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Singular Values of Compact Pseudodifferential Operators*

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This paper investigates the asymptotic decay of the singular values of compact operators arising from the Weyl correspondence. The motivating problem is to find sufficient conditions on a symbol which ensure that the corresponding operator has singular values with a prescribed rate of decay. The problem is approached by using a Gabor frame expansion of the symbol to construct an approximating finite rank operator. This establishes a variety of sufficient conditions for the associated operator to be in a particular Schatten class. In particular, an improvement of a sufficient condition of Daubechies for an operator to be trace-class is obtained. In addition, a new development and improvement of the Calderón–Vaillancourt theorem in the context of the Weyl correspondence is given. Additional results of this type are then obtained by interpolation. © 1997 Academic Press

1. INTRODUCTION

In this paper, we investigate the decay of the singular values of compact operators on $L^2(\mathbf{R}^n)$ arising from the Weyl correspondence. The Weyl

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correspondence is a formalism that bijectively associates to any continuous linear operator $L\colon \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ a distributional symbol $\sigma \in \mathcal{S}'(\mathbf{R}^{2n})$. The Weyl correspondence plays an important role in a variety of contexts, including quantum mechanics and partial differential equations [How80]. Our interest stemmed from recent results in the signal processing literature, where the decay properties of singular values have been proposed as a tool to determine the quality of time-frequency filters [Fla88, HK94, RT93, RT94]. The motivating problem is to find sufficient conditions on the symbol which ensure that the corresponding operator has singular values with a prescribed rate of decay. Our analysis leads us further to consider when the corresponding operators are bounded. These questions have a long and venerable history [Fol89], as does a related question which we do not treat here: which symbols give rise to positive operators? This latter question also connects to signal processing applications, where it is related to the positivity of time-frequency distributions [Jan84].

Let H be a separable Hilbert space and let $\mathcal{B}(H)$ be the space of all bounded operators mapping H into itself. $\mathcal{B}(H)$ is a Banach space under the operator norm $\|\cdot\|_{\mathcal{B}(H)}$. The singular values $\{s_k(L)\}_{k=1}^{\infty}$ of a compact operator $L \in \mathcal{B}(H)$ are defined via spectral theory. Since L is compact, the non-negative operator L^*L has a discrete spectrum tending towards zero. The singular value $s_k(L)$ coincides with the square root of the kth largest eigenvalue of L^*L, i.e., $s_k(L) = \lambda_k(L^*L)^{1/2}$. Alternatively, since H is a Hilbert space, the singular values of such a compact operator L coincide with the approximation numbers of L, i.e.,

$$s_k(L) = a_k(L) = \inf\{\|L - T\|_{\mathcal{B}(H)} : \operatorname{rank}(T) < k\}. \tag{1.1}$$

One way of quantifying the rate of decay of the singular values of a compact operator L is by determining the l^p class to which they belong. This leads to the definition of the *Schatten class* $\mathcal{I}_p$ as the set of all compact operators $L \in \mathcal{B}(H)$ for which the sequence of singular values $\{s_k(L)\}$ is in l^p. In particular, $\mathcal{I}_\infty$ is the space of all compact operators on H. Other useful identifications are that $\mathcal{I}_1$ is the space of all trace-class operators on H and that $\mathcal{I}_2$ is the space of all Hilbert–Schmidt operators on H. The Schatten class $\mathcal{I}_p$ is a Banach space under the norm

$$\|L\|_{\mathcal{I}_p} = \|\{s_k(L)\}\|_{l^p} = \begin{cases} \left(\sum_k s_k(L)^p\right)^{1/p}, & 1 \leqslant p < \infty, \\ \sup_k s_k(L) = s_1(L) = \|L\|_{\mathcal{B}(H)}, & p = \infty. \end{cases}$$

The singular values of two compact operators L_1, L_2 obey the inequality $s_{k+l+1}(L_1 + L_2) \leqslant s_{k+1}(L_1) + s_{l+1}(L_2)$ [DS88, p. 1089]. As a consequence, the following refinement of Eq. (1.1) holds for Hilbert–Schmidt operators $L \in \mathscr{I}_2$:

$$\sum_{k>N} s_k(L)^2 \leqslant \inf \{ \|L - T\|^2_{\mathscr{I}_2} : \operatorname{rank}(T) \leqslant N \}. \tag{1.2}$$

This inequality will play a key role in our later estimates.

Useful variants of the Schatten classes are obtained by replacing the l^p norm of the singular values by the Lorentz space $l^{p,q}$ quasi-norm. We define $\mathscr{I}_{p,q}$ to be the space of all compact operators $L \in B(H)$ such that

$$\|L\|_{\mathscr{I}_{p,q}} = \|\{s_k(L)\}\|_{l^{p,q}} = \begin{cases} \left(\sum_k (k^{1/p - 1/q} s_k(L))^q \right)^{1/q}, & 1 \leqslant q < \infty, \\ \sup_k k^{1/p} s_k(L), & q = \infty, \end{cases}$$

is finite. Although $\|\cdot\|_{\mathscr{I}_{p,q}}$ is only a quasi-norm and not a norm, the spaces $\mathscr{I}_{p,q}$ have been well-studied, and their behavior under interpolation is known.

A recurring theme in the study of integral operators L is that the rate of decay of the singular values can be controlled by the smoothness and decay of the associated kernel k. This theme has a rich history, much of which has been recorded in the books by König [Kön86] and Pietsch [Pie87]. A typical strategy in this type of problem is to approximate the kernel k of L by a suitable kernel k_N whose associated operator L_N is of finite rank. Equations (1.1) or (1.2) then yield a bound on the singular values of L, provided that the error between the operator L and its finite rank approximation L_N is controlled by the error between the kernel k and its approximation k_N. Such an approach can be used to prove that a compact operator is within a particular Schatten p-class.

In the Weyl pseudodifferential calculus, the operator L is defined via its *symbol* σ instead of its kernel k. We review the fundamentals of the Weyl correspondence in Section 2. Operators with reasonable symbols may give rise to kernels that are defined only in the sense of distributions and conversely. As a consequence, it is natural to attempt the approximation strategy outlined above on the symbol side instead of the kernel side. This is the approach we take in this paper. We expand the symbol σ in terms of a *Gabor frame* generated by a Gaussian function and use the fact that the partial sums of this expansion are naturally associated to finite rank operators. Frames, Gabor frames, and frame expansions of the symbol σ are covered in Section 3. This method enables us to establish a variety of sufficient conditions for an operator to be in a particular Schatten class. These conditions involve the rate of decay of the L^2-norm of the symbol σ and its Fourier transform $\hat{\sigma}$ outside of large balls B_r centered at the origin. Our main result in this direction is the following.

THEOREM 1.1. *If $\sigma \in L^2(\mathbf{R}^{2n})$ then there exists constants $\varepsilon, C_1, C_2 > 0$ such that the singular values of the Weyl transform L_σ satisfy*

$$\sum_{k>N} s_k(L_\sigma)^2 \leqslant C_1 S_\varepsilon(C_2 N^{1/2n}), \tag{1.3}$$

for every $N > 0$, where

$$S_\varepsilon(r) = \|\sigma \cdot \chi_{B_r^C}\|_{L^2}^2 + \|\hat{\sigma} \cdot \chi_{B_r^C}\|_{L^2}^2 + e^{-\varepsilon r^2}(\|\sigma \cdot \chi_{B_r}\|_{L^2}^2 + \|\hat{\sigma} \cdot \chi_{B_r}\|_{L^2}^2). \tag{1.4}$$

In particular,

$$s_{2k}(L_\sigma)^2 \leqslant \frac{C_1 S_\varepsilon(C_2 k^{1/2n})}{k}. \tag{1.5}$$

A *Gabor system* has the form $\{e^{2\pi i q x}\phi(x+p)\}_{(p,q)\in\Lambda}$, where Λ is a lattice in $\mathbf{R}^{2n}$ and $\phi \in L^2(\mathbf{R}^n)$. Such a system of time-frequency shifts of ϕ is uniquely suited to analysis in the context of the Weyl correspondence. However, no such Gabor system can be a basis if ϕ is simultaneously well-localized in both time and frequency. Our technique specifically requires such simultaneous localization, and in fact we take ϕ to be a Gaussian function. However, by taking the lattice Λ with sufficiently high density, this Gabor system is a *frame*, i.e., there is a norm equivalence between $\|f\|_{L^2}$ and $\|\{\langle f, e^{2\pi i q x}\phi(x+p)\rangle\}\|_{l^2}$. Moreover, there is a basis-like expansion of f in terms of the frame elements. The expansion coefficients are not necessarily unique, but this nonuniqueness is irrelevant to our purposes.

The proof of Theorem 1.1 is given in Section 4, and implications of this result are discussed in Section 5. For example, Theorem 1.1 leads immediately to an improvement of a sufficient condition of Daubechies for an operator to be trace-class [Dau80], namely, we show that $L_\sigma \in \mathscr{I}_1$ if both σ and $\hat{\sigma}$ lie in a Sobolev space $H^{n+\varepsilon}$, rather than $H^{2n+\varepsilon}$. In fact, we show that $L_\sigma \in \mathscr{I}_{2n/(2n+\varepsilon),\infty}$ if $\sigma, \hat{\sigma} \in H^{n+\varepsilon}$.

In Section 6, we again employ Gabor frame expansions of the symbol to give a new development and improvement of the *Calderón–Vaillancourt theorem* in the context of the Weyl correspondence. The usual Calderón–Vaillancourt theorem states that L_σ is a bounded operator on $L^2(\mathbf{R}^n)$ if $\sigma \in C^{2n+1}(\mathbf{R}^{2n})$, i.e., σ and all derivatives of order $2n+1$ or less are bounded, continuous functions on $\mathbf{R}^{2n}$. We obtain the following improvement, stated in terms of the Hölder–Zygmund classes $\Lambda^s(\mathbf{R}^{2n})$.

THEOREM 1.2. *If $\sigma \in \Lambda^s(\mathbf{R}^{2n})$ with $s > 2n$, then L_σ is a bounded operator on $L^2(\mathbf{R}^n)$.*

Finally, in Section 7 we use interpolation to obtain an extension of Theorem 1.2.

Notation. The usual dot product of two points $x, y \in \mathbf{R}^n$ is denoted by a simple juxtaposition, i.e., $xy = x_1 y_1 + \cdots + x_n y_n$. The length of x is $|x| = \sqrt{x^2} = \sqrt{xx}$. The cardinality of a finite set F is $|F|$, and the Lebesgue measure of a subset $E \subset \mathbf{R}^n$ is also denoted $|E|$. The characteristic function of $E \subset \mathbf{R}^n$ is χ_E, and the complement of E is E^{C}. The ball in $\mathbf{R}^n$ of radius r centered at x is $B_r(x)$. When $x = 0$ we write $B_r = B_r(0)$. The translation of a function f by $y \in \mathbf{R}^n$ is $\tau_y f(x) = f(x - y)$.

The space $L^p(\mathbf{R}^n)$ consists of complex-valued functions f on $\mathbf{R}^n$ with norm $\|f\|_{L^p} = (\int |f(x)|^p \, dx)^{1/p}$. The inner product of f, $g \in L^2(\mathbf{R}^n)$ is $\langle f, g \rangle = \int f(x) \overline{g(x)} \, dx$. $C(\mathbf{R}^n)$ is the space of bounded, continuous functions on $\mathbf{R}^n$. $C_c(\mathbf{R}^n)$ is the space of continuous functions with compact support. $C_0(\mathbf{R}^n)$ is the space of continuous functions on $\mathbf{R}^n$ vanishing at infinity. $C^k(\mathbf{R}^n)$ is the space of bounded, continuous functions possessing bounded, continuous derivatives up to order k. $\mathcal{S}(\mathbf{R}^n)$ denotes the Schwartz space of all infinitely differentiable functions on $\mathbf{R}^n$ decaying rapidly at infinity, and $\mathcal{S}'(\mathbf{R}^n)$ is its topological dual, the space of tempered distributions.

The Fourier transform of $f \in L^1(\mathbf{R}^n)$ is $\hat{f}(\gamma) = \mathcal{F} f(\gamma) = \int f(x) \, e^{-2\pi i \gamma x} \, dx$; the inverse Fourier transform is $\check{f}(\gamma) = \mathcal{F}^{-1} f(\gamma) = \hat{f}(-\gamma)$. The Fourier transform maps $\mathcal{S}(\mathbf{R}^n)$ onto itself, and extends to $\mathcal{S}'(\mathbf{R}^n)$ by duality.

The Sobolev space $H^s(\mathbf{R}^n)$ is defined by the norm

$$\|f\|_{H^s}^2 = \|\hat{f}(\gamma)(1+\gamma^2)^{s/2}\|_{L^2}^2 = \int |\hat{f}(\gamma)|^2 \, (1+\gamma^2)^s \, d\gamma.$$

The Hölder–Zygmund class are denoted by $\Lambda^s(\mathbf{R}^n)$. For noninteger $s > 0$, say $s = k + \varepsilon$, $\Lambda(\mathbf{R}^n)$ consists of functions $f \in C^k(\mathbf{R}^n)$ such that for each multi-index α with order $|\alpha| = k$, the derivative $f^{(\alpha)}$ satisfies a Hölder condition $|D^\alpha f(x) - D^\alpha(y)| \leqslant K \, |x - y|^\varepsilon$.

The Besov spaces on $\mathbf{R}^n$ are denoted $B^s_{p,q}(\mathbf{R}^n)$. We have $\Lambda^s(\mathbf{R}^n) = B^s_{\infty,\infty}(\mathbf{R}^n)$ when $s > 0$ and $H^s(\mathbf{R}^n) = B^s_{2,2}(\mathbf{R}^n)$ when $s > 0$.

2. BACKGROUND: THE WEYL CORRESPONDENCE

In Sections 2.1–2.3 we review some basic facts on the Weyl correspondence as a tool for constructing pseudodifferential operators. We follow the book of Folland [Fol89] closely.

2.1. *The Schrödinger Representation*

The Schrödinger representation ρ of the Heisenberg group $\mathbf{H}^n = \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ is defined by

$$\rho(p, q, t)f(x) = e^{2\pi it}\, e^{\pi ipq}\, e^{2\pi iqx} f(x+p).$$

Each $\rho(p, q, t)$ is a unitary mapping of $L^2(\mathbf{R}^n)$ onto itself. The composition of two such operators follows the rule

$$\rho(p, q, t)\, \rho(p', q', t') = \rho(p+p', q+q', t+t' + \tfrac{1}{2}(pq' - p'q)).$$

This rule determines the group law on $\mathbf{H}^n$. In many considerations the t-variable is unimportant, so for $\alpha = (p, q) \in \mathbf{R}^{2n}$ we define

$$\rho(\alpha)f(x) = \rho(p, q)f(x) = \rho(p, q, 0)f(x) = e^{\pi ipq}\, e^{2\pi iqx} f(x+p).$$

We refer to $\rho(\alpha)f = \rho(p, q)f$ as a *time-frequency shift* of f. In particular, $\rho(p, 0)f = \tau_{-p}f$ is a translate of f. We have the formula $(\rho(p, q)f)^\wedge = \rho(-q, p)\hat{f}$.

2.2. *The Ambiguity Function and the Wigner Distribution*

The (cross)-*ambiguity function*, or *Fourier–Wigner transform*, of $f, g \in L^2(\mathbf{R}^n)$ is

$$\begin{aligned} A(f, g)(p, q) &= \langle \rho(p, q)f, g \rangle \\ &= \int e^{\pi ipq}\, e^{2\pi iqx} f(x+p)\, \overline{g(x)}\, dx \\ &= \int e^{2\pi iqx} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)}\, dx. \end{aligned}$$

Regarded as is a bilinear mapping $L^2(\mathbf{R}^n) \times L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^{2n})$, the ambiguity function extends in the obvious way to a linear mapping $\tilde{A}$ on the tensor product $L^2(\mathbf{R}^n) \otimes L^2(\mathbf{R}^n)$, which is naturally isomorphic to $L^2(\mathbf{R}^{2n})$. In particular, $\tilde{A}(F)(p, q) = \int e^{2\pi iqx} F(x + p/2, x - p/2)\, dx$ is a unitary mapping of $L^2(\mathbf{R}^{2n})$ onto itself which also maps $\mathcal{S}(\mathbf{R}^{2n})$ onto itself and extends to a continuous bijection of $\mathcal{S}'(\mathbf{R}^{2n})$ onto itself. Analogues of these facts transfer back to A on $L^2(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$, $\mathcal{S}(\mathbf{R}^n) \times \mathcal{S}(\mathbf{R}^n)$, and $\mathcal{S}'(\mathbf{R}^n) \times \mathcal{S}'(\mathbf{R}^n)$.

The (cross-) *Wigner distribution* of $f, g \in L^2(\mathbf{R}^n)$ is the Fourier transform of the ambiguity function of f and g,

$$W(f, g)(\xi, x) = A(f, g)\,\hat{}\,(\xi, x)$$
$$= \iint A(f, g)(p, q)\, e^{-2\pi i(p\xi + qx)}\, dp\, dq$$
$$= \int e^{-2\pi i p\xi} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)}\, dp.$$

As with the ambiguity function, the Wigner distribution extends from a bilinear map $W: L^2(\mathbf{R}^n) \times L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^{2n})$ to a unitary map $\tilde{W}$ of $L^2(\mathbf{R}^{2n})$ onto itself which is also a continuous bijection of $\mathscr{S}(\mathbf{R}^{2n})$ and $\mathscr{S}'(\mathbf{R}^{2n})$ onto themselves. Transferring these facts back to W, and combining them with other elementary calculations, we obtain the following useful facts about the Wigner distribution.

PROPOSITION 2.1. *Let $f, g \in L^2(\mathbf{R}^n)$ and let $a, b, c, d \in \mathbf{R}^n$. Then*

(a) $W(f, g) \in L^2(\mathbf{R}^{2n})$, *with* $\|W(f, g)\|_{L^2} = \|f\|_{L^2} \|g\|_{L^2}$.

(b) $W(f, g) \in C_0(\mathbf{R}^{2n})$, *and* $\|W(f, g)\|_{L^\infty} \leqslant \|f\|_{L^2} \|g\|_{L^2}$.

(c) $W(g, f) = \overline{W(f, g)}$.

(d) $W(\hat{f}, \hat{g})(\xi, x) = W(f, g)(x, -\xi)$.

(e) $W(\rho(a, b)f, \rho(c, d)g)(\xi, x)$
$= e^{\pi i(bc - ad)}\, e^{2\pi i((a-c)\xi + (b-d)x)}\, W(f, g)(\xi - (b+d)/2, x + (a+c)/2)$.

(f) (*Moyal's Identity*) $\langle W(f_1, g_1), W(f_2, g_2)\rangle = \langle f_1, f_2\rangle\langle g_2, g_1\rangle$.

Let us define the linear transformation $M: \mathbf{R}^{2n} \times \mathbf{R}^{2n} \to \mathbf{R}^{4n}$ by the formula

$$M(\alpha, \beta) = M(\alpha_1, \alpha_2, \beta_1, \beta_2) = \left(-\frac{\alpha_2 + \beta_2}{2}, \frac{\alpha_1 + \beta_1}{2}, \alpha_1 - \beta_1, \alpha_2 - \beta_2\right) \tag{2.1}$$

for $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbf{R}^{4n}$. Then the identity in Proposition 2.1(e) can be restated

$$W(\rho(\alpha)f, \rho(\beta)g) = \rho(M(\alpha, \beta))\, W(f, g), \tag{2.2}$$

where ρ on the left-hand side of (2.2) is the Schrödinger representation of $\mathbf{H}^n$ while ρ on the right-hand side is the Schrödinger representation of $\mathbf{H}^{2n}$.

2.3. *The Weyl Correspondence*

A continuous linear operator $L: \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}'(\mathbf{R}^n)$ is a *pseudodifferential operator*. The *Weyl correspondence* employs the Wigner distribution to define a 1-1 correspondence between tempered distributions $\sigma \in \mathscr{S}'(\mathbf{R}^{2n})$ and pseudodifferential operators $L_\sigma: \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}'(\mathbf{R}^n)$. The distribution σ is the *symbol* of the operator L_σ, and L_σ is the *Weyl transform* of σ. The Weyl transform L_σ is defined implicitly by the equation

$$\begin{aligned}\langle L_\sigma f, g\rangle &= \langle \sigma, W(g, f)\rangle \\ &= \langle \sigma, \overline{W(f, g)}\rangle \\ &= \iint \sigma(\xi, x)\, W(f, g)(\xi, x)\, d\xi\, dx, \end{aligned} \tag{2.3}$$

or explicitly in kernel form by

$$L_\sigma f(x) = \iint \sigma\left(\xi, \frac{x+y}{2}\right) e^{2\pi i(x-y)\xi} f(y)\, dy\, d\xi.$$

Of course, operators L_σ arising from distributional symbols $\sigma \in \mathscr{S}'(\mathbf{R}^{2n})$ will be defined *a priori* only on Schwartz-class functions and will take values in the space of tempered distributions $\mathscr{S}'(\mathbf{R}^n)$. It is therefore natural to ask when a given symbol is associated with a bounded operator on $L^2(\mathbf{R}^n)$. The following theorem summarizes some known facts along these lines.

THEOREM 2.2. *Given $1 \leqslant p \leqslant 2$, let p' satisfy $1/p + 1/p' = 1$. Then the Weyl correspondence is a continuous mapping of symbols $\sigma \in L^p(\mathbf{R}^{2n})$ to operators $L_\sigma \in \mathscr{I}_{p'}$, i.e., there exists a constant C_p so that*

$$\forall \sigma \in L^p(\mathbf{R}^{2n}), \qquad \|L_\sigma\|_{\mathscr{I}_{p'}} \leqslant C_p\, \|\sigma\|_{L^p}.$$

Moreover, for $p = 2$ the Weyl correspondence is a unitary bijection of $L^2(\mathbf{R}^{2n})$ onto $\mathscr{I}_2$. In particular,

$$\forall \sigma \in L^2(\mathbf{R}^{2n}), \qquad \|L_\sigma\|_{\mathscr{I}_2} = \|\sigma\|_{L^2}.$$

Observe that Theorem 2.2 implies

$$\sigma \in L^1(\mathbf{R}^{2n}) \quad \Rightarrow \quad L_\sigma \text{ is compact},$$

$$\sigma \in L^2(\mathbf{R}^{2n}) \quad \Leftrightarrow \quad L_\sigma \text{ is Hilbert–Schmidt}.$$

The proof of Theorem 2.2 for the case $p = 1$ can be found in [Fol89]. The case $p = 2$ is due to Pool [Poo66]. The case $1 < p < 2$ is due to Howe [How80], and follows by interpolating between the $p = 1$ and $p = 2$ cases.

3. FRAMES AND APPROXIMATION OF SYMBOLS

In Sections 3.1–3.3 we review basic properties of frames, and show how a Gabor frame expansion of the symbol σ can be used to construct finite-rank approximations to the Weyl transform L_σ.

3.1. *Frames*

Let H be a separable Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and let I be a countable index set. Then a sequence $\{f_i\}_{i\in I}$ of elements of H is a *frame* for H if there exists constants $A, B>0$ so that the following *approximate Plancherel formula* holds:

$$\forall f\in H, \qquad A\,\|f\|^2 \leqslant \sum_i |\langle f, f_i\rangle|^2 \leqslant B\,\|f\|^2. \tag{3.1}$$

The numbers A, B are *frame bounds*. The frame is *tight* if $A=B$. The frame is *exact* if it ceases to be a frame when any one of its elements is deleted.

Frames were introduced by Duffin and Schaeffer [DS52] in the context of nonharmonic Fourier series. Frame analysis has seen a recent resurgence with the advent of wavelet theory and the continuing development of Gabor analysis. Expository treatments of frames can be found in [HW89], [Dau92].

The class of exact frames for H coincides with the class of Riesz bases for H, which coincides with the class of bounded unconditional bases for H. Inexact frames are not bases, yet lead to basis-like expansions of elements of the Hilbert space in terms of the frame elements. The utility of inexact frames lies in the fact that it is sometimes possible to construct inexact frames whose elements satisfy some desirable property even though this property is denied to the elements of any Riesz basis for H. Such is the case for the specific Gabor frames for $L^2(\mathbf{R}^n)$ that we will consider in Section 3.2.

The following result summarizes useful properties of frames.

PROPOSITION 3.1. *Let $\{f_i\}_{i\in I}$ be a frame for H with frame bounds A, B.*

(a) *The* coefficient mapping *$V: H\to l^2$ defined by $Vf=\{\langle f, f_i\rangle\}$ is continuous and injective, with $\|V\|^2\leqslant B$.*

(b) *The adjoint $V^*: l^2\to H$ is the continuous map defined by $V^*\{c_i\}=\sum c_i f_i$, and satisfies $\|V^*\|^2\leqslant B$. In particular,*

$$\forall \{c_i\}\in l^2, \qquad \left\|\sum_i c_i f_i\right\|^2 \leqslant B\sum_i |c_i|^2. \tag{3.2}$$

(c) *The* frame operator $Sf = V^*Vf = \sum \langle f, f_i \rangle f_i$ *is a positive, continuous, and invertible mapping of* H *onto itself. The frame definition* (3.1) *is equivalent to the property* $AI \leqslant S \leqslant BI$.

(d) *Define* $\tilde{f}_i = S^{-1} f_i$. *Then the* dual frame $\{\tilde{f}_i\}$ *is a frame for* H *with frame bounds* B^{-1}, A^{-1}.

(e) *The following series converge unconditionally in the norm of* H:

$$\forall f \in H, \qquad f = \sum_i \langle f, \tilde{f}_i \rangle f_i = \sum_i \langle f, f_i \rangle \tilde{f}_i. \tag{3.3}$$

The frame is exact if and only if Eq. (3.3) *is the unique representation of* f *as* $f = \sum c_i f_i$ *or* $f = \sum d_i \tilde{f}_i$.

We also require the following facts regarding frames.

LEMMA 3.2. *Let* $\{f_i\}_{i \in I}$ *be a frame for* H *with frame bounds* A, B, *frame operator* S, *and dual frame* $\{\tilde{f}_i\}_{i \in I}$.

(a) *If* $T: H \to H$ *is a continuous bijection then* $\{Tf_i\}_{i \in I}$ *is a frame for* H *with frame bounds* $A \, \|T^{-1}\|^{-2}$, $B \, \|T\|^2$, *frame operator* TST^*, *and dual frame* $\{(T^*)^{-1} \tilde{f}_i\}_{i \in I}$.

(b) $\{f_i \otimes f_j\}_{(i, j) \in I \times I}$ *is a frame for* $H \otimes H$ *with frame bounds* A^2, B^2, *frame operator* $S \otimes S$, *and dual frame* $\{\tilde{f}_i \otimes \tilde{f}_j\}_{(i, j) \in I \times I}$.

Proof. (a) First note that TST^* is a continuous, self-adjoint bijection of H onto itself, satisfying $TST^*f = \sum \langle f, Tf_i \rangle Tf_i$, $(TST^*)^{-1} (Tf_i) = (T^*)^{-1} \tilde{f}_i$ and $\langle TST^*f, f \rangle = \langle S(T^*f), (T^*f) \rangle$. Therefore $A \, \|T^*f\|^2 \leqslant \langle TST^*f, f \rangle \leqslant B \, \|T^*f\|^2$. The result then follows from calculation $\|T^{-1}\|^{-1} \, \|f\| = \|(T^*)^{-1}\|^{-1} \, \|f\| \leqslant \|T^*f\| \leqslant \|T^*\| \, \|f\| = \|T\| \, \|f\|$.

(b) Since S is a positive, invertible operator, it has an invertible square root $S^{1/2}$. We compute $f = S^{-1/2} S S^{-1/2} f = \sum \langle f, S^{-1/2} f_i \rangle S^{-1/2} f_i$. Therefore, by Proposition 3.1(c), $\{S^{-1/2} f_i\}$ is a tight frame for H with frame bound 1.

Note that $S^{1/2} \otimes S^{1/2}$ is a continuous, self-adjoint bijection of $H \otimes H$ onto itself, and $(S^{1/2} \otimes S^{1/2})^{-1} = S^{-1/2} \otimes S^{-1/2}$. If $F = \sum_{k=1}^{l} (g_k \otimes h_k) \in H \otimes H$ is any finite linear combination of simple tensors, then

$$\begin{aligned}
&\sum_{i, j} \langle F, (S^{-1/2} \otimes S^{-1/2})(f_i \otimes f_j) \rangle (S^{-1/2} \otimes S^{-1/2})(f_i \otimes f_j) \\
&\quad = \sum_{k=1}^{l} \left(\sum_i \langle g_k, S^{-1/2} f_i \rangle S^{-1/2} f_i \right) \otimes \left(\sum_j \langle h_k, S^{-1/2} f_j \rangle S^{-1/2} f_j \right) \\
&\quad = \sum_{k=1}^{l} (g_k \otimes h_k) = F.
\end{aligned}$$

This extends by continuity to all $F \in H \otimes H$. Therefore $\{(S^{-1/2} \otimes S^{-1/2})(f_i \otimes f_j)\}$ is a tight frame for $H \otimes H$ with frame bound 1. By part (a), $\{f_i \otimes f_j\}$ is therefore a frame for $H \otimes H$ with frame bounds $\|S^{-1/2} \otimes S^{-1/2}\|^{-2}$, $\|S^{1/2} \otimes S^{1/2}\|^2$. Since $\|S^{-1/2} \otimes S^{-1/2}\| \leqslant \|S^{-1/2}\|^2 \leqslant A^{-1}$ and $\|S^{1/2} \otimes S^{1/2}\| \leqslant \|S^{1/2}\|^2 \leqslant B$, the result follows. ∎

3.2. *Gabor Frames*

A current survey of Gabor theory and related issues can be found in [BHW95].

A subset $\Lambda \subset \mathbf{R}^{2n}$ is a *rectangular lattice* if it has the form $\Lambda = a_1 \mathbf{Z} \times \cdots \times a_{2n} \mathbf{Z}$. The *density* of Λ is $d(\Lambda) = 1/(a_1 \cdots a_{2n})$. The lattice is square if $a_1 = \cdots = a_{2n}$.

If $g \in L^2(\mathbf{R}^n)$ and $\Lambda \subset \mathbf{R}^{2n}$ is a rectangular lattice then the *Gabor system* generated by g and Λ is the collection $\{\rho(\alpha)\, g\}_{\alpha \in \Lambda}$ of time-frequency shifts of g along Λ. Gabor's fundamental work [Gab46] proposed using a Gabor system generated by the Gaussian function

$$\phi(x) = 2^{n/4} e^{-\pi x^2}$$

and a lattice Λ with density $d(\Lambda) = 1$. This Gabor system is complete in $L^2(\mathbf{R}^n)$, but it is not a frame. In fact, when the lattice Λ as density 1, any Gabor system that is a frame must be an exact frame, and the *Balian–Low theorem* implies that the generator g of any Gabor system that is an exact frame cannot be well-localized in both time and frequency. Seip and Wallstén [Sei92], [SW92] established, for the one-dimensional case $n = 1$, that the Gaussian function ϕ will generate a Gabor frame for $L^2(\mathbf{R}^n)$ for any lattice Λ with density $d(\Lambda) > 1$. Such a Gabor system must be inexact. Since $L^2(\mathbf{R}^{m+n}) = L^2(\mathbf{R}^m) \otimes L^2(\mathbf{R}^n)$, this construction extends to higher dimensions by Lemma 3.2(b).

THEOREM 3.3. *If Λ is a rectangular lattice in $\mathbf{R}^{2n}$ with density $d(\Lambda) > 1$, then the Gabor system $\{\rho(\alpha)\phi\}_{\alpha \in \Lambda}$ is a frame for $L^2(\mathbf{R}^n)$.*

For simplicity of notation, we will write

$$\phi_\alpha = \rho(\alpha)\phi.$$

We will let A_Λ, B_Λ denote the frame bounds for the frame $\{\phi_\alpha\}_{\alpha \in \Lambda}$ in $L^2(\mathbf{R}^n)$.

The dual frame of the Gabor frame $\{\phi_\alpha\}_{\alpha \in \Lambda}$ is itself a Gabor frame $\{\tilde{\phi}_\alpha\}_{\alpha \in \Lambda}$ using the same lattice but generated by a different function $\tilde{\phi} \in L^2(\mathbf{R}^n)$. In fact, $\tilde{\phi} = S^{-1}\phi$, where S is the frame operator for $\{\phi_\alpha\}_{\alpha \in \Lambda}$.

By Lemma 3.2(b), $\{\phi_\alpha(x)\, \overline{\phi_\beta(y)}\}_{(\alpha, \beta) \in \Gamma}$ forms a frame for $L^2(\mathbf{R}^{2n}) = L^2(\mathbf{R}^n) \otimes L^2(\mathbf{R}^n)$, where $\Gamma = \Lambda \times \Lambda$. By the discussion in Section 2.2, the Wigner distribution of ϕ_α and ϕ_β satisfies $W(\phi_\alpha, \phi_\beta) = \widetilde{W}(\phi_\alpha(x)\, \overline{\phi_\beta(y)})$, where $\tilde{W}$ is a unitary mapping of $L^2(\mathbf{R}^{2n})$ onto itself. Since frames are preserved by unitary mappings, we have the following fact.

LEMMA 3.4. *Let Λ be a rectangular lattice in $\mathbf{R}^{2n}$ with density $d(\Lambda) > 1$. Define*

$$\Phi_{\alpha,\beta} = W(\phi_\alpha, \phi_\beta) \qquad \text{and} \qquad \tilde{\Phi}_{\alpha,\beta} = W(\tilde{\phi}_\alpha, \tilde{\phi}_\beta)$$

and set

$$\Gamma = \Lambda \times \Lambda.$$

Then $\{\Phi_{\alpha,\beta}\}_{(\alpha,\beta)\in\Gamma}$ is a frame for $L^2(\mathbf{R}^{2n})$ with frame bounds A_Λ^2, B_Λ^2, dual frame $\{\tilde{\Phi}_{\alpha,\beta}\}_{(\alpha,\beta)\in\Gamma}$, and dual frame bounds B_Λ^{-2}, A_Λ^{-2}.

For later use, define

$$\Phi(\xi, x) = \Phi_{0,0}(\xi, x) = W(\phi, \phi)(\xi, x) = 2^n e^{-2\pi(\xi^2 + x^2)}.$$

Then, by equation (2.2),

$$\Phi_{\alpha,\beta} = W(\phi_\alpha, \phi_\beta) = \rho(M(\alpha, \beta))\Phi. \tag{3.4}$$

Note that while ϕ_α denotes a time-frequency shift of $\phi \in L^2(\mathbf{R}^n)$ by $\alpha \in \mathbf{R}^{2n}$, the notation $\Phi_{\alpha,\beta}$ describes a time-frequency shift of $\Phi \in L^2(\mathbf{R}^{2n})$ by $M(\alpha, \beta) \in \mathbf{R}^{4n}$.

3.3. *Approximation of Symbols*

Given any symbol $\sigma \in L^2(\mathbf{R}^{2n})$, we can use Proposition 3.1(e) to expand σ in terms of the frame $\{\Phi_{\alpha,\beta}\}$:

$$\sigma = \sum_{(\alpha,\beta)\in\Gamma} \langle \sigma, \Phi_{\alpha,\beta}\rangle \, \tilde{\Phi}_{\alpha,\beta}. \tag{3.5}$$

This series converges unconditionally in L^2-norm. Given $f, g \in L^2(\mathbf{R}^n)$, we can therefore perform the following calculation on the Weyl transform L_σ:

$$\begin{aligned}
\langle L_\sigma f, g\rangle &= \langle \sigma, W(g, f)\rangle \\
&= \sum_{(\alpha,\beta)\in\Gamma} \langle \sigma, \Phi_{\alpha,\beta}\rangle \langle \tilde{\Phi}_{\alpha,\beta}, W(g, f)\rangle \\
&= \sum_{(\alpha,\beta)\in\Gamma} \langle \sigma, \Phi_{\alpha,\beta}\rangle \langle W(\tilde{\phi}_\alpha, \tilde{\phi}_\beta), W(g, f)\rangle \\
&= \sum_{(\alpha,\beta)\in\Gamma} \langle \sigma, \Phi_{\alpha,\beta}\rangle \langle f, \tilde{\phi}_\beta\rangle \langle \tilde{\phi}_\alpha, g\rangle, \qquad (3.6)
\end{aligned}$$

the last equality following from Moyal's identity (Proposition 2.1(f)). Therefore

$$L_\sigma f = \sum_{(\alpha,\beta)\in\Gamma} \langle \sigma, \Phi_{\alpha,\beta}\rangle \langle f, \tilde{\phi}_\beta\rangle\, \tilde{\phi}_\alpha. \tag{3.7}$$

The partial sums formed by truncating the expansions in Eqs. (3.5) and (3.7) can be used to construct finite-rank approximations of L_σ. We design the truncation to facilitate estimates that we will make in later sections. Let B_r denote the ball in $\mathbf{R}^{2n}$ of radius r centered at the origin, and define

$$\Gamma_N = \Gamma \cap M^{-1}(B_N \times B_N).$$

Then set

$$\sigma_N = \sum_{(\alpha,\beta)\in\Gamma_N} \langle \sigma, \Phi_{\alpha,\beta}\rangle\, \tilde{\Phi}_{\alpha,\beta}. \tag{3.8}$$

Since $\{\Phi_{\alpha,\beta}\}$ is not an exact frame, Eq. (3.8) is not the frame expansion of σ_N. However, a calculation similar to the one in Eq. (3.6) shows that the Weyl transform L_{σ_N} of σ_N is given by

$$L_{\sigma_N} f = \sum_{(\alpha,\beta)\in\Gamma_N} \langle \sigma, \Phi_{\alpha,\beta}\rangle \langle f, \tilde{\phi}_\beta\rangle\, \tilde{\phi}_\alpha. \tag{3.9}$$

This L_{σ_N} has finite rank, which we estimate as follows.

LEMMA 3.5. *There exists a constant* R *so that* $\operatorname{rank}(L_{\sigma_N}) \leqslant RN^{2n}$ *for every* N.

Proof. From Eq. (3.9) we have $\operatorname{rank}(L_{\sigma_N}) \leqslant |\{\alpha \in \mathbf{R}^{2n} : (\alpha,\beta)\in\Gamma_N\}|$. By definition, $\Gamma_N \subset M^{-1}(B_N \times B_N) \subset B_{2N} \times B_{2N}$, so $\{\alpha \in \mathbf{R}^{2n} : (\alpha,\beta)\in\Gamma_N\} \subset \Lambda \cap B_{2N}$. The result then follows from the observation that

$$\lim_{N\to\infty} \frac{|\Lambda \cap B_{2N}|}{|B_{2N}|} = d(\Lambda),$$

and the fact that the volume of B_{2N} is $|B_{2N}| = (2N)^{2n}\, |B_1|$. ∎

By Eq. (1.2), the singular values of L_σ are controlled by the error between L_σ and L_{σ_N} in Hilbert–Schmidt norm. By Theorem 2.2, this is controlled in turn by the error between σ and σ_N in L^2-norm, which is further controlled by properties of the frame expansion of σ. Specifically, we have the following result, which will form the key step leading to Theorem 1.1.

LEMMA 3.6.

$$\sum_{k > \operatorname{rank}(L_{\sigma_N})} s_k(L_\sigma)^2 \leqslant A_\Lambda^{-2} \sum_{(\alpha, \beta) \in \Gamma \backslash \Gamma_N} |\langle \sigma, \Phi_{\alpha, \beta} \rangle|^2. \tag{3.10}$$

Proof. Note that $\sigma - \sigma_N = \sum_{(\alpha, \beta) \in \Gamma \backslash \Gamma_N} \langle \sigma, \Phi_{\alpha, \beta} \rangle \tilde{\Phi}_{\alpha, \beta}$. Although this is not the frame expansion of $\sigma - \sigma_N$, Eq. (3.2) allows us to estimate the norm of this quantity in terms of the coefficients $\langle \sigma, \Phi_{\alpha, \beta} \rangle$. Since $\{\tilde{\Phi}_{\alpha, \beta}\}$ has frame bounds B_Λ^{-2}, A_Λ^{-2}, we compute:

$$\begin{aligned}
\sum_{k > \operatorname{rank}(L_{\sigma_N})} s_k(L_\sigma)^2 &\leqslant \|L_\sigma - L_{\sigma_N}\|_{\mathcal{I}_2}^2 && \text{by (1.2)} \\
&= \|\sigma - \sigma_N\|_{L^2}^2 && \text{by Theorem 2.2} \\
&= \left\| \sum_{(\alpha, \beta) \in \Gamma \backslash \Gamma_N} \langle \sigma, \Phi_{\alpha, \beta} \rangle \tilde{\Phi}_{\alpha, \beta} \right\|_{L^2}^2 \\
&\leqslant A_\Lambda^{-2} \sum_{(\alpha, \beta) \in \Gamma \backslash \Gamma_N} |\langle \sigma, \Phi_{\alpha, \beta} \rangle|^2 && \text{by (3.2).} \quad \blacksquare
\end{aligned}$$

The estimate in Eq. (3.10) will be crucial in providing bounds on the rate of decay of the singular values of L_σ.

4. SINGULAR VALUES OF L_σ

We will prove Theorem 1.1 in this section. Our main task is to estimate the quantity on the right-hand side of Eq. (3.10). By Eq. (3.4) and the fact that the linear transformation M defined in Eq. (2.1) is invertible, we have

$$\begin{aligned}
\sum_{(\alpha, \beta) \in \Gamma \backslash \Gamma_N} |\langle \sigma, \Phi_{\alpha, \beta} \rangle|^2 &= \sum_{(\alpha, \beta) \in \Gamma \backslash \Gamma_N} |\langle \sigma, \rho(M(\alpha, \beta)) \Phi \rangle|^2 \\
&= \sum_{(\mu, \nu) \in M(\Gamma \backslash \Gamma_N)} |\langle \sigma, \rho(\mu, \nu) \Phi \rangle|^2.
\end{aligned} \tag{4.1}$$

We assume now that the lattice Λ is square. Then

$$M(\Gamma_N) = M(\Gamma) \cap (B_N \times B_N) \qquad \text{and} \qquad M(\Gamma) \subset \tfrac{1}{2}\Gamma = \tfrac{1}{2}\Lambda \times \tfrac{1}{2}\Lambda,$$

so

$$M(\Gamma \backslash \Gamma_N) \subset \tfrac{1}{2}\Gamma \cap (B_N \times B_N)^{\mathrm{C}} = (\tfrac{1}{2}\Lambda \cap B_N^{\mathrm{C}}) \times (\tfrac{1}{2}\Lambda \cap B_N^{\mathrm{C}}).$$

Therefore,

$$\begin{aligned}\sum_{(\mu,\nu)\in M(\Gamma\backslash\Gamma_N)} &|\langle\sigma,\rho(\mu,\nu)\Phi\rangle|^2\\ &\leqslant \sum_{\mu\in\frac{1}{2}\Lambda\cap B_N^{\mathrm{C}}}\ \sum_{\nu\in\frac{1}{2}\Lambda\cap B_N^{\mathrm{C}}} |\langle\sigma,\rho(\mu,\nu)\Phi\rangle|^2\\ &\leqslant \sum_{\mu\in\frac{1}{2}\Lambda\cap B_N^{\mathrm{C}}}\ \sum_{\nu\in\frac{1}{2}\Lambda} |\langle\sigma,\rho(\mu,\nu)\Phi\rangle|^2\\ &\quad+\sum_{\nu\in\frac{1}{2}\Lambda\cap B_N^{\mathrm{C}}}\ \sum_{\mu\in\frac{1}{2}\Lambda} |\langle\sigma,\rho(\mu,\nu)\Phi\rangle|^2.\end{aligned}\tag{4.2}$$

We now estimate each of the sums in Eq. (4.2). First, note that

$$\Phi(\xi,x)=2^n e^{-2\pi(\xi^2+x^2)} \qquad\text{and}\qquad \hat{\Phi}(p,q)=e^{-\pi(p^2+q^2)/2}.$$

Define

$$G(\xi,x)=2^{n/2}e^{-\pi(\xi^2+x^2)} \qquad\text{and}\qquad H(p,q)=e^{-\pi(p^2+q^2)/4},$$

so that

$$G^2=\Phi \qquad\text{and}\qquad H^2=\hat{\Phi}$$

and

$$\rho(\mu,\nu)\Phi=\rho(\mu,\nu)G\cdot\tau_{-\mu}G \qquad\text{and}\qquad \rho(\nu,\mu)\hat{\Phi}=\rho(\nu,\mu)H\cdot\tau_{-\nu}H,$$

where τ_β is the translation operator $\tau_\beta f(\alpha)=f(\alpha-\beta)$.

A Gabor system generated by any Gaussian function on any arbitrary rectangular lattice is always a *Bessel sequence*, i.e., at least an upper frame bound is satisfied, even if there is no lower frame bound. Let B_G and B_H denote the upper frame bounds for the Bessel sequence $\{\rho(\mu,\nu)G\}_{(\mu,\nu)\in\frac{1}{2}\Gamma}$ and $\{\rho(\nu,\mu)H\}_{(\mu,\nu)\in\frac{1}{2}\Gamma}$, respectively. Then we can make the following initial estimates.

LEMMA 4.1.

(a) *If* $\mu\in\frac{1}{2}\Lambda$ *then* $\sum_{\nu\in\frac{1}{2}\Lambda}|\langle\sigma,\rho(\mu,\nu)\Phi\rangle|^2\leqslant B_G\,\|\sigma\cdot\tau_{-\mu}G\|_{L^2}^2$.

(b) *If* $\nu\in\frac{1}{2}\Lambda$ *then* $\sum_{\mu\in\frac{1}{2}\Lambda}|\langle\sigma,\rho(\mu,\nu)\Phi\rangle|^2\leqslant B_H\,\|\hat{\sigma}\cdot\tau_\nu H\|_{L^2}^2$.

Proof. (a) Suppose $\mu\in\frac{1}{2}\Lambda$. Then, since G is real-valued, we can compute

$$\begin{aligned}\sum_{\nu\in\frac{1}{2}\Lambda}|\langle\sigma,\rho(\mu,\nu)\Phi\rangle|^2&=\sum_{\nu\in\frac{1}{2}\Lambda}|\langle\sigma\cdot\tau_{-\mu}G,\rho(\mu,\nu)G\rangle|^2\\ &\leqslant B_G\,\|\sigma\cdot\tau_{-\mu}G\|_{L^2}^2.\end{aligned}$$

(b) Now suppose that $v \in \frac{1}{2}\Lambda$. Then

$$\begin{aligned}\sum_{\mu \in \frac{1}{2}\Lambda} |\langle \sigma, \rho(\mu, v)\Phi \rangle|^2 &= \sum_{\mu \in \frac{1}{2}\Lambda} |\langle \hat{\sigma}, \rho(-v, \mu)\hat{\Phi} \rangle|^2 \\ &= \sum_{\mu \in \frac{1}{2}\Lambda} |\langle \hat{\sigma} \cdot \tau_v H, \rho(-v, \mu) H \rangle|^2 \\ &\leqslant B_H \, \|\sigma \cdot \tau_v H\|_{L^2}^2. \quad \blacksquare\end{aligned}$$

LEMMA 4.2. *There exist constants* ε, C_1, $C_2 > 0$ *such that*

(a) $\sum_{\mu \in \frac{1}{2}\Lambda \cap B_N^C} \|\sigma \cdot \tau_{-\mu} G\|_{L^2}^2 \leqslant C_1 \, \|\sigma \cdot \chi_{B_{N/2}^C}\|_{L^2}^2 + C_2 e^{-\varepsilon N^2} \, \|\sigma \cdot \chi_{B_{N/2}}\|_{L^2}^2$,

(b) $\sum_{v \in \frac{1}{2}\Lambda \cap B_N^C} \|\hat{\sigma} \cdot \tau_v H\|_{L^2}^2 \leqslant C_1 \, \|\hat{\sigma} \cdot \chi_{B_{N/2}^C}\|_{L^2}^2 + C_2 e^{-\varepsilon N^2} \, \|\hat{\sigma} \cdot \chi_{B_{N/2}}\|_{L^2}^2$.

Proof. We prove only (a) as (b) is similar. Define

$$G_N = \sum_{\mu \in \frac{1}{2}\Lambda \cap B_N^C} (\tau_{-\mu} G)^2 = \sum_{\mu \in \frac{1}{2}\Lambda \cap B_N^C} \tau_{-\mu} \Phi.$$

By Tonelli's Theorem,

$$\sum_{\mu \in \frac{1}{2}\Lambda \cap B_N^C} \|\sigma \cdot \tau_{-\mu} G\|_{L^2}^2 = \iint |\sigma(\xi, x)|^2 \, G_N(\xi, x) \, d\xi \, dx. \tag{4.3}$$

Since G is a Gaussian function, we have $C_1 = \sup \|G_N\|_{L^\infty} < \infty$. Therefore,

$$\iint_{B_{N/2}^C} |\sigma(\xi, x)|^2 \, G_N(\xi, x) \, d\xi \, dx \leqslant C_1 \, \|\sigma \cdot \chi_{B_{N/2}^C}\|_{L^2}^2. \tag{4.4}$$

Further,

$$\iint_{B_{N/2}} |\sigma(\xi, x)|^2 \, G_N(\xi, x) \, d\xi \, dx \leqslant \|G_N \cdot \chi_{B_{N/2}}\|_{L^\infty} \, \|\sigma \cdot \chi_{B_{N/2}}\|_{L^2}^2. \tag{4.5}$$

However, if $(\xi, x) \in B_{N/2}$ and $|\mu| \geqslant N$ then $|(\xi, x) - \mu| \geqslant |\mu| - N/2$, so

$$\begin{aligned}G_N(\xi, x) &= \sum_{\mu \in \frac{1}{2}\Lambda \cap B_N^C} 2^n e^{-2\pi((\xi, x) - \mu)^2} \\ &\leqslant 2^n \sum_{\mu \in \frac{1}{2}\Lambda \cap B_N^C} e^{-2\pi(|\mu| - N/2)^2} \\ &\leqslant C_2 \int_{|x| \geqslant N/2} e^{-2\pi x^2} \, dx \\ &\leqslant C_2 e^{-(\pi n/4) N^2}.\end{aligned} \tag{4.6}$$

The result follows upon combining (4.3)–(4.6). $\blacksquare$

We combine these lemmas to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Combining Lemma 3.6, Eqs. (4.1) and (4.2), and Lemmas 4.1 and 4.2, we see that there exist constants ε and C so that

$$\sum_{k > \operatorname{rank}(L_{\sigma_N})} s_k(L_\sigma)^2 \leqslant C\, S_\varepsilon(N/2),$$

where $S_\varepsilon(r)$ is defined in Eq. (1.4). By Lemma 3.5 there exists constant R so that $\operatorname{rank}(L_{\sigma_N}) \leqslant RN^{2n}$ for every N. Therefore

$$\sum_{k > RN^{2n}} s_k(L_\sigma)^2 \leqslant C\, S_\varepsilon(N/2). \tag{4.7}$$

Eq. (1.3) then follows upon reparametrizing Eq. (4.7).

Finally, the estimate in Eq. (1.5) follows from Eq. (1.3) because the singular values are arranged in decreasing order, so

$$N s_{2N}(L_\sigma)^2 \leqslant \sum_{k=N+1}^{2N} s_k(L_\sigma)^2 \leqslant C_1\, S_\varepsilon(C_2 N^{1/2n}). \quad \blacksquare$$

5. SCHATTEN–CLASS APPLICATIONS

In this section we apply Theorem 1.1 to derive conditions on the symbol σ which imply that the Weyl transform L_σ lies in a particular Schatten class.

Recall that the Sobolev space $H^s(\mathbf{R}^{2n})$ is defined by the norm

$$\|f\|_{H^s}^2 = \|\hat{f}(p, q)(1 + p^2 + q^2)^{s/2}\|_{L^2}^2 = \iint |\hat{f}(p, q)|^2\, (1 + p^2 + q^2)^s\, dp\, dq.$$

In particular, $H^s(\mathbf{R}^{2n}) \subset L^2(\mathbf{R}^{2n})$ when $s \geqslant 0$. Let $\mathscr{F}$ denote the Fourier transform operator, i.e., $\mathscr{F}f = \hat{f}$, and let $\check{f} = \mathscr{F}^{-1} f$. Then $H^s(\mathbf{R}^{2n}) \cap \mathscr{F}H^t(\mathbf{R}^{2n})$ is a Banach space with norm

$$\|f\|_{s,t} = \max\{\|f\|_{H^s}, \|\check{f}\|_{H^t}\}.$$

Theorem 1.1 leads immediately to the following result bounding the decay of the singular values of L_σ in terms of the $H^s \cap \mathscr{F}H^t$ norm of the symbol σ.

PROPOSITION 5.1. *Assume* $\sigma \in H^s(\mathbf{R}^{2n}) \cap \mathscr{F}H^t(\mathbf{R}^{2n})$ *with* $\gamma = \min\{s, t\} \geqslant 0$. *Then there exists a constant* $C > 0$ *such that*

$$\forall k > 0, \qquad s_k(L_\sigma) \leqslant C\, \|\sigma\|_{s,t}\, k^{-(\gamma/2n) - (1/2)}.$$

Proof. Since $\sigma \in L^2(\mathbf{R}^{2n})$, Theorem 1.1 implies that there exist constants ε, C_1, $C_2 > 0$ such that $s_{2k}(L_\sigma)^2 \leqslant C_1 S_\varepsilon(C_2 k^{1/2n})/k$. Now, if $r > 0$, then

$$(1+r^2)^t \iint_{B_r^C} |\sigma(\xi, x)|^2 \, d\xi \, dx \leqslant \iint_{B_r^C} |\sigma(\xi, x)|^2 \, (1+\xi^2+x^2)^t \, d\xi \, dx$$
$$\leqslant \|\check{\sigma}\|_{H^t}^2 \leqslant \|\sigma\|_{s,t}^2.$$

Similarly,

$$(1+r^2)^s \iint_{B_r^C} |\hat{\sigma}(p, q)|^2 \, dp \, dq \leqslant \|\sigma\|_{s,t}^2.$$

Therefore,

$$S_\varepsilon(r) \leqslant \frac{\|\sigma\|_{s,t}^2}{r^{2s}} + \frac{\|\sigma\|_{s,t}^2}{r^{2t}} + 2e^{-\varepsilon r^2} \|\sigma\|_{L^2}^2 \leqslant C \, \|\sigma\|_{s,t}^2 \, r^{-2\gamma}$$

for some constant C independent of r. Hence

$$s_{2k}(L_\sigma)^2 \leqslant C \, \|\sigma\|_{s,t}^2 \, \frac{C_1 (C_2 k^{1/2n})^{-2\gamma}}{k} = C C_1 C_2^{-2\gamma} \, \|\sigma\|_{s,t}^2 \, k^{-(\gamma/n)-1}.$$

The result then follows upon reindexing and taking square roots. ∎

We next give a version of Proposition 5.1 that is "rotationally invariant in phase space." Let $\mathscr{L}$ denote the Hamiltonian for the simple harmonic oscillator on $\mathbf{R}^{2n}$, i.e.,

$$\mathscr{L} = -\frac{1}{4\pi^2} \frac{d^2}{d\xi^2} - \frac{1}{4\pi^2} \frac{d^2}{dx^2} + \xi^2 + x^2.$$

Since $\mathscr{L}$ is a positive, self-adjoint operator, we can define a Hilbert space $\mathscr{H}^s(\mathbf{R}^{2n})$ by the norm

$$\|\sigma\|_{\mathscr{H}^s} = \langle \mathscr{L}^s \sigma, \sigma \rangle^{1/2}.$$

We clearly have $\mathscr{H}^1(\mathbf{R}^{2n}) = H^1(\mathbf{R}^{2n}) \cap \mathscr{F} H^1(\mathbf{R}^{2n})$. In fact, this extends to all values of s.

Lemma 5.2. *If* $s \geqslant 0$ *then* $\mathscr{H}^s(\mathbf{R}^{2n}) = H^s(\mathbf{R}^{2n}) \cap \mathscr{F} H^s(\mathbf{R}^{2n})$, *with equivalence of norms.*

Applying this fact to Proposition 5.1 for the case $s = t$ gives the following result.

PROPOSITION 5.3. *If $\sigma \in \mathscr{H}^s(\mathbf{R}^{2n})$ with $s \geqslant 0$ then there exists a constant $C > 0$ such that*

$$\forall k > 0, \qquad s_k(L_\sigma) \leqslant C \, \|\sigma\|_{\mathscr{H}^s} \, k^{-(s/2n)-(1/2)}. \tag{5.1}$$

Daubechies [Dau80] proved that L_σ is trace-class (i.e., in $\mathscr{I}_1$) if $\sigma \in \mathscr{H}^s(\mathbf{R}^{2n})$ with $s > 2n$. We can use Proposition 5.3 to improve this result.

PROPOSITION 5.4. *If $\sigma \in \mathscr{H}^s(\mathbf{R}^{2n})$ with $s \geqslant 0$ then $L_\sigma \in \mathscr{I}_{2n/(n+s),\,\infty} \subset \mathscr{I}_p$ for each $p > 2n/(n+s)$. In particular, L_σ is trace-class if $s > n$.*

Proof. We have

$$k^{(n+s)/2n} s_k(L_\sigma) \leqslant C \, \|\sigma\|_{\mathscr{H}^s} \, k^{(n+s)/2n} k^{-(s/2n)-(1/2)} = C \, \|\sigma\|_{\mathscr{H}^s}. \quad \blacksquare$$

It also follows easily from Proposition 5.3 that $L_\sigma \in \mathscr{I}_{2n/(n+s)+\epsilon,\,2}$ if $\sigma \in \mathscr{H}^s(\mathbf{R}^{2n})$ with $s \geqslant 0$. However, we can refine this latter result by using interpolation.

THEOREM 5.5. *The mapping $\sigma \mapsto L_\sigma$ is a bounded operator from $\mathscr{H}^s(\mathbf{R}^{2n})$ to $\mathscr{I}_{2n/(n+s),\,2}$ for each $s \geqslant 0$.*

Proof. We apply the technique of real interpolation to the Banach spaces $\mathscr{H}^s(\mathbf{R}^{2n})$ and the Schatten quasi-ideals $\mathscr{I}_{p,q}$. First, standard interpolation results (e.g., [BL76, Theorem 6.2.4]) imply that

$$\begin{aligned} (H^{s_1}(\mathbf{R}^{2n}), H^{s_2}(\mathbf{R}^{2n}))_{\theta,2} &= B^s_{2,2}(\mathbf{R}^{2n}) \\ &= H^s(\mathbf{R}^{2n}), \qquad \begin{cases} s_1 \neq s_2, \\ 0 < \theta < 1, \\ s = (1-\theta)\, s_1 + \theta s_2. \end{cases} \end{aligned} \tag{5.2}$$

Therefore $(\mathscr{H}^{s_1}(\mathbf{R}^{2n}), \mathscr{H}^{s_2}(\mathbf{R}^{2n}))_{\theta,2} = \mathscr{H}^s(\mathbf{R}^{2n})$ as well. Also, by [Kön86, Prop. 2.c.6], we have

$$(\mathscr{I}_{p_1,\infty}, \mathscr{I}_{p_2,\infty})_{\theta,2} = \mathscr{I}_{p,2}, \qquad \begin{cases} 0 < p_1 < p_2 < \infty, \\ 0 < \theta < 1, \\ \dfrac{1}{p} = \dfrac{1-\theta}{p_1} + \dfrac{\theta}{p_2}. \end{cases} \tag{5.3}$$

Now choose $s > 0$, and define

$$\theta = 1/2, \qquad s_1 = s - \varepsilon, \qquad s_2 = s + \varepsilon,$$

$$p = \frac{2n}{n+s}, \qquad p_1 = \frac{2n}{n+s-\varepsilon}, \qquad p_2 = \frac{2n}{n+s+\varepsilon}.$$

Then Proposition 5.4 implies that the mapping $\sigma \mapsto L_\sigma$ generates a bounded operator from $\mathscr{H}^{s_i}(\mathbf{R}^{2n})$ to $\mathscr{I}_{p_i,\infty}$ for $i=1, 2$, and therefore by the interpolation statements in Eqs. (5.2) and (5.3) it also generates a bounded operator from $\mathscr{H}^s(\mathbf{R}^{2n})$ to $\mathscr{I}_{p,2}$. ∎

Let $\sigma \in \mathscr{H}^s(\mathbf{R}^{2n})$. Note that Eq. (5.1) states that $\sup(k^{s/2n+1/2} s_k(L_\sigma)) < \infty$. However, Theorem 5.5 implies $\{s_k(L_\sigma)\} \in \mathscr{I}_{2n/(n+s),2}$, i.e., $\sum k^{s/n} s_k(L_\sigma)^2 < \infty$. Therefore, we cannot have, for example, $\inf(k^{s/2n+1/2} s_k(L_\sigma)) > 0$.

EXAMPLE 5.6. Let $n=1$, and set $\sigma = \chi_{B_r}$ where B_r is a sphere in $\mathbf{R}^2$. Then $\sigma \in \mathscr{H}^s(\mathbf{R}^2)$ for $s < 1/2$ but not for $s = 1/2$. Proposition 5.3 therefore implies $s_k(L_\sigma) = \mathcal{O}(k^{-t})$ for $t < 3/4$. The same is true for $\sigma = \chi_A$ where $A = B_{r_1} \backslash B_{r_2}$ is an annulus in $\mathbf{R}^2$. However, in this latter case it is known that $t = 3/4$ is the optimal exponent, i.e., that $0 < \limsup k^{3/4} s_k(L_\sigma) < \infty$ [RT93].

6. THE CALDERÓN–VAILLANCOURT THEOREM

The usual Calderón–Vaillancourt Theorem for the Weyl correspondence states that L_σ is a bounded operator on $L^2(\mathbf{R}^n)$ if $\sigma \in C^{2n+1}(\mathbf{R}^{2n})$. Various improvements and related results are known, including nearly sharp results in the context of the Kohn–Nirenberg correspondence [Fol89]. In this section we will prove Theorem 1.2, which states that L_σ is bounded on $L^2(\mathbf{R}^n)$ if σ is in the Hölder–Zygmund class $\Lambda^{2n+\varepsilon}(\mathbf{R}^{2n})$.

First, however, we briefly sketch the idea of the proof, which uses Gabor frame expansions in a different manner than previous sections. As before, we approximate the symbol σ by another symbol σ_N, but now these approximations are not obtained by truncating the frame expansion of σ. Instead, in order that σ_N share the smoothness properties of σ yet be an element of $L^2(\mathbf{R}^{2n})$, we choose a smooth, compactly supported, non-negative function $m \in C_c^\infty(\mathbf{R}^{2n})$ satisfying $m(\xi, x) = 1$ if $|(\xi, x)| \leqslant 1$, and define

$$\sigma_N(\xi, x) = m(\xi/N, x/N)\, \sigma(\xi, x). \tag{6.1}$$

We then consider the frame expansions of σ and σ_N simultaneously. By Lemma 6.1 below, the frame coefficients $\langle \sigma_N, \Phi_{\alpha,\beta} \rangle$ can be realized as a Fourier transform evaluated at $\alpha - \beta$, specifically, $\langle \sigma_N, \Phi_{\alpha,\beta} \rangle = c_{\alpha,\beta} (\sigma_N \cdot \tau_\eta \Phi)^\wedge(\alpha - \beta)$, where $c_{\alpha,\beta}$ is a scalar with modulus 1 and η is determined by (α, β). If it was the case that there was a single sequence $k \in l^1(\Lambda)$, independent of N and such that $|\langle \sigma_N, \Phi_{\alpha,\beta} \rangle| \leqslant k(\alpha - \beta)$, then we could use the uniform pointwise convergence of σ_N to σ on compact sets to derive a weak convergence of L_{σ_N} to L_σ (Proposition 6.2). With sufficient control on the convergence, we could then conclude from the boundedness of each L_{σ_N} on $L^2(\mathbf{R}^n)$ that L_σ itself is bounded on $L^2(\mathbf{R}^n)$. The smoothness of σ is the key to constructing such a sequence k. In particular, since each $\sigma_N \in \Lambda^s(\mathbf{R}^{2n})$, we expect decay of $(\sigma_N \cdot \tau_\eta \Phi)^\wedge(\alpha)$ as $|\alpha| \to \infty$. We show in Proposition 6.3 that $(\sigma_N \cdot \tau_\eta \Phi)^\wedge(\alpha)$ decays like $|\alpha|^{-(s-\varepsilon)}$ with constants independent of N and η. The final step is therefore to set $k(\alpha) = |\alpha|^{-(s-\varepsilon)}$, and to observe that $k \in l^1(\Lambda)$ if $s > 2n$ since Λ is a rectangular lattice in $\mathbf{R}^{2n}$.

We now proceed with the technical details of the proof. First, Lemma 6.1 establishes the desired form of $\langle \sigma_N, \Phi_{\alpha,\beta} \rangle$.

Lemma 6.1. *$|\langle \sigma_N, \Phi_{\alpha,\beta} \rangle| = |(\sigma_N \cdot \tau_\eta \Phi)^\wedge(\alpha - \beta)|$, where $\eta = ((\alpha_2 + \beta_2)/2, -(\alpha_1 + \beta_1)/2)$.*

Proof. By Eqs. (3.4) and (2.1), we can compute as follows:

$$\begin{aligned}
\langle \sigma_N, \Phi_{\alpha,\beta} \rangle &= \langle \sigma_N, \rho(M(\alpha, \beta))\Phi \rangle \\
&= C \iint \sigma_N(\xi, x)\, e^{-2\pi i((\alpha_1 - \beta_1)\,\xi + (\alpha_2 - \beta_2)\,x)} \\
&\quad \times \Phi\left(\xi - \frac{\alpha_2 + \beta_2}{2}, x + \frac{\alpha_1 + \beta_1}{2}\right) d\xi\, dx \\
&= C \iint \sigma_N(\xi, x)\, e^{-2\pi i(\alpha - \beta)(\xi, x)}\, \tau_\eta \Phi(\xi, x)\, d\xi\, dx \\
&= C\, (\sigma_N \cdot \tau_\eta \Phi)^\wedge\, (\alpha - \beta),
\end{aligned}$$

where $C = e^{-\pi i(\alpha_2\beta_1 - \alpha_1\beta_2)}$. ▮

Next we show how existence of a bounding sequence k would lead to weak convergence of L_{σ_N} to L_σ and therefore to boundedness of L_σ on $L^2(\mathbf{R}^n)$. We state this result in terms of general approximation symbols σ_N, although we shall only apply the proposition to σ_N defined by Eq. (6.1).

Proposition 6.2. *Let $\sigma \in \mathscr{S}'(\mathbf{R}^{2n})$ and let $k \in l^1(\Lambda)$ be a nonnegative sequence. Assume that $\sigma_N \in L^2(\mathbf{R}^{2n})$ are such that*

$$\forall \alpha, \beta \in \Lambda, \qquad |\langle \sigma_N, \Phi_{\alpha,\beta} \rangle| \leqslant k(\alpha - \beta),$$

and suppose that $L_{\sigma_N} \to L_\sigma$ weakly as operators from $\mathscr{S}(\mathbf{R}^n)$ to $\mathscr{S}'(\mathbf{R}^n)$, i.e., $\langle L_{\sigma_N} f, g \rangle \to \langle L_\sigma f, g \rangle$ for $f, g \in \mathscr{S}(\mathbf{R}^n)$. Then the operator L_σ is bounded on $L^2(\mathbf{R}^n)$, and its operator norm satisfies

$$\|L_\sigma\|_{\mathscr{B}(L^2)} \leqslant A_\Lambda^{-1}\, \|k\|_{l^1}.$$

Proof. Each L_{σ_N} is a bounded operator on $L^2(\mathbf{R}^n)$, and, by Eq. (3.6), we have

$$\langle L_{\sigma_N} f, g\rangle = \sum_{\alpha, \beta \in \Lambda} \langle \sigma_N, \Phi_{\alpha, \beta}\rangle \langle f, \tilde{\phi}_\beta\rangle \langle \tilde{\phi}_\alpha, g\rangle.$$

Let $u = \{|\langle f, \tilde{\phi}_\alpha\rangle|\}$ and $v = \{|\langle \tilde{\phi}_\alpha, g\rangle|\}$. Then for each $f, g \in L^2(\mathbf{R}^n)$ we have

$$\begin{aligned} |\langle L_{\sigma_N} f, g\rangle| &\leqslant \sum_{\alpha, \beta \in \Lambda} |\langle \sigma_N, \Phi_{\alpha, \beta}\rangle| \, |\langle f, \tilde{\phi}_\beta\rangle| \, |\langle \tilde{\phi}_\alpha, g\rangle| \\ &\leqslant \sum_{\alpha, \beta \in \Lambda} k(\alpha - \beta)\, u(\beta)\, v(\alpha) \\ &= \langle k * u, v\rangle_{l^2} \\ &\leqslant \|k\|_{l^1} \, \|u\|_{l^2} \, \|v\|_{l^2} \\ &\leqslant A_\Lambda^{-1} \, \|k\|_{l^1} \, \|f\|_{L^2} \, \|g\|_{L^2}, \end{aligned}$$

where we have used Young's convolution inequality and the fact that $\{\tilde{\phi}_\alpha\}$ is a frame with frame bounds B_Λ^{-1}, A_Λ^{-1}. It follows immediately from this that L_σ is bounded on $L^2(\mathbf{R}^n)$ if $L_{\sigma_N} \to L_\sigma$ weakly as operators from $\mathscr{S}(\mathbf{R}^n)$ to $\mathscr{S}'(\mathbf{R}^n)$. ∎

In particular, if $\sigma \in C(\mathbf{R}^{2n})$ and σ_N is defined by Eq. (6.1), then the σ_N are uniformly bounded in L^∞ norm and converge to σ uniformly on compact sets. Since for each $f, g \in \mathscr{S}(\mathbf{R}^n)$ we have $\langle L_\sigma f, g\rangle - \langle L_{\sigma_N} f, g\rangle = \langle \sigma - \sigma_N, W(g, f)\rangle$ with $W(g, f) \in \mathscr{S}(\mathbf{R}^{2n})$, it follows that $L_{\sigma_N} \to L_\sigma$ weakly.

In order to establish that a bounding sequence k does exist, our next task is to estimate the decay of $(\sigma_N \cdot \tau_\eta \Phi)^\wedge(\alpha)$ independently of N and η.

Proposition 6.3. *Let $f \in \Lambda^s(\mathbf{R}^n)$ with $s > 0$, and let $t = s - \varepsilon > 0$ with $\varepsilon < 1$. Let $\psi \in \mathscr{S}(\mathbf{R}^n)$. Then there exists a constant C, depending only on n, ψ, and f, such that*

$$\forall \gamma, b \in \mathbf{R}^n, \qquad |(f \cdot \tau_b \psi)^\wedge(\gamma)| \leqslant C\,(1 + \gamma^2)^{-t/2}.$$

Proof. We will use the standard identification of $\Lambda^s(\mathbf{R}^n)$ with the Besov space $B^s_{\infty\infty}(\mathbf{R}^n)$ [Tri92, p. 28]. This provides us with an equivalent norm for $\Lambda^s(\mathbf{R}^n)$ via a smooth dyadic partition of unity in the transform domain, as follows. Let $v_0 \in C_c^\infty(\mathbf{R}^n)$ be any function such that $\operatorname{supp}(v_0) \subset \{\gamma \in \mathbf{R}^n : |\gamma| < 2\}$ and such that $v_0(\gamma) = 1$ if $|\gamma| \leqslant 1$. For each $j > 0$ define $v_j(\gamma) = v_0(2^{-j}\gamma) - v_0(2^{-(j-1)}\gamma)$. Then $\{v_j\}_{j=0}^\infty$ is a smooth dyadic partition of unity, i.e.,

(a) if $j>0$ then $\operatorname{supp}(v_j) \subset \{\gamma \in \mathbf{R}^n : 2^{j-1} \leqslant |\gamma| \leqslant 2^{j+1}\}$,
(b) $\sum_{j=0}^{\infty} v_j(\gamma) = 1$, and
(c) for each multi-index α, $\sup_{j \geqslant 0}(2^{j|\alpha|} \|D^\alpha v_j\|_{L^\infty}) < \infty$.

Moreover, the following equivalence of norms holds:

$$\|g\|_{\Lambda^s} \sim \sup_{j \geqslant 0} 2^{sj} \|\check{v}_j * g\|_{L^\infty}. \tag{6.2}$$

For our purposes, we will impose additional restrictions on v_0, namely that $1/2 \leqslant v_0(\gamma) \leqslant 1$ when $1 \leqslant |\gamma| \leqslant 3/2$ and that $0 \leqslant v_0(\gamma) \leqslant 1/2$ when $3/2 \leqslant |\gamma| \leqslant 2$. Then, since $v_j(\gamma) = v_1(2^{-(j-1)}\gamma)$ for each $j>0$, we have

$$\forall j>0, \qquad 3 \cdot 2^{j-2} \leqslant |\gamma| \leqslant 3 \cdot 2^{j-1} \quad \Rightarrow \quad \tfrac{1}{2} \leqslant v_j(\gamma) \leqslant 1.$$

Now, $|\gamma|$ is comparable to 2^j if $\gamma \in \operatorname{supp}(v_j)$, and $v_j(\gamma)$ is comparable to 1 if γ is in the annulus $3 \cdot 2^{j-2} \leqslant |\gamma| \leqslant 3 \cdot 2^{j-1}$, which is contained in $\operatorname{supp}(v_j)$. Since these annuli cover all of $\mathbf{R}^n$, it will be enough to prove that there is a constant C independent of j and b such that

$$\|\check{v}_j * (f \cdot \tau_b \psi)\|_{L^1} \leqslant C 2^{-tj}. \tag{6.3}$$

For, if (6.3) is established and γ is given, then $3 \cdot 2^{j-2} \leqslant |\gamma| \leqslant 3 \cdot 2^{j-1}$ for some j, so

$$\begin{aligned} |(f \cdot \tau_b \psi)^\wedge(\gamma)| &\leqslant \frac{\|v_j \cdot (f \cdot \tau_b \psi)^\wedge\|_{L^\infty}}{v_j(\gamma)} \leqslant 2 \|\check{v}_j * (f \cdot \tau_b \psi)\|_{L^1} \\ &\leqslant 2C 2^{-tj} \leqslant 2C \left(\frac{4|\gamma|}{3}\right)^{-t}. \end{aligned}$$

Hence, we seek to establish Eq. (6.3). Note that there exists a constant C_1 independent of b such that $\|f \cdot \tau_b \psi\|_{\Lambda^s} \leqslant C_1 \|f\|_{\Lambda^s}$. Therefore, by the norm equivalence in Eq. (6.2),

$$C_2 = \sup_b \sup_{j \geqslant 0} 2^{sj} \|\check{v}_j * (f \cdot \tau_b \psi)\|_{L^\infty} < \infty. \tag{6.4}$$

Our goal is to obtain a similar result with L^∞ replaced by L^1 and s replaced by t. Fix b and j, and define $B = B_{2^{\varepsilon j/n}}(b)$, the ball of radius $2^{\varepsilon j/n}$ centered at b. Then, by Eq. (6.4),

$$\begin{aligned} \int_B |(\check{v}_j * (f \cdot \tau_b \psi))(x)|\, dx &\leqslant |B|\, \|\check{v}_j * (f \cdot \tau_b \psi)\|_{L^\infty} \\ &\leqslant |B_1|\, (2^{\delta j})^n\, C_2 2^{-sj} = C_3 2^{-tj}, \end{aligned}$$

where B_1 is the ball of radius 1.

In order to derive a similar estimate for the integral outside B, define

$$F_1 = f \cdot \tau_b \psi \cdot \chi_{\frac{1}{2}B} \qquad \text{and} \qquad F_2 = f \cdot \tau_b \psi \cdot (1 - \chi_{\frac{1}{2}B}),$$

so that $\check{v}_j * (f \cdot \tau_b \psi) = \check{v}_j * F_1 + \check{v}_j * F_2$. Then it suffices to show that $\|\check{v}_j * F_i\|_{L^1} \leqslant C 2^{-tj}$ for some constant C. Note that $\check{v}_j(x) = 2^{j-1} \check{v}_1(2^{j-1}x)$ and that ψ and $\check{v}_1$ are both Schwartz-class functions, hence decay faster than any polynomial. In particular, for each M there exists a constant $K = K(M)$ so that $|\psi(x)|, |\check{v}_1(x)| \leqslant K |x|^{-M}$. Hence,

$$\forall x \in \mathbf{R}^n, \qquad |\check{v}_j(x)| \leqslant K\, 2^{j-1}\, |2^{j-1}x|^{-M} \leqslant K\, 2^{-(M+1)-(M+1)j}\, |x|^{-M}.$$

Now, if $x \notin B$ and $y \in \frac{1}{2}B$, then $|x - y| \geqslant |x - b|/2$. Hence, for such x,

$$\begin{aligned}
|(\check{v}_j * F_1)(x)| &\leqslant \int_{\frac{1}{2}B} |\check{v}_j(x - y)\, F_1(y)|\, dy \\
&\leqslant \|f\|_{L^\infty}\, \|\psi\|_{L^\infty}\, K\, 2^{-(M+1)-(M+1)j} \int_{\frac{1}{2}B} |x - y|^{-M}\, dy \\
&\leqslant \|f\|_{L^\infty}\, \|\psi\|_{L^\infty}\, K\, 2^{-(M+1)-(M+1)j}\, |\tfrac{1}{2}B| \left(\frac{|x - b|}{2} \right)^{-M} \\
&= C_3 2^{-(1+M-\varepsilon)j}\, |x - b|^{-M},
\end{aligned}$$

with C_4 depending on M, but not on b or x. Therefore, taking $M > n$ and $M > t + \varepsilon - 1$, we have

$$\int_{B^C} |\check{v}_j * F_1(x)|\, dx \leqslant C_5 2^{-(1+M-\varepsilon)j} \leqslant C_5 2^{-tj},$$

with C_5 depending only on M. Finally, if we also take $M > tn/\varepsilon$, then

$$\begin{aligned}
\int_{B^C} |\check{v}_j * F_2(x)|\, dx &\leqslant \|\check{v}_j\|_{L^1}\, \|F_2 \cdot \chi_{B^C}\|_{L^\infty} \\
&\leqslant \|\check{v}_1\|_{L^1}\, \|f\|_{L^\infty}\, \|\tau_b \psi \cdot \chi_{B^C}\|_{L^\infty} \\
&\leqslant \|\check{v}_1\|_{L^1}\, \|f\|_{L^\infty}\, K \sup_{x \notin B} |x - b|^{-M} \\
&\leqslant C_6 2^{-\varepsilon j M/n} \\
&\leqslant C_6 2^{-tj},
\end{aligned}$$

with C_6 depending only on M. ∎

The above results can now be combined to obtain a proof of Theorem 1.2.

Proof of Theorem 1.2. Assume that $\sigma \in \Lambda^s(\mathbf{R}^n)$, and let σ_N be defined by Eq. (6.1). Fix $t = s - \varepsilon > 2n$. Then by Lemma 6.1 and Proposition 6.3, there exists a constant C, independent of N, such that $|\langle \sigma_N, \Phi_{\alpha,\beta} \rangle| \leqslant C(1 + (\alpha - \beta)^2)^{-t/2}$. Define $k(\alpha) = (1 + \alpha^2)^{-t/2}$. Then $k \in l^1(\Lambda)$ since $t > 2n$ and Λ is a rectangular lattice in $\mathbf{R}^{2n}$. Since $\sigma_N \to \sigma$ uniformly on compact sets, the Weyl transforms L_{σ_N} converge weakly to L_σ. Hence the conditions of Proposition 6.2 are fulfilled, and therefore L_σ extends to a bounded operator on $L^2(\mathbf{R}^n)$. ∎

7. EXTENSIONS

In this final section, we connect Theorem 1.2 (Calderón–Vaillancourt) with the result of Pool that the Weyl transform $\sigma \mapsto L_\sigma$ is a unitary mapping of $L^2(\mathbf{R}^n)$ onto $\mathcal{I}_2$. Our motivation is the recent result of Simon [Sim92] that there are no estimates on the operator norm of L_σ of the form $\|L_\sigma\|_{\mathcal{B}(L^2)} \leqslant C \|\sigma\|_{L^p}$ when $p > 2$. In particular, since the operator norm is dominated by any of the Schatten norms, this implies that no estimate of the form $\|L_\sigma\|_{\mathcal{I}_q} \leqslant C \|\sigma\|_{L^p}$ is possible for any $p > 2$ and $q \geqslant 1$.

Note that $L^p(\mathbf{R}^{2n}) = B^0_{pp}(\mathbf{R}^{2n})$. Hence Pool's result is that the Weyl transform is a bounded mapping of $B^0_{22}(\mathbf{R}^{2n})$ onto $\mathcal{I}_2$. Our version of the Calderón–Vaillancourt Theorem states that the Weyl transform maps $\Lambda^s(\mathbf{R}^{2n}) = B^s_{\infty\infty}(\mathbf{R}^{2n})$ into the space of bounded operators $\mathcal{B}(L^2(\mathbf{R}^n))$. Although $\mathcal{I}_\infty$ is only a proper subspace of $\mathcal{B}(L^2(\mathbf{R}^n))$, we can interpolate between these results on $B^0_{22}(\mathbf{R}^{2n})$ and $B^s_{\infty\infty}(\mathbf{R}^{2n})$ to obtain a result which states that the Weyl transform maps $B^t_{pp}(\mathbf{R}^{2n})$ into $\mathcal{I}_p$ when $p > 2$ and t is large enough. Compare this to Simon's result, that the Weyl transform does not map $B^0_{pp}(\mathbf{R}^{2n})$ into any $\mathcal{I}_q$.

THEOREM 7.1. *The Weyl transform* $\sigma \mapsto L_\sigma$ *is a bounded mapping of* $B^t_{pp}(\mathbf{R}^{2n})$ *into* $\mathcal{I}_p$ *for each* $2 < p < \infty$ *and* $2n(1 - 2/p) < t < \infty$.

Proof. Choose any p and t such that $2 < p < \infty$ and $2n(1 - 2/p) < t < \infty$. Set $\theta = 1 - 2/p$, and note that $0 < \theta < 1$. Define $s = t/\theta$, and note that $s > 2n$. Then, by Theorem 1.2, we know that the Weyl correspondence is a bounded mapping of $\Lambda^s(\mathbf{R}^{2n}) = B^s_{\infty\infty}(\mathbf{R}^{2n})$ into $\mathcal{B}(L^2(\mathbf{R}^n))$. Moreover, Pool's theorem states that the Weyl transform is a bounded mapping of $B^0_{22}(\mathbf{R}^{2n}) = L^2(\mathbf{R}^{2n})$ onto $\mathcal{I}_2$. By standard results on interpolation of Besov spaces, e.g., [BL76, Theorem 6.4.5], we have that

$$(B^0_{22}(\mathbf{R}^{2n}), B^s_{\infty\infty}(\mathbf{R}^{2n}))_{\theta,p} = B^t_{pp}(\mathbf{R}^{2n}).$$

Moreover, by the proof of and the remark following Theorem 2.c.6 in [Kön86],

$$(\mathscr{I}_2, \mathscr{B}(L^2(\mathbf{R}^n)))_{\theta, p} = \mathscr{I}_p.$$

It therefore follows that the Weyl correspondence is a bounded mapping of $B^t_{pp}(\mathbf{R}^{2n})$ into $\mathscr{I}_p$. ∎

Note added in proof. Following submission of this manuscript, we learned of the paper of Tachizawa [Tac94], which derives results on pseudodifferential operators by using a technique somewhat similar to the one used here. In particular, Tachizawa expands the symbol into a Wilson basis, rather than a Gabor frame. Both Wilson bases and Gabor frames are defined in terms of time-frequency translates of functions. However, the results in [Tac94] are distinct from ours. A recent preprint by Rochberg and Tachizawa [RT97] also uses Gabor frame expansions of the symbol to obtain results on pseudodifferential operators. As pointed out by the referee, the idea of using expansions based on some kind of time-frequency shifts has a long history in the study of integral operators. For example, some of the atomic decompositions employed by Janson, Peetre, and Rochberg in [JPR87] are of this type.

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