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A CONVERGENT FAMILY OF APPROXIMATE INERTIAL MANIFOLDS

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Abstract—A new method of construction of approximate inertial manifolds (AIMs) is derived for a very general class of evolution partial differential equations. We construct a family $(\mathcal{M}_N)_{N\in\mathbb{N}}$ and show that when the spectral gap condition holds, it converges to an exact inertial manifold. When it does not hold, we prove that the attractor—when it exists—is contained in a thin neighborhood of the AIM $\mathcal{M}_N$ and when N is large, the thinness decreases exponentially with respect to the dimension of $\mathcal{M}_N$.

1. INTRODUCTION

Let $\mathcal{E} \subset E \subset F$ be three Banach spaces. We consider an abstract evolution equation in E:

$$\begin{aligned} \frac{du}{dt} + Au &= f(u), \\ u(0) &= u_0. \end{aligned} \tag{1}$$

Here, A is a densely defined linear unbounded operator in $\mathcal{E}$, it generates a strongly continuous semigroup $(e^{-At})_{t\geq 0}$; and f denotes a nonlinear function from E to F. Under appropriate asumptions, (1) defines a continuous semigroup $(S(t))_{t\geq 0}$ in E.

An inertial manifold (IM) for $(S(t))_{t\geq 0}$ is a finite-dimensional Lipschitz manifold $\mathcal{M}$ that attracts exponentially all orbits and is positively invariant.

The question of existence of IMs has been addressed in several articles (see [1] or also [2] for further references), and it is known that it exists under a restriction on A: the spectral gap condition. We shall give, in Section 2, a theorem that gathers all the results of existence.

Also, C. Foias, O. Manley and R. Temam [3] developed the notion of approximate inertial manifold (AIM). A set $\mathcal{M}$ is an AIM of order η if it is a finite-dimensional smooth manifold such that for all u_0 in E:

$$d(S(t)\,u_0, \mathcal{M}) \leq \eta, \qquad \text{for } t \geq T(u_0).$$

Again, numerous works have been devoted to the existence of AIMs (see [2] for references), and for many equations, families of AIMs have been constructed.

In [4] and [5], it is explained how to construct a family $(\mathcal{M}_N)_{N\in\mathbb{N}}$ such that the order of $\mathcal{M}_N$ is:

$$\eta_N = \frac{K_N}{\lambda_n^N}, \tag{2}$$

where n is the dimension of $\mathcal{M}_N$ and λ_n is the n^{th} eigenvalue of A.

In [6], another type of family is given. The order of the N^{th} AIM is:

$$\eta_N = c_1 \left(\frac{1}{2^N} + e^{-\delta\lambda_n} \right), \tag{3}$$

but unlike all the other constructions, these AIMs are not graphs.

A natural question is: what is the behaviour of $\mathcal{M}_N$ when N increases. For the first type of family, the bound (2) is not fully satisfactory since K_N is not known explicitly. The bound (3) is better, η_N clearly goes to $c_1 e^{-\delta\lambda_n}$, which is very small. It shows that for large N, $\mathcal{M}_N$ has an exponential order, i.e., its order decreases exponentially with its dimension N. (See [3] where it is shown that there exist AIMs with exponential order, but the proof there is not constuctive.) But it is not known if the family converges to an exact IM when it exists.

Our aim in the following sections is to construct a family $(\mathcal{M}_N)_{N\in\mathbb{N}}$ of AIMs such that: they are graphs; $\mathcal{M}_N$ converges to an exact IM when it is known to exist; $\mathcal{M}_N$ has an exponential order for large N otherwise.

2. CONSTRUCTION OF THE FAMILY

Let us first give the existence result for the existence of IMs. We assume that the function f is globally Lipschitz and that there exists a sequence of eigenprojectors $(P_n)_{n\in\mathbb{N}}$ of A associated to two sequences of numbers $(\Lambda_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ such that:

$$\Lambda_n \geq \lambda_n \geq 0, \qquad \text{for all } n \geq 0,$$

and there exist K_1, K_2 positive, α in $[0,1)$:

$$\left.\begin{aligned} |e^{-At}P_n|_{\mathcal{L}(E)} &\leq K_1 e^{-\lambda_n t}, \\ |e^{-At}P_n|_{\mathcal{L}(F,E)} &\leq K_1 \lambda_n^\alpha e^{-\lambda_n t}, \end{aligned}\right. \qquad \text{for } t \leq 0, \tag{4}$$

$$\left.\begin{aligned} |e^{-At}Q_n|_{\mathcal{L}(F,E)} &\leq K_2\left(\frac{1}{|t|^\alpha} + \Lambda_n^\alpha\right) e^{-\Lambda_n t}, \\ |A^{-1}e^{-At}Q_n|_{\mathcal{L}(F,E)} &\leq K_2 \Lambda_n^{\alpha-1} e^{-\Lambda_n t}, \end{aligned}\right. \qquad \text{for } t > 0, \tag{5}$$

where we have written: $Q_n = I - P_n$. Then we have:

THEOREM 1. *There exists a constant C_0 depending on f, K_1, K_2 such that, if $\Lambda_n - \lambda_n \geq C_0(\Lambda_N^\alpha + \lambda_n^\alpha)$, then the semigroup defined by (1) possesses an IM which is a graph of a C^1 globally Lipschitz function from $P_n E$ to $Q_n E$.*

To prove this theorem, we consider the mapping $\mathcal{T}$ defined on:

$$\mathcal{F}_{b,l} = \left\{\Psi : P_n E \to Q_n E / \sup_{y\in P_n E} \frac{|\Psi(y)|_E}{1+|y|_E} \leq b,\ \operatorname{Lip}\Psi \leq l\right\},$$

by:

$$\mathcal{T}\Psi(y_0) = \int_{-\infty}^0 e^{As} Q_n\, f(y(s) + \Psi(y(s)))\, ds, \tag{6}$$

where y is the solution of:

$$\left.\begin{aligned} \frac{dy}{dt} + Ay &= P_n f(y + \Psi(y)), \\ y(0) &= y_0; \end{aligned}\right. \tag{7}$$

and we show that $\mathcal{T}$ is a strict contraction on $\mathcal{F}_{b,l}$.

The idea to construct AIMs is to approximate $\mathcal{T}$ by explicit mappings. We first use a numerical scheme to compute an approximation of the solution of (2):

$$y_{k+1} = R_\tau y_k + S_\tau P_n f(y_k + \Psi(y_k)). \tag{8}$$

We expect y_k to be close to $y(-k\tau)$, τ being the time step, and y the solution of (2). To ensure stability, we assume certains bounds on the operators R_τ and S_τ that are similar to (2). Then, a quadrature formula is used to replace (6):

$$\mathcal{T}_N^\tau \Psi(y_0) = A^{-1} \sum_{k=0}^{N-1} e^{kA\tau} Q_n f(y_k + \Psi(y_k)) + A^{-1} e^{-NA\tau} Q_n f(y_N + \Psi(y_N)). \tag{9}$$

We have a two parameter family $\mathcal{T}_N^\tau$.

We choose a sequence $(\tau_N)_{N\in\mathbb{N}}$ and define:

$$\Phi_{N+1} = \mathcal{T}_N^{\tau_N}(\Phi_N), \qquad \Phi_0 = 0.$$

The N^{th} manifold $\mathcal{M}_N$ is the graph of Φ_N.

3. APPROXIMATION RESULTS

When equation (1) possesses a global attractor $\mathcal{A}$, we can assume that the function f is bounded and that the projection on P_nE of a solution on $\mathcal{A}$ satisfies (8) with a consistency error equal at most to τ times a constant that does not depend on k. This assumption enables us to prove:

THEOREM 2. *There exist C_1, C_2, C_3 such that, if $C_2 \leq \tau_N(N+1)\lambda_n^\alpha \leq C_1$, then for all N, Φ_N is bounded and Lipschitz and:*

$$d(\mathcal{A}, \mathcal{M}_N) \leq \epsilon(N, \tau_N) + C_3\Lambda_n^{\alpha-1}e^{-\Lambda_n^{1-\alpha}},$$

where $\epsilon(N,\tau)$ goes to zero when $N \to \infty$ and $\tau \to 0$.

We deduce that for N large, $\mathcal{M}_N$ has exponential order.

We now assume that the spectral condition holds, therefore, Theorem 1 provides an IM $\mathcal{M}$. To have the convergence of the family to $\mathcal{M}$ it is clear from (9), that we need:

$$N\tau_N \to \infty, \qquad \text{when } N \to \infty. \tag{10}$$

Then under a consistency assumption on (8) that is easy to check, we show:

THEOREM 3. *There exists C_4 depending on f, b, l such that if $\Lambda_n - \lambda_n \geq C_4(\Lambda_n^\alpha + \lambda_n^\alpha)$ then for all N, Φ_N is in $\mathcal{F}_{b,l}$, and $\Phi_N \to \Phi$ in the C^0 topology. Moreover, if f has a compact support, then Φ_N converges to Φ in the C^1 topology.*

4. APPLICATIONS

The first example is the case of a parabolic semilinear partial differential equation with a symmetric linear part. In this case, A is a selfadjoint operator in a Hilbert space $\mathcal{E} = H$ of the type $(-\Delta)^r$ and f is a nonlinear function from $E = D(A^\beta)$ to $F = D(A^{\beta-\alpha})$. Assumptions (2),(2) are easily checked if P_n is the eigenprojector of A corresponding to the n first eigenvalues, and λ_n, Λ_n are the n^{th} and $(n+1)^{\text{th}}$ eigenvalues. We assume that the equation has an attractor in E, then f can be truncated so that it is bounded, Lipschitz, and has a compact support. We consider a simple Euler scheme for (8):

$$R_\tau = I + \tau A, \qquad S_\tau = \tau I.$$

Theorem 2 always holds while Theorems 1 and 3 are true when the spectral gap condition is satisfied.

Numerous equations enter in this setting; they include Cahn-Hilliard, Kuramoto-Sivashinsky, or Navier-Stokes equations.

A second application is provided by the theory of slow manifolds in meteorology. The operator A is the sum of a selfadjoint operator and a lower order perturbation. The same scheme as above is considered and similar results hold (see [7]).

The last type of equations is a complex amplitude equation. Now A is the sum of a seladjoint operator and of a skewsymmetric operator. The approximation (8) is:

$$R_\tau = e^{\tau A}, \qquad S_\tau = A^{-1}(e^{\tau A} - I).$$

This setting applies to the Ginzburg-Landau equation. It has a global attractor and its nonlinearity can be truncated. All the results apply when the spectral gap condition holds. On the contrary, the Laser equation does not possess absorbing balls. Since the spectral gap condition is satisfied but the nonlinear term does not have a compact support, Theorem 1 and only the first part of Theorem 3 apply.

REFERENCES

1. C. Foias, G.R. Sell, R. Temam, Inertial manifolds for nonlinear evolutionary equations, *J. Diff. Eqs.* **73**, 309–353 (1988).
2. A. Debusche, R. Temam, Convergent families of approximate inertial manifolds, *J. Math. Pures. Appl.* (to appear).
3. C. Foias, O. Manley, R. Temam, On the interaction of small and large eddies in two-dimensional turbulent flows, *Math. Modelling and Numerical Analysis, M2AN* **22**, 93–114 (1988).
4. A. Debussche, M. Marion, On the construction of families of approximate inertial manifolds, *J. Diff. Eqs.* (to appear).
5. R. Temam, Attractors for the Navier-Stokes equations: Localization and approximation, *J. Fac. Sci. Tokyo* **36** (Sec. 1A), 629–647 (1989).
6. C. Foias, R. Temam, Approximation of attractors by algebraic and analytic sets, The Institute for Scientific Computing, Bloomington, Indiana, Preprint # 9004.
7. A. Debussche, R. Temam, Inertial manifolds and the slow manifolds in meteorology, *Diff. and Int. Equ.* **4** (5), 897–931 (1991).