2.7.

(1) A bloc b with entry N is positive if $N \geqslant 0$.
(2) Suppose b is positive. $\varepsilon^*(b) = \varepsilon(b)$ unless $N = 0$ and b is locally maximal. In this case we set $\varepsilon^*(b)$ equal to 0.
(3) $\gamma_3(a) = \sum \varepsilon^*(b)$, the sum ranging over the positive blocs of a.

We now compute $(1-T)^2 \sum \gamma_3(a(k))T^k$. The sum is evidently a sum of contributions, one from each bloc of a. Consider first a bloc with entry 2 or 1. The contribution of this bloc is $\varepsilon(b)(1-T)^2 \cdot (1+T+T^2+\cdots) = \varepsilon(b)(1-T)$. Next consider a bloc with entry 0. If the block is locally minimal it gives a contribution of $(-1)(1-T)^2(1+T+T^2+\cdots) = \varepsilon(b)(1-T)$, while if it is locally maximal, the contribution is $(1)(1-T)^2(T+T^2+T^3+\cdots) = T - T^2 = \varepsilon(b)\cdot(1-T) - (1-T)^2$.

More generally, a locally maximal bloc with entry $-n$, $n \equiv 0$ (3), provides a contribution of $\varepsilon(b)T^{\frac{n}{3}}(1-T) - T^{\frac{n}{3}}(1-T)^2$, while in all other cases (i.e. when $n \equiv \pm 1$ (3) or the bloc is not locally maximal) the contribution is $\varepsilon(b)T^{\frac{n+2}{3}}(1-T)$, $\varepsilon(b)T^{\frac{n+1}{3}}(1-T)$, or $\varepsilon(b)T^{\frac{n}{3}}(1-T)$ according as $n \equiv 1$, 2 or 0 mod 3. We'll express this result in a different way.

Definition 2.8. Suppose the distinct entries of a are the integers $-n_i$. Then:

(1) s_i is $\sum \varepsilon(b)$, the sum extending over all the blocs of a with entry $-n_i$.
(2) If $n_i \equiv 0$ (3), B_i is the number of locally maximal blocs with entry $-n_i$.

The discussion preceding the definition shows:

Theorem 2.9. *$P_3(a) = (1-T)^2 \sum \gamma_3(a(k))T^k$ is a sum of contributions, one from each n_i. The contribution from n_i is:*

$$\begin{aligned} T^{\frac{n_i+2}{3}}(s_i - s_iT) \quad &\textit{if } n_i \equiv 1 \quad (3), \\ T^{\frac{n_i+1}{3}}(s_i - s_iT) \quad &\textit{if } n_i \equiv 2 \quad (3), \\ T^{\frac{n_i}{3}}\big(s_i - s_iT - B_i(1-T)^2\big) \quad &\textit{if } n_i \equiv 0 \quad (3). \end{aligned}$$

We next derive an alternative description of $\gamma_1(a)+\gamma_3(a)$ in terms of "positive parts of a".

Definition 2.10. A positive part, p, of a consists of consecutive entries of a all of which are $\geqslant 0$; if a has a negative entry we further require that the entry of a preceding the first entry of p and the entry of a following the last entry of p are <0. (Note that any positive part of a is a union of consecutive positive blocs.)

Definition 2.11.

(1) $\theta(p)=l(p)$ if p consists of a single bloc of zeroes.
(2) $\theta(p)=l(p)$ if $l(p)=r$.
(3) In all other cases, $\theta(p)=1+l(p)$.

Definition 2.12. $\theta(a)=\sum\theta(p)$, the sum extending over the positive parts of a.

Lemma 2.13. *If p is a positive part of a, $\theta(p)=l(p)+\sum\varepsilon^*(b)$, the sum extending over the blocs in p.*

Proof. If p contains a bloc with $\varepsilon^*\neq\varepsilon$, then since this bloc is locally maximal with entry 0 it is the only bloc in p and we use (1) of Definition 2.11. So we may assume that $\varepsilon^*=\varepsilon$ for each bloc in p. If $l(p)=r$, $\sum\varepsilon^*(b)=\sum\varepsilon(b)$, which is 0 by Remark 2.6, and we use (2) of Definition 2.11. Suppose finally that $l(p)<r$. There is at least one bloc in p with $\varepsilon\neq 0$. The first and last blocs appearing in p with $\varepsilon\neq 0$ are evidently locally maximal. The first sentence of Remark 2.6 then shows that $\sum\varepsilon(b)$, the sum running over the blocs contained in p, is 1. Definition 2.11(3), now gives the result. $\square$

Summing the result of Lemma 2.13 over the positive parts of a we find:

Corollary 2.14. $\theta(a)=\gamma_1(a)+\gamma_3(a)$.

Theorem 2.15. *Let $\gamma_4(a)=(\sum\max(a_i+1,0))-\theta(a)$ with $\theta(a)$ as in Definition 2.12. Let $P_4(a)$ be $(1-T)^2\cdot\sum\gamma_4(a(k))T^k$. Then $P_4(a)$ is a sum of contributions, one from each n_i, where the $-n_i$ are the distinct entries of a. In the notation of Lemma 2.3 and Definition 2.8, the contribution from n_i is:*

$$\begin{aligned} T^{\frac{n_i+2}{3}}\big((2r_i-s_i)+(r_i+s_i)T\big) &\quad \text{if } n_i\equiv 1 \quad (3),\\ T^{\frac{n_i+1}{3}}\big((r_i-s_i)+(2r_i+s_i)T\big) &\quad \text{if } n_i\equiv 2 \quad (3),\\ T^{\frac{n_i}{3}}\big(-s_i+(3r_i+s_i)T+B_i(1-T)^2\big) &\quad \text{if } n_i\equiv 0 \quad (3). \end{aligned}$$

Proof. Combining Corollary 2.14 with the sentence following Definition 2.2 we find that $\gamma_4=(\gamma_1+\gamma_2)-(\gamma_1+\gamma_3)=\gamma_2-\gamma_3$. Applying this to $a(k)$, multiplying by T^k and summing over k we find that $P_4(a)=P_2(a)-P_3(a)$. Lemma 2.3 and Theorem 2.9 conclude the proof. $\square$

3. Results for arbitrary *W* and *J*

A locally free sheaf of rank >0 is "indecomposable" if it is not a direct sum of two subsheaves of rank >0. Indecomposable locally free W on the nodal cubic C have been classified—see Burban [4] and the references given there. I'll summarize results from the classification.

(1) Suppose $r>0$, a is an aperiodic cycle of length r, $m\geqslant 1$ and λ is in F^*. One may attach to the triple a,m,λ an indecomposable locally free sheaf $W=\mathcal{B}(a,m,\lambda)$.
(2) The pull-back of W to $X=\mathbb{P}^1$ is the direct sum of the $(\mathcal{O}_X(a_i))^m$ where the entries of a are the a_i. In particular, the rank of W is mr, and the degree is $m\sum a_i$.

(3) If $W = \mathcal{B}(a, m, \lambda)$, then $W(k)$ is isomorphic to $\mathcal{B}(a(k), m, \lambda)$ with $a(k)$ as in Definition 2.1.
(4) When F is algebraically closed (as it is throughout this paper) every indecomposable locally free sheaf on C is isomorphic to some $\mathcal{B}(a, m, \lambda)$.

In Theorem 2.2 of [5], Drozd, Greuel and Kashuba give a formula for $h^0(W)$ when $W = \mathcal{B}(a, m, \lambda)$. (As we're dealing with a nodal cubic rather than a cycle of projective lines, we take the s in the statement of that theorem to be 1.) In particular they show:

Theorem 3.1. *Suppose $W = \mathcal{B}(a, m, \lambda)$ with $r > 1$. Then in the notation of our Section 2, $h^0(W) = m \cdot ((\sum \max(a_i + 1, 0)) - \theta(a)) = m(\gamma_4(a))$.*

Corollary 3.2. *Situation as in Theorem 3.1. Then $(1 - T)^{-1}\,\text{poincaré}(W) = m(1 - T)^2 \sum \gamma_4(a(k))T^k$.*

Applying Theorem 2.15 we find:

Theorem 3.3. *Situation as in Theorem 3.1. Suppose the distinct entries in a are $-n_i$. Then $(1 - T)^{-1}\,\text{poincaré}(W)$ is the sum of the following contributions, one from each n_i:*

$$\begin{aligned}
T^{\frac{n_i+2}{3}}\big((2mr_i - ms_i) + (mr_i + ms_i)T\big) \quad &\textit{if } n_i \equiv 1 \quad (3),\\
T^{\frac{n_i+1}{3}}\big((mr_i - ms_i) + (2mr_i + ms_i)T\big) \quad &\textit{if } n_i \equiv 2 \quad (3),\\
T^{\frac{n_i}{3}}\big(-ms_i + (3mr_i + ms_i)T + mB_i(1 - T)^2\big) \quad &\textit{if } n_i \equiv 0 \quad (3),
\end{aligned}$$

where r_i is the number of times $-n_i$ appears in a, and s_i and B_i are obtained from a as in Definition 2.8.

We now make use of the following key result of Burban [4]: if $W = \mathcal{B}(a, m, \lambda)$ then $W^{[q]}$ is isomorphic to $\mathcal{B}(qa, m, \lambda^q)$ where qa is obtained from a by multiplying each cycle entry, a_i, by q.

Theorem 3.4. *Let W be a locally free sheaf on C. Suppose the pull-back of W to $X = \mathbb{P}^1$ is the direct sum of $(\mathcal{O}_X(-n_i))^{r_i}$ where the n_i are distinct and each $r_i > 0$. Then one can assign to each n_i an s_i (with $|s_i| \leqslant r_i$), and to each $n_i \equiv 0$ (3) a B_i, so that the following holds:*

For each q (when $p = 3$, for each $q > 1$), $(1 - T)^{-1}\,\text{poincaré}(W^{[q]})$ is the sum of the following contributions, one for each n_i:

$$\begin{aligned}
T^{\frac{qn_i+2}{3}}\big((2r_i - s_i) + (r_i + s_i)T\big) \quad &\textit{if } qn_i \equiv 2 \quad (3),\\
T^{\frac{qn_i+1}{3}}\big((r_i - s_i) + (2r_i + s_i)T\big) \quad &\textit{if } qn_i \equiv 1 \quad (3),\\
T^{\frac{qn_i}{3}}\big(-s_i + (3r_i + s_i)T + B_i(1 - T)^2\big) \quad &\textit{if } qn_i \equiv 0 \quad (3).
\end{aligned}$$

Proof. It suffices to prove the result for indecomposable W. So we may assume that W is $\mathcal{B}(a, m, \lambda)$. Suppose first that the length of the cycle a is > 1. Then $W^{[q]}$ is isomorphic to $\mathcal{B}(qa, m, \lambda^q)$; furthermore the pull-back of $W^{[q]}$ to $X = \mathbb{P}^1$ is the direct sum of the $(\mathcal{O}_X(-qn_i))^{mr_i}$.

Now replace W by $W^{[q]}$ in Theorem 3.3. The effect of this is to replace n_i by qn_i and leave m unchanged. The result we desire would follow if we could show that the s_i and B_i attached to the cycle qa and its cycle entry $-qn_i$ are independent of the choice of q (when $p = 3$ we need to show that this independence holds for $q \geqslant 3$). But as there is an obvious 1 to 1 correspondence between the blocs of a and the blocs of qa, and this correspondence preserves ε, this is clear.

When the cycle a consists of a single entry, $-n_1$, we can make a much simpler argument. In this case W has a filtration with m isomorphic quotients, each a line bundle of degree $-n_1$, and it's easy to calculate $(1-T)^{-1}\,\text{poincaré}(W^{[q]})$. Now $r_1 = m$, and we find that Theorem 3.4 holds for W with $s_1 = 0$, and when $n_1 \equiv 0$ (3), $B_1 = 1$ if $\lambda = 1$ and $B_1 = 0$ otherwise. $\square$

Suppose now that W is the kernel bundle attached to an ideal J and generators $g_1, \dots, g_s$ of J. Let $d_i = \deg g_i$, and set $e_n = \dim A/(J^{[q]}, h)$ where $q = p^n$. Theorem 3.4 attaches to W certain integers n_i, r_i, s_i and B_i. We'll use the argument given at the end of Section 1 to express each e_n (when $p = 3$, each e_n with $n > 0$) in terms of n_i, r_i, s_i, B_i and $\sum d_i^2$.

Definition 3.5. $\mu = \frac{1}{6}\sum r_i n_i^2 - \frac{3}{2}\sum d_i^2$, $\alpha = \frac{1}{3}\sum s_i n_i$.

The general result of Brenner [1] concerning Hilbert–Kunz multiplicities in graded dimension 2 shows that $e_n = \mu q^2 + O(q)$. We'll show that when $p = 3$ (and $n > 0$) $e_n = \mu q^2 + \alpha q - R$ for constant R. And when $p \neq 3$, $e_n = \mu q^2 + \alpha q - R(q)$ where $R(q)$ only depends on q mod 3.

Theorem 3.6. *Suppose $p = 3$. Let $R = \sum (r_i - B_i)$. Then for $n > 0$, $e_n = \mu q^2 + \alpha q - R$.*

Proof. Let $u_n = (1-T)^{-1}\,\text{poincaré}(W^{[q]})$ and $v_n = (1+T+T^2)\cdot(-1+\sum T^{d_i q})$. As we saw in Section 1, $2e_n = u_n''(1) - v_n''(1)$; see Lemma 1.5 and the proof of Corollary 1.10. Now $v_n''(1) = -2 +$ a sum of terms $(d_i q)(d_i q - 1) + (d_i q + 1)(d_i q) + (d_i q + 2)(d_i q + 1)$. Expanding we find that $v_n''(1) = (3\sum d_i^2)q^2 + (3\sum d_i)q + 2s - 2$, where s is the number of d_i. Since W has degree $-\sum r_i n_i$ and rank $\sum r_i$ we find:

$$v_n''(1) = \Bigl(3\sum d_i^2\Bigr)q^2 + \Bigl(\sum r_i n_i\Bigr)q + 2\sum r_i. \qquad (*)$$

Now as $p = 3$ and $q > 1$, each $qn_i \equiv 0$ (3). Theorem 3.4 then shows that u_n is a sum of terms $T^{\frac{qn_i}{3}}(-s_i + (3r_i + s_i)T + B_i(1-T)^2)$. So $u_n''(1)$ is a sum of terms $\frac{qn_i}{3}\cdot\frac{qn_i - 3}{3}\cdot(-s_i) + \frac{qn_i+3}{3}\cdot\frac{qn_i}{3}\cdot(3r_i + s_i) + 2B_i$. This term simplifies to $\frac{qn_i}{3}(qr_i n_i + 3r_i + 2s_i) + 2B_i$, and so:

$$u_n''(1) = \Bigl(\frac{1}{3}\sum r_i n_i^2\Bigr)q^2 + \Bigl(\sum r_i n_i\Bigr)q + \Bigl(\frac{2}{3}\sum s_i n_i\Bigr)q + 2\sum B_i. \qquad (**)$$

Combining $(*)$ and $(**)$ we find that $2e_n = u_n''(1) - v_n''(1) = 2\mu q^2 + 2\alpha q + 2\sum(B_i - r_i)$, giving the theorem. $\square$

Theorem 3.7. *Suppose $p \neq 3$. Set*

$$R(q) = \sum_{qn_i \equiv 1\ (3)} \Bigl(\frac{2r_i - 2s_i}{3}\Bigr) + \sum_{qn_i \equiv 2\ (3)} \Bigl(\frac{2r_i - s_i}{3}\Bigr) + \sum_{qn_i \equiv 0\ (3)} (r_i - B_i).$$

Note that $R(q)$ only depends on q mod 3. *Then $e_n = \mu q^2 + \alpha q - R(q)$.*

Proof. We argue as in the proof of Theorem 3.6. $(*)$ remains valid, but now $u_n''(1)$ is a more complicated sum of terms. When $qn_i \equiv 0$ (3), the term once again is $\frac{qn_i}{3}(qr_i n_i + 3r_i + 2s_i) + 2B_i$. But when $qn_i \equiv 1$ (3) this term is replaced by $\frac{qn_i+2}{3}(qr_i n_i + r_i + 2s_i)$; that is to say by $\frac{qn_i}{3}(qr_i n_i + 3r_i + 2s_i) + \frac{2r_i + 4s_i}{3}$. And when $qn_i \equiv 2$ (3), it is replaced by $\frac{qn_i+1}{3}(qr_i n_i + 2r_i + 2s_i)$; that is to say by $\frac{qn_i}{3}(qr_i n_i + 3r_i + 2s_i) + \frac{2r_i + 2s_i}{3}$. So:

$$u_n''(1) = \Bigl(\frac{1}{3}\sum r_i n_i^2\Bigr)q^2 + \Bigl(\sum r_i n_i\Bigr)q + \sum_{qn_i \equiv 1\ (3)} \frac{2r_i + 4s_i}{3} + \sum_{qn_i \equiv 2\ (3)} \frac{2r_i + 2s_i}{3} + 2\sum B_i.$$

Combining the above result with $(*)$ we find that $2e_n = u_n''(1) - v_n''(1) = 2\mu q^2 + 2\alpha q + \sum_{qn_i\equiv 1\ (3)} \frac{4s_i-4r_i}{3} + \sum_{qn_i\equiv 2\ (3)} \frac{2s_i-4r_i}{3} + 2\sum_{qn_i\equiv 0\ (3)}(B_i - r_i) = 2\mu q^2 + 2\alpha q - 2R(q)$. $\square$

Theorems 3.6 and 3.7 differ from similar results in [2] and [7] in that they allow practical calculation of all the e_n. (The eventually periodic terms that occur in the results of [2] and [7] arise from dynamical systems acting on the rational points of certain moduli spaces—in practice they cannot be calculated.) The following examples show how easy it is to apply Theorems 3.6 and 3.7.

Example 3.8. Suppose $p = 2$ and $h = x^3 + y^3 + xyz$. Let J be generated by $g_1, \dots, g_8$ where the g_i are x^3, y^3, z^3, x^2y, x^2z, xz^2, y^2z and yz^2. If W is the kernel bundle arising from these g_i, then $(1-T)^{-1}\,\text{poincaré}(W^{[8]}) = (1-T)^{-1}\,\text{poincaré}(A/(J^{[8]}, h)) - (1-T^3)(1-8T^{24})$. This is calculated immediately using Macaulay 2 which shows:

$$(1-T)^{-1}\,\text{poincaré}\big(W^{[8]}\big) = 3T^{27} + 12T^{28} + 6T^{30} = T^{27}(3 + 12T) + T^{30}(6 + 0T).$$

We'll use this information to determine all the e_n.

(a) $n_1 = \lfloor \frac{3\cdot 27}{8} \rfloor = 10$, $n_2 = \lfloor \frac{3\cdot 30}{8} \rfloor = 11$.
(b) Since $8n_1 \equiv 2\ (3)$, $r_1 - s_1 = 3$ and $2r_1 + s_1 = 12$. It follows that $r_1 = 5$, $s_1 = 2$. Similarly, since $8n_2 \equiv 1\ (3)$, $2r_2 - s_2 = 6$ and $r_2 + s_2 = 0$. So $r_2 = 2$, $s_2 = -2$.
(c) $\mu = \frac{1}{6}(5\cdot 100 + 2\cdot 121) - \frac{3}{2}(\sum_1^8 9) = \frac{47}{3}$, $\alpha = \frac{1}{3}(2\cdot 10 - 2\cdot 11) = -\frac{2}{3}$.
(d) Since $n_1 \equiv 1\ (3)$ and $n_2 \equiv 2\ (3)$, $R(1) = \frac{2r_1-2s_1}{3} + \frac{2r_2-s_2}{3} = \frac{6}{3} + \frac{6}{3} = 4$, $R(2) = \frac{2r_1-s_1}{3} + \frac{2r_2-2s_2}{3} = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$.

Theorem 3.7 now tells us that $e_n = \frac{47}{3}q^2 - \frac{2}{3}q - 4$ for even n and $\frac{47}{3}q^2 - \frac{2}{3}q - \frac{16}{3}$ for odd n.

Example 3.9. Take the g_i and h as in the above example but with $p = 3$. Now Macaulay 2 gives:

$$\begin{aligned}(1-T)^{-1}\,\text{poincaré}\big(W^{[9]}\big) &= 13T^{31} + 2T^{32} + 2T^{33} + 4T^{34}\\ &= T^{30}\big(0 + 13T + 2T^2\big) + T^{33}\big(2 + 4T + 0T^2\big).\end{aligned}$$

It follows that $n_1 = \frac{30\cdot 3}{9} = 10$, and we find that $r_1 = 5$, $s_1 = 2$, $B_1 = 2$. Similarly, $n_2 = \frac{33\cdot 3}{9} = 11$, and $r_2 = 2$, $s_2 = -2$, $B_2 = 0$. The μ and α are once again $\frac{47}{3}$ and $-\frac{2}{3}$, but now $R = (5-2) + (2-0) = 5$. We conclude from Theorem 3.6 that $e_n = \frac{47}{3}q^2 - \frac{2}{3}q - 5$ for $n > 0$.

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