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Extension of functions defined on products of pseudocompact spaces and continuity of the inverse in pseudocompact groups

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Abstract

It is proved that a Tychonoff pseudocompact group with continuous multiplication is a topological group. It is also proved that a Tychonoff countably compact group with separately continuous multiplication is a topological group. A continuous function defined on a product $X \times Y$ of pseudocompact spaces is extended to a separately continuous function defined on the product $\beta(X) \times \beta(Y)$ of their Stone–Čech compactifications.

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0. Introduction

This paper is concerned with the problem which may generally be formulated as follows.

Problem 0.1. Let a group or similar to group structure and a topological structure on set G be given. What restrictions on these structures and relationships between them ensure that G is a topological group?

Recall [3] that a group G with a topology is called
(a) a semitopological group if multiplication in G is separately continuous, and inversion $x \mapsto x^{-1}$ is continuous;
(b) a paratopological group if multiplication in G is continuous.

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In [18] a group G with a topology is called an m-topological group if multiplication in G is separately continuous.

Ellis [6] has proved in 1957 that a locally compact group with separately continuous multiplication is a topological group. Korovin [18] (see also [16,17]) has proved that if group G belongs to class $\mathscr{N}$ (for a definition of this class see [18] or directly after Definition 1.7 in this paper) and is either semitopological or an Abelian m-topological group then G is a topological group, and that if a group G from $\mathscr{N}$ is an m-topological group then G is a paratopological group. Class $\mathscr{N}$ contains countably compact spaces, pseudocompact spaces with countable tightness and some other "good" classes of spaces [18].

Grant [12] proved that if a pseudocompact paratopological group G satisfies some additional conditions (for example, if G^2 is pseudocompact) then G is a topological group. Pfister [22] obtained a result which can be formulated in the following way: a locally strongly countably complete in Frolic's sense [9] regular paratopological group is a topological group. In particular, a (locally) countably compact regular paratopological group is a topological group. The Pfister theorem and results obtained by Korovin (see also Corollary 2.7 in this paper) imply the following theorem:

Theorem 0.2. *A countable compact Tychonoff m-topological group is a topological group.*

Korovin has constructed an Abelian pseudocompact group of period 2 with separately continuous multiplication which is not a topological group [18,16]. Therefore, a semitopological pseudocompact group need not be a topological group.

We remind of Wallace's problem [25, see also Problem 523 in 21]:

Problem 0.3. It it true that every countably compact Hausdorff topological semigroup S with two-sides cancellations is a topological group?

The argument of [10] responds affirmatively when S is compact. According to Mukhurjea and Tserpes [19] the answer is affirmative for S that are additionally assumed to be first countable. In fact, it is also proved in [19] that a sequentially compact Hausdorff cancellative semigroup is a paratopological group. Therefore, the Pfister theorem implies the following:

Theorem 0.4. *A sequentially compact regular topological semigroup with two-sides cancellations is a topological group.*

Yur'eva [26] has recently showed that a sequential countably compact Hausdorff semigroup with two-sides cancellations is a topological group.

The following theorem is, probably, known.

Theorem 0.5. *If S is a compact Hausdorff semigroup with separately continuous multiplication and two-sides cancellations then S is a topological group.*

Proof. There is an idempotent in any compact left-topological semigroup [7]. As S is a cancellative semigroup, idempotent $e \in S$ is the unit of S (and it is unique). Let $x \in S$. The set $xS = \{xy: y \in S\}$ is a closed subsemigroup of S and, therefore, contains an idempotent. Since an idempotent e is unique in S, we have $e \in xS$, and for some $y \in S$ the equality $xy = e$ holds. Therefore, for every element of the semigroup S there exists the inverse element, and S is a group. By the Ellis Theorem [6] S is a topological group. □

It is proved in this paper that

(a) an m-topological group from class $\mathscr{N}$ is a topological group (Theorem 2.5);

(b) a pseudocompact paratopological group is a topological group (Theorem 2.6).

One of the main results of this paper is also Theorem 1.6: any continuous function defined on the product $X \times Y$ of pseudocompact spaces can be extended to a separately continuous function defined on $\beta(X) \times \beta(Y)$. As it is shown in this paper, the Grothendieck Theorem [13] (for every countably compact X any countably compact subspace Y of $C_p(X)$ is compact) is, in fact, equivalent to the statement: any separately continuous function defined on the product $X \times Y$ of countably compact spaces can be extended to a separately continuous function defined on $\beta(X) \times \beta(Y)$.

Tkachuk noticed that the Shakhmatov Example [24] shows that the Grothendieck Theorem cannot be extended on the class of all pseudocompact spaces, i.e., there exist pseudocompact spaces X_0 and Y_0 and a separately continuous function defined on $X_0 \times Y_0$ that cannot be extended to a separately continuous function defined on $\beta(X_0) \times \beta(Y_0)$. Recall the Gliksberg Theorem [11]: for pseudocompact spaces X and Y any continuous function can be extended to a continuous function defined on $\beta(X) \times \beta(Y)$ if and only if $X \times Y$ is a pseudocompact space.

All topological spaces are supposed to be Tychonoff spaces and are called simply "spaces" below. For spaces X, Y, and metric space Z, $C(X, Z)$ is the set of all continuous maps of X to Z, $C(X) = C(X, \mathbb{R})$, $SC(X \times Y)$ is the set of all separately continuous real-valued functions defined on $X \times Y$; $C^*(X, Z)$ is the set of all continuous bounded maps of X to Z; $C^*(X) = C^*(X, \mathbb{R})$, $SC^*(X \times Y)$ is the set of all bounded separately continuous real-valued functions on $X \times Y$.

We will consider two topologies on the spaces specified above: the topology of pointwise convergence and uniform convergence. The topology of pointwise convergence will be denoted by "p", and the topology of uniform convergence by "u"; for example, $C_p(X)$ is $C(X)$ with the topology of pointwise convergence, and $C_u^*(X)$ is $C^*(X)$ with the topology of uniform convergence.

1. Functions on products of spaces

Let X and Y be sets and $\Lambda_{X,Y}$ be the exponential map [8] of $\mathbb{R}^{X\times Y}$ into $(\mathbb{R}^Y)^X$ defined as

$$\Lambda_{X,Y}(f)(x)(y)=f(x, y)$$

for $f\in\mathbb{R}^{X\times Y}$, $x\in X$, $y\in Y$. The map $\Lambda_{X,Y}$ is a one-to-one correspondence. It is easy to check the following assertion:

Assertion 1.1. *Let X and Y be spaces. Then*

$$\Lambda_{X,Y}(\mathrm{SC}(X\times Y))=C(X, C_{\mathrm{p}}(Y)),$$
$$\Lambda_{X,Y}(C^*(X\times Y))\supset C^*(X, C^*_{\mathrm{u}}(Y)).$$

Assertion 1.2. *Let X and Y be spaces, $f\in\mathrm{SC}(X\times Y)$, $\varphi=\Lambda_{X,Y}(f)$. Then the following conditions are equivalent:*

(a) *the function f can be extended to a separately continuous function*

$$\hat{f}:\beta(X)\times Y\to\mathbb{R};$$

(b) *the closure of $\varphi(X)$ in $C_{\mathrm{p}}(Y)$ is compact.*

Proof. (a) $\Rightarrow$ (b) Assertion 1.1 implies that map $\hat{\varphi}=\Lambda_{\beta(X),Y}(\hat{f}):\beta(X)\to C_{\mathrm{p}}(X)$ is continuous. So, the set $\varphi(X)$ is contained in the compact space $\hat{\varphi}(\beta(X))\subset C_{\mathrm{p}}(X)$, and its closure is compact.

(b) $\Rightarrow$ (a) It follows from the fact that the closure of $\varphi(X)$ in $C_{\mathrm{p}}(Y)$ is compact that the map φ can be extended to a continuous map $\hat{\varphi}:\beta(X)\to C_{\mathrm{p}}(Y)$. Assertion 1.1 implies that function

$$\hat{f}=\Lambda^{-1}_{\beta(X),Y}(\hat{\varphi}):\beta(X)\times Y\to\mathbb{R}$$

is separately continuous. Clearly, $\hat{f}$ is an extension of f. □

Assertion 1.3. *Let X and Y be spaces, $f\in C^*(X\times Y)$, $\varphi=\Lambda_{X,Y}(f)$. Then*

(a) *function f can be extended to continuous function*

$$\hat{f}:\beta(X)\times\beta(Y)\to\mathbb{R};$$

(b) *$\varphi\in C(X, C^*_{\mathrm{u}}(Y))$ and the closure of $\varphi(X)$ in $C^*_{\mathrm{u}}(Y)$ is compact.*

Proof. (a) $\Rightarrow$ (b) The map

$$\hat{\varphi}=\Lambda_{\beta(X),\beta(Y)}(\hat{f}):\beta(X)\to C_{\mathrm{u}}(\beta(Y))$$

is continuous [5, Theorem XII.5.3, p. 265]. Bearing in mind that $C_{\mathrm{u}}(\beta(Y))$ is canonically homeomorphic to $C^*_{\mathrm{u}}(Y)$ we may assume that $\hat{\varphi}$ continuously maps $\beta(X)$ into $C^*_{\mathrm{u}}(Y)$. As $\hat{\varphi}$ is an extension of φ, the map φ is continuous and the closure in $C^*_{\mathrm{u}}(Y)$ of set $\varphi(X)\subset\varphi(\beta(X))\subset C^*_{\mathrm{u}}(Y)$ is compact.

(b) ⇒ (a) Since the map φ is continuous and the closure of $\varphi(X)$ in $C_u^*(Y)$ is compact, φ can be extended to a continuous map $\hat{\varphi}: \beta(X) \to C_u^*(Y)$. Bearing in mind the fact that $C_u(\beta(Y))$ is canonically homeomorphic to $C_u^*(Y)$, we may assume that $\hat{\varphi}$ continuously maps $\beta(X)$ into $C_u^*(\beta(Y))$. The function

$$\hat{f} = \Lambda_{\beta(X),\beta(Y)}^{-1}(\hat{\varphi}) : \beta(X) \times \beta(Y) \to \mathbb{R}$$

is an extension of the function f, and by Assertion 1.1 $\hat{f}$ is a continuous function. □

Assertions 1.2 and 1.3 may be made more specific if X and Y are pseudocompact.

In order to prove the following assertion we need the Haydon Theorem [14]: if Y is a pseudocompact space then the compact subspaces of $C_p(Y)$ and $C_p(\beta(Y))$ are the same (here we imply that the algebraic rings $C(Y)$ and $C(\beta(Y))$ are identified under the canonical isomorphism).

Recall that Eberlein compacta are compact spaces that can be embedded into Banach spaces with the weak* topology. A compact space X is an Eberlein compact if and only if X can be embedded into $C_p(Y)$, where Y is a compact space. It follows from the Haydon Theorem that for a pseudocompact space Y all compact subspaces of $C_p(Y)$ are Eberlein compacta. And, finally, there is the Preiss–Simon Theorem [23], which can be reformulated in the following way: any pseudocompact subspace of an Eberlein compact is a compact space.

Assertion 1.4. *Let X and Y be pseudocompact spaces, $f \in \mathrm{SC}(X \times Y)$, $\varphi = \Lambda_{X,Y}(f)$. Then the following conditions are equivalent:*

(a) *f can be extended to a separately continuous function on $\beta X \times Y$;*

(b) *f can be extended to a separately continuous function on $X \times \beta Y$;*

(c) *f can be extended to a separately continuous function*

$$\hat{f}: \beta(X) \times \beta(Y) \to \mathbb{R};$$

(d) *the closure of $\varphi(X)$ in $C_p(Y)$ is compact;*

(e) *$\varphi(X)$ is a compact subspace of $C_p(Y)$.*

Proof. Implications (c) ⇒ (a), (c) ⇒ (b), and (e) ⇒ (d) are evident. Implications (a) ⇔ (d) follow from Assertion 1.2.

(d) ⇒ (e) The Haydon Theorem implies that the closure of $\varphi(X)$ in $C_p(Y)$ is compact in $C_p(\beta(X))$ and is an Eberlein compact. Therefore the pseudocompact space $\varphi(X)$ lies in an Eberlein compact. It follows from the Preiss–Simon Theorem [23] that $\varphi(X)$ is a compact space.

(e) ⇒ (c) The Haydon Theorem implies that $\varphi(X)$ is compact in $C_p(\beta(Y))$, since we may presume that φ continuously maps X into $C_p(\beta(Y))$. It follows from

Assertion 1.2 that the function $f_1 = \Lambda^{-1}_{X,\beta(Y)}(\varphi)$ can be extended to a separately continuous function

$$\hat{f}: \beta(X) \times \beta(Y) \to \mathbb{R}.$$

(b) ⇒ (c) follows from (a) ⇒ (c). □

Assertion 1.5. *Let X and Y be pseudocompact spaces, $f \in C(X \times Y)$, $\varphi = \Lambda_{X,Y}(f)$. Then the following conditions are equivalent:*

(a) *f can be extended to a continuous function on $\beta X \times Y$;*

(b) *f can be extended to a continuous function on $X \times \beta Y$;*

(c) *f can be extended to a continuous function on $\beta(X) \times \beta(Y)$;*

(d) *the map $\varphi : X \to C_u(Y)$ is continuous;*

(e) *the set $\varphi(X)$ is compact in the space $C_u(Y)$;*

(f) *$[\varphi(X)]_{C_u(Y)}$ is compact in space $C_u(Y)$.*

Proof. The implications (c) ⇒ (a), (c) ⇒ (b) and (e) ⇒ (f) are evident.

(a) ⇒ (c) Let f_1 be the continuous extension of f to $\beta(X) \times Y$. The product $\beta(X) \times Y$ is pseudocompact as the product of a compact space and a pseudocompact space [8, Corollary 3.10.27]. It follows from the Glicksberg Theorem [11,4] that the function f_1 can be extended to a continuous function on $\beta(X) \times \beta(Y)$.

(b) ⇒ (c) follows from (a) ⇒ (c).

(c) ⇒ (d) follows from Assertion 1.3.

(d) ⇒ (c) Since the map φ is continuous, $\varphi(X)$ is a pseudocompact subspace of the metric space $C_u(Y)$, hence $\varphi(X)$ is compact in $C_u(Y)$. It follows from Assertion 1.3 that the function f can be extended to a continuous function on $\beta(X) \times \beta(Y)$.

(d) ⇒ (e) The set $\varphi(X)$ is pseudocompact and metrizable in $C_u(Y)$ as a continuous image of the pseudocompact space X. Therefore, $\varphi(X)$ is a compact subspace of $C_u(Y)$.

(f) ⇒ (d) The identity map of $C_u(Y)$ onto $C_p(Y)$ is continuous. Hence the fact that the set $F = [\varphi(X)]_{C_u(Y)}$ is compact with respect to the topology of uniform convergence implies that the topologies of pointwise convergence and uniform convergence coincide on F and, all the more, on $\varphi(X)$. Assertion 1.1 implies that map $\varphi : X \to C_p(Y)$ is continuous. Therefore, the map $\varphi : X \to C_u(Y)$ is also continuous. □

Theorem 1.6. *Let X and Y be pseudocompact spaces, and f be a continuous real-valued function on $X \times Y$. Then f can be extended to a separately continuous function on $\beta(X) \times \beta(Y)$.*

Proof. By Assertion 1.4, it is sufficient to show that f can be extended to a separately continuous function $\hat{f}: X \times \beta(Y) \to \mathbb{R}$.

Let $\varphi = \Lambda_{X,Y}(f)$. As Y is a pseudocompact space, for every $x \in X$ function $\varphi(x)$ is bounded and can be extended to a continuous function $g_x : \beta(Y) \to \mathbb{R}$ uniquely. Put $\hat{f}(x, y) = g_x(y)$ for $x \in X$ and $y \in Y$. The function $\hat{f}: X \times \beta(Y) \to \mathbb{R}$ is an extension of f.

Show that $\hat{f}$ is separately continuous. Suppose the contrary. Then for some $y_* \in \beta(Y)$ the function $h_* : X \to \mathbb{R}$ defined by the rule $x \mapsto \hat{f}(x, y_*)$ is not continuous, i.e., it has a break at some point $x_* \in X$. Without loss of generality we may assume that $h_*(x_*) = 0$ and

$$x_* \in \overline{\{x \in X: h_*(x) \geqslant 1\}}.$$

By induction on $n \in \mathbb{N}^+$ we will construct sequences $\{x_n: n \in \mathbb{N}^+\} \subset X$, $\{W_n: W_n$ is open in $\beta(Y)$, $n \in \mathbb{N}^+\}$, $\{U_n: U_n$ is open in $\beta(Y)$, $n \in \mathbb{N}^+\}$ and $\{V_n: V_n$ is open in X, $n \in \mathbb{N}^+\}$ in such a way that for every $n \in \mathbb{N}^+$ the following conditions will hold:

(1) $y_* \in W_{n+1} \subset \overline{W_{n+1}} \subset W_n$;

(2) $x_* \in V_{n+1} \subset \overline{V}_{n+1} \subset V_n$;

(3) $h_*(x_n) \geqslant 1$;

(4) $x_n \in V_{n+1}$;

(5) $g_{x_n}(W_n) \subset (2/3, +\infty)$;

(6) $U_n \subset W_n$;

(7) $\hat{f}(V_n \times U_n) \subset (-\infty, 1/3)$.

Choose a point $x_1 \in X$ such that $h_*(x_1) \geqslant 1$ and a neighborhood $W_1 \subset \beta(Y)$ of x_* such that $g_{x_1}(W_1) \subset (2/3, +\infty)$. Let $y \in W_1 \cap Y$ be some point for which

$$\hat{f}(x_*, y) = g_{x_*}(y) < 1/3.$$

From the continuity of f it follows that there exist open neighborhoods $U_1 \subset W_1$ of x_* and $V_1 \subset \beta(Y)$ of y_* for which

$$\hat{f}(V_1 \times U_1) \subset (-\infty, 1/3).$$

Assume that $x_n \in X$ and open sets $W_n, U_n \subset \beta(Y)$ and $V_n \subset Y$ are constructed. Choose a point $x_{n+1} \in X$ such that conditions $h_*(x_{n+1}) \geqslant 1$ and $x_{n+1} \in V_n$ hold. Let W_{n+1} be a neighborhood of y_* for which $\overline{W}_{n+1} \subset W_n$ and $g_{x_{n+1}}(W_{n+1}) \subset (2/3, +\infty)$.

Let $y \in W_{n+1} \cap Y$ be a point such that $f(x_*, y) = g_{x_*}(y) < 1/3$. From continuity of f it follows that there exist neighborhood $U_{n+1} \subset W_{n+1}$ of y and neighborhood $V_{n+1} \subset \overline{V}_{n+1} \subset V_n$ of x_* for which $\hat{f}(V_n \times U_n) \subset (-\infty, 1/3)$.

The construction is completed.

Since space Y is pseudocompact, the family $\{U_n \cap Y: n \in \mathbb{N}^+\}$ of open subsets of Y accumulates to some point $y_0 \in Y$. Define a function $h_0 : X \to \mathbb{R}$ by $h_0(x) = f(x, y_0)$. It follows from (1) and (6) that $y_0 \in \bigcap_{n \in \mathbb{N}^+} W_n$. Condition (5) implies that $h_0(x_n) = g_{x_n}(y_0) > 2/3$ for every $n \in \mathbb{N}^+$. As the function h_0 is continuous, for every natural number n there exists a neighborhood O_n of x_{n+1} for which $h_0(O_n) \subset (2/3, +\infty)$ and (see (4)) $O_n \subset V_n$. As X is a pseudocompact space, the family $\{O_n: n \in \mathbb{N}^+\}$ accumulates to some point $x_0 \in X$. It follows from continuity of h_0 that $h_0(x_0) \geqslant 2/3$. On the other hand, condition (2) implies that $x_0 \in \bigcap_{n \in \mathbb{N}^+} V_n$, and conditions (2) and (7) imply that $h_0(x_0) \leqslant 1/3$. Contradictory. □

Definition 1.7. Let X and Y be spaces. We will say that (X, Y) is a *Grothendieck pair* if every continuous image of X in $C_p(Y)$ has compact closure in $C_p(Y)$.

Clearly, if (X, Y) is a Grothendieck pair then X is pseudocompact. In [18] some classes of spaces are introduced. Using Definition 1.7 we may define these classes as follows:

$$\mathscr{L} = \{X\colon\ X \text{ is pseudocompact and for any pseudocompact } Y,\ (Y, X) \text{ is a Grothendieck pair}\},$$
$$\mathscr{N} = \{X\colon\ (X, X) \text{ is a Grothendieck pair}\}.$$

It appears that the widest subclass of class $\mathscr{L}$ is provided by the Arhangel'skiĭ Theorem [1].

Theorem 1.8 [1]. *Let X be a space functionally generated by the family of its σ countably pracompact subspaces. Then the closure of any pseudocompact subset of* $C_p(X)$ *is compact.*

Necessary definitions can be found in [1,2]. A space X is called countably pracompact in [1,2] if there exists a dense subspace Y of X such that any infinite subset of Y has a limit point in X. The Arhangel'skiĭ Theorem is a generalization of a series of results concerned with this theme. Formulating these results in a form convenient for our purposes yields the following corollary:

Corollary 1.9. *Let the pseudocompact space X have one of the following properties:*
(a) *countable compactness;*
(b) *countable tightness;*
(c) *separability;*
(d) *the property of being a k-space.*
Then X belongs to class $\mathscr{L}$.

Assertion 1.4 yields the following proposition:

Proposition 1.10. *Let X and Y be pseudocompact spaces. Then* (X, Y) *is a Grothendieck pair if and only if* (Y, X) *is a Grothendieck pair.*

Proposition 1.11. *Let* $f: X \times Y \to Z$ *be a separately continuous map, X and Y be pseudocompact spaces and* (X, Y) *be a Grothendieck pair. Then f can be extended to a separately continuous map*

$$\hat{f}: \beta(X) \times \beta(Y) \to \beta(Z).$$

Proof. Let I be the unit segment of the real line:

$$I = \{x \in \mathbb{R}: 0 \leqslant x \leqslant 1\}.$$

It is sufficient to show that for any continuous function $h: Z \to I$ the function

$$g = f \circ h : X \times Y \to I$$

can be extended to a separately continuous function defined on $\beta(X) \times \beta(Y)$.

Since f is a separately continuous mapping, g is a separately continuous function. It follows from Assertion 1.4 that g can be extended to a separately continuous function on $\beta(X) \times \beta(X)$. □

From Theorem 1.6 using arguments similar to arguments in the proof of the Proposition 1.11, the following proposition can be obtained:

Proposition 1.12. *let $f: X \times Y \to Z$ be a continuous map and X, Y pseudocompact spaces. Then f can be extended to a separately continuous map*

$$\hat{f}: \beta(X) \times \beta(Y) \to \beta(Z).$$

Proposition 1.13. *Let X be a pseudocompact space and K a compact subspace of $C_p(X)$. Then there exists an element f of K at which restrictions to K of the topologies of pointwise and uniform convergence coincide.*

Proof. Identify $C(X)$ and $C(\beta(X))$ in a natural way. Then by the Haydon Theorem [14] restrictions to K of the topologies of $C_p(X)$ and $C_p(\beta(X))$ coincide. Spaces $C_u(X)$ and $C_u(\beta(X))$ are naturally isomorphic. Hence it suffices to prove the proposition for compact X, and for compact X this proposition is exactly the same as the Namioka Theorem [20, Theorem 2.3]. □

2. Pseudocompact groups with separately and jointly continuous operations

Assertion 2.1. *Let $(K, *)$ be a compact space with continuous binary operation $*$, G be dense in K, G be closed with respect to operation $*$ and $(G, *)$ be a group. Then $(K, *)$ and $(G, *)$ are topological groups.*

Proof. Continuity of $*$ and density of G in K imply that $*$ is an associative operation on K and the unit element of G is the unit element of K. Hence $(K, *)$ is a compact topological semigroup with unity. For $g \in G$ we have $G * g = g * G = G$, therefore $g * K = K * g = K$. Since G is dense in K and K is a compact semigroup, the following condition holds:

$$(*) \qquad K * x = x * K = K \quad \text{for any } x \in K.$$

It is shown in the first part of the proof of Theorem 2.9.16 in [15] that a compact semigroup with unit satisfying condition $(*)$ is a topological group. □

The following theorem is a strengthening of Theorem 1 from [18] (see also [16]):

Theorem 2.2. *Let G and X be pseudocompact spaces, $*$ a binary operation defined on G and $(G, *)$ a group with respect to this operation. Let*

$$d : G \times X \to X$$

be an action of G on X such that d can be extended to a separately continuous map

$$\hat{d} : \beta(G) \times \beta(X) \to \beta(X).$$

Then the map $\hat{d}$ is continuous.

Proof. It is sufficient to show that for any continuous function $q \in C(X)$ the function $f = d \circ q : G \times X \to \mathbb{R}$ can be extended to a continuous function defined on $\beta(G) \times \beta(X)$.

The fact that the action d can be extended to separately continuous map $\hat{d} : \beta(G) \times \beta(X) \to \beta(X)$ implies that f can be extended to a separately continuous function on $\beta(G) \times \beta(X)$. Assertion 1.4 yields the continuity of map

$$\varphi = \Lambda_{G,X}(f) : G \to C_p(X)$$

and compactness of $\varphi(G)$ in $C_p(X)$.

As G acts on X, G acts on $\mathbb{R}^X$ in a natural way; denote this action by d_f:

$$d_f(g, p)(x) = p\big(d(g^{-1}, x)\big)$$

for $x \in X$, $p \in C(X)$, and $g \in G$. It may be verified directly that $\varphi(G) = d_f(G, q)$.

Let $g \in G$. Since d is separately continuous, $d(g^{-1}, -)$ is a continuous map. Bearing in mind that $d(g, -) \circ d(g^{-1}, -)$ is the identity map on X we get that $d(g^{-1}, -)$ is an autohomeomorphism. Hence $d_f(g, -)$ is an autohomeomorphism of $C(X)$ with respect to both the topology of uniform convergence and that of pointwise convergence on X. Since $\varphi(G)$ is compact, by Proposition 1.13 there exists a function $q_0 \in \varphi(G)$ such that the restrictions to $\varphi(G)$ of the topologies of pointwise and uniform convergence coincide at point q_0. The action of G on $\varphi(G) = d_f(G, q)$ is transitive, i.e., for any point $q_1 \in \varphi(G)$ there exists $g_1 \in G$ for which $d_f(g_1, q_0) = q_1$. Since

$$d_f(g_1, -)\big|_{\varphi(G)}$$

is an autohomeomorphism of set $\varphi(G)$ with respect to the topology of uniform convergence as well as with respect to the topology of pointwise convergence, and as the restrictions of these topologies to $\varphi(G)$ coincide at q_0, they also coincide at the point q_1.

The point $q_1 \in \varphi(G)$ had been chosen arbitrarily, hence the restrictions to $\varphi(G)$ of the topologies of pointwise and uniform convergence coincide. Therefore, the map

$$\varphi : G \to C_u(X)$$

is continuous. Assertion 1.5 implies that f can be extended to a continuous function defined on $\beta(G) \times \beta(X)$. □

Theorem 2.2 and Proposition 1.11 yield the following theorem:

Theorem 2.3. *Let G and X be pseudocompact spaces, (G, X) a Grothendieck pair, and let there be a group structure on G,*

$$d: G \times X \to X$$

an action which is a separately continuous map. Then d can be extended to a continuous map

$$\hat{d}: \beta(G) \times \beta(X) \to \beta(X).$$

Theorem 2.2 and Proposition 1.12 imply the following:

Theorem 2.4. *Let G, X be pseudocompact spaces and there be a group structure on G. Let*

$$d: G \times X \to X$$

be an action on X, and map d be continuous. Then d can be extended to a continuous map

$$\hat{d}: \beta(G) \times \beta(X) \to \beta(X).$$

Any group acts on itself through right multiplication; hence Proposition 2.1 and Theorems 2.3 and 2.4 imply the following two theorems:

Theorem 2.5. *Let $(G, *)$ be a group with separately continuous multiplication and (G, G) be a Grothendieck pair. Then $(G, *)$ is a topological group.*

Theorem 2.6. *Let $(G, *)$ be a group with continuous multiplication (i.e., paratopological group), and let the space G be pseudocompact. Then $(G, *)$ is a topological group.*

Theorem 2.5 and Corollary 1.9 imply the following:

Corollary 2.7. *Let $(G, *)$ be a pseudocompact group with separately continuous multiplication, and G belong to one of the classes below:*
(a) *countably compact spaces;*
(b) *spaces with countable tightness;*
(c) *separable spaces;*
(d) *k-spaces.*
Then $(G, *)$ is a topological group.

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