Representing Weak Maps of Oriented Matroids

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This paper gives new characterizations of weak maps of oriented matroids: as maps of covector complexes, as maps of vector complexes, and as homotopy equivalences of spheres.

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An oriented matroid can be represented in several equivalent ways, both combinatorial and topological. This paper will deal with oriented matroids represented either by their poset of covectors or by their poset of vectors. One of the most fundamental results of oriented matroid theory is the Topological Representation Theorem of Folkman and Lawrence [6]. This theorem gives yet another representation of a rank \( n \) oriented matroid: as an arrangement of oriented pseudospheres on \( S^{n-1} \). Such an arrangement of oriented pseudospheres gives a cell decomposition of \( S^n \), whose poset of cells is canonically isomorphic to the poset of non-zero covectors of \( M^n \). The non-zero vectors of \( M^n \) correspond in the same way to the cells in a dual representation of \( M^n \) as an arrangement of pseudospheres on \( S^{n-k-1} \), where \( k \) is the number of elements in \( M^n \). (All of this will be reviewed in more detail in Section 1.)

A weak map of oriented matroids is a combinatorial analog to moving an arrangement of oriented hyperplanes into more special position. Weak maps give a partial order on oriented matroids that is important in a variety of areas, including extension spaces and matroid bundle oriented matroids as a map between their sets of covectors or between their sets of vectors. In this paper, we will focus on weak maps as a topological map between the corresponding cell decompositions of the sphere.

Throughout the following the set \([+, 0, -] \) will always have the partial order \(+ > 0, - > 0, + \neq - , - \neq +\). For any set \( E \), the set \([+, 0, -] \) will have the product order. We will denote the set of vectors of an oriented matroid \( M \) by \( V(M) \) and the set of covectors by \( V^*(M) \). For any poset \( P \), let \( \Delta P \) denote its order complex, i.e., the abstract simplicial complex of all chains in \( P \).

**THEOREM 0.1.** Let \( M_1 \) and \( M_2 \) be oriented matroids on the same ground set. Then:

1. \( M_1 \) weak maps to \( M_2 \) if and only if there is a surjective poset map \( g^* : V^*(M_1) \rightarrow V^*(M_2) \) such that \( g^*(X) \leq X \) for every \( X \in V^*(M_2) \).
   
   If \( M_1 \) weak maps to \( M_2 \) and both oriented matroids have the same rank, then there exists such a \( g^* \) with \( (g^*)^{-1}(0) = \{0\} \).

2. \( M_1 \) weak maps to \( M_2 \) if and only if there is a poset map \( g : V(M_1) \setminus \{0\} \rightarrow V(M_2) \setminus \{0\} \) such that \( g(X) \leq X \) for every \( X \in V(M_1) \).
   
   If \( M_1 \) weak maps to \( M_2 \) and both oriented matroids have the same rank, then there exists such a \( g \) which is surjective.

The maps \( g \) and \( g^* \) are not unique, as the results of Section 3 will show. However, for every weak map there are unique maximal \( g \) and \( g^* \). They are described explicitly in Section 2. We will call these unique maximal \( g \) and \( g^* \) the poset maps representing the weak map.
Note that a map \( f : P_1 \to P_2 \) of posets induces a map \( \Delta f : \Delta P_1 \to \Delta P_2 \) of order complexes.

**Theorem 0.2.** Let \( M_1 \) and \( M_2 \) be oriented matroids of the same rank. If \( M_1 \) weak maps to \( M_2 \) and \( g : V(M_1) \setminus \{0\} \to V(M_2) \setminus \{0\} \) and \( g^* : V^*(M_1) \to V^*(M_2) \) are the poset maps representing the weak map, then \((g^*)^{-1}(0) = \{0\}\) and the induced maps \( \Delta g : \Delta V(M_1) \setminus \{0\} \to \Delta V(M_2) \setminus \{0\} \) and \( \Delta g^* : \Delta V^*(M_1) \setminus \{0\} \to \Delta V^*(M_2) \setminus \{0\} \) are homotopy equivalences.

The above theorems suggest a functorial perspective on weak maps. The category of rank \( n \) oriented matroids and weak maps is called the MacPhersonian \( \text{MacP}_n \), and is of importance in the theory of matroid bundles (cf. [1]). Let \( \text{S}_n \) be the category of \((n-1)\)-dimensional simplicial spheres and simplicial maps. Theorem 0.2 might lead one to hope for a functorial relationship between \( \text{MacP}_n \) and \( \text{S}_n \). The Topological Representation Theorem gives a map from the objects of \( \text{MacP}_n \) to the objects of \( \text{S}_n \), sending each \( M \) to \( \Delta V^*(M) \). The question is then whether this map extends to a functor from \( \text{MacP}_n \) to \( \text{S}_n \), i.e., whether weak maps of oriented matroids translate to simplicial maps of spheres in a way that is both compatible with the Topological Representation Theorem and functorial. The obvious candidate for such a functor would associate to any weak map the simplicial map \( \Delta g^* \) representing it. Unfortunately, this association is not functorial. In fact, Section 3 will show that there is no ‘good’ functor between these categories. We get a similar result on \( \Delta g \) by restricting to the category \( \text{MacP}(n, k) \) of rank \( n \) oriented matroids with elements \([1, 2, \ldots, k]\). Proposition 3.2 will show that \( g \) and \( g^* \) give ‘functors up to homotopy’ between \( \text{S}_n \) and \( \text{MacP}(k, n) \) resp. \( \text{S}_{k-n} \) and \( \text{MacP}(k, n) \).

The map \( g^* \) representing a weak map first arose in topology, in work of Davis and the author on matroid bundles [3]. The existence of \( g^* \) is a key ingredient in the construction of spherical quasifibrations associated to matroid bundles. The failure of \( g^* \) to give a functor from \( \text{MacP}_n \) to \( \text{S}_n \) appears to present a substantial obstacle to associating topological sphere bundles to matroid bundles.

### 1. Background: Oriented Matroids and Weak Maps

If \( X \) and \( Y \) are two elements of \([+, 0, -]^E\), define their composition \( X \circ Y \) by \( X \circ Y(e) = \max\{X(e), Y(e)\} \) for each \( e \in E \). Write \( X \) for \( \{e \in E : X(e) \neq 0\} \), \( X^- \) for \( \{e \in E : X(e) = 0\} \), and \( X^+ \) for \( \{e \in E : X(e) = +\} \). For any two sets \( S \) and \( T \) write \( S \cap T \) for \( \{e \in S : e \notin T\} \). Write \( A^+ B^- C^0 \) for the signed set \( X^+ = A, X^- = B, \) and \( C = E \setminus (A \cup B) \).

**Definition 1.1 (from [5]).** Let \( E \) be a finite set and \( V^*(M) \) a subset of \([+, 0, -]^E\) such that

1. \( 0 \in V^*(M) \),
2. (symmetry) \( V^*(M) = -V^*(M) \),
3. (composition) If \( X, Y \in V^*(M) \), then \( X \circ Y \in V^*(M) \),
4. (elimination) For all \( X, Y \in V^*(M) \) and \( e \in X^+ \cap Y^- \) there is a \( Z \in V^*(M) \) such that

\[
Z^+ \subseteq (X^+ \cup Y^+) \setminus e,
\]
\[
Z^- \subseteq (X^- \cup Y^-) \setminus e,
\]

and \((X \setminus Y) \cup (Y \setminus X) \cup (X^+ \cap Y^+) \cup (X^- \cap Y^-) \subseteq Z\).

Then we say \( V^*(M) \) is the set of covectors of an oriented matroid \( M \) on \( E \). The elements of \( E \) are called the elements of the oriented matroid. A loop of \( M \) is an element \( e \) such that \( X(e) = 0 \) for every \( X \in V^*(M) \).
DEFINITION 1.2. Let $M$ be an oriented matroid with elements $E$. A subset $I$ of $E$ is independent in $M$ if for every $e \in I$,

$$\{X \in V^*(M) | X(I) = 0\} \neq \{X \in V^*(M) | X(I \setminus \{e\}) = 0\}.$$ 

The rank of $M$ is the maximum order of a set of independent elements of $M$.

The motivating example: consider a rank $n$ finite set $\{v_e | e \in E\}$ of vectors in $\mathbb{R}^n$. To each point $x$ in $\mathbb{R}^n$ we can associate a sign vector $\phi(x) \in \{+, 0, -\}^E$ by $\phi(x)(e) = (v_e, x)$. The set $\{\phi(x) | x \in \mathbb{R}^n\}$ is the set of covectors of a rank $n$ oriented matroid. A set $I \subset E$ is independent in this oriented matroid if and only if $\{v_e | e \in I\}$ is independent in $\mathbb{R}^n$.

Let $A = \{v_e^* | e \in E\}$ be the arrangement of oriented hyperplanes corresponding to this vector arrangement. (We allow the degenerate hyperplane $0^* = \mathbb{R}^n$.) Note that the elements of $A$ subdivide the unit sphere $S^{n-1}$ into convex cells, and the poset of non-zero covectors is naturally isomorphic to the poset of these closed cells, ordered by inclusion. Thus this cell decomposition of the sphere can be viewed as a topological representation of the oriented matroid.

Not every oriented matroid arises from an arrangement of hyperplanes in this way. However, for any rank $n$ oriented matroid, the set of non-zero covectors can be represented by a cell decomposition of the sphere. To make this statement more specific, we need some definitions.

DEFINITION 1.3. A pseudosphere in the sphere $S^n$ is a subset $S \subset S^n$ such that some automorphism of $S^n$ takes $S$ to an equator. An oriented pseudosphere is a pseudosphere together with a distinguished connected component $S^+$ of $S^n \setminus S$. An arrangement of oriented pseudospheres is a finite multiset $\{S_e | e \in E\}$ of oriented pseudospheres in $S^n$ such that for any $A \subset E$, the intersection $S_A = \bigcap_{e \in A} S_e$ is a topological sphere, and if $e$ is an element of $E$ with $S_A \not\subset S_e$, then $S_A \cap S_e^\circ$ is a connected component of $S_A \setminus S_e$.

Let $A = \{S_e | e \in E\}$ be an arrangement of oriented pseudospheres. Then $A$ decomposes $S^{n-1}$ into regular cells, which we can identify with signed sets in $\{+, 0, -\}^E$ just as we did with arrangements of hyperplanes. Let $V^*(A)$ be the family of all such signed sets.

TOPOLOGICAL REPRESENTATION THEOREM (FROM [6]).

(1) If $A = \{S_e | e \in E\}$ is an arrangement of oriented pseudospheres on $S^{n-1}$ such that $\bigcap_{e \in E} S_e = \emptyset$, then $V^*(A)$ is the family of covectors of a rank $n$ oriented matroid on $E$.

(2) If $V^*(M)$ is the set of covectors of a rank $n$ oriented matroid with no loops, then there exists an arrangement of oriented pseudospheres $A$ in $S^{n-1}$ such that $V^*(M) = V^*(A)$.

(3) $V^*(A) = V^*(A')$ for two arrangements $A$ and $A'$ if and only if there exists a homeomorphism $h : S^{n-1} \rightarrow S^{n-1}$ such that $h(A) = A'$.

Recall [5, Section 3.4] that associated to every rank $n$ oriented matroid $M$ with ground set $E$ there is a rank $|E| - n$ oriented matroid $M^*$ with elements $E$, called the dual to $M$. We may define the poset of vectors $V(M)$ of $M$ to be $V^*(M^*)$.

Let $M_1$ and $M_2$ be oriented matroids on the same ground set. Recall that there is a weak map from $M_1$ to $M_2$, written $M_1 \rightsquigarrow M_2$, if the following two equivalent properties hold:

(1) For every $X \in V(M_1) \setminus \{0\}$ there exists $Y \in V(M_2) \setminus \{0\}$ such that $Y \leq X$.

(2) For every $Y \in V^*(M_2)$ there exists $X \in V^*(M_1)$ such that $Y \leq X$.  


For instance, if \( \{v_e \mid e \in E\} \) and \( \{w_e \mid e \in E\} \) are vector arrangements giving oriented matroids \( M_v \) and \( M_w \), let \( A_v \) and \( A_w \) be the \( n \times |E| \) matrices with columns \( \{v_e \mid e \in E\} \) resp. \( \{w_e \mid e \in E\} \). Then \( M_v \leadsto M_w \) if and only if the sign of each maximal minor of \( A_v \) is less than or equal to the sign of the corresponding minor of \( A_w \). Thus \( M_v \leadsto M_w \) if \( M_w \) is obtained from \( M_v \) by moving the vectors \( v_e \) into ‘more special position’.

**Proposition 1.4 ([5, Corollary 7.7.7]).**

1. If \( M_1 \leadsto M_2 \), then \( \text{rank}(M_1) \geq \text{rank}(M_2) \).
2. If there is a rank-preserving weak map \( M_1 \leadsto M_2 \), then \( M_1^* \leadsto M_2^* \) is a rank-preserving weak map as well.

**Definition 1.5.** If \( X \in \{+,-,0\}^E \) and \( A \subset E \), define \( X \setminus A \) to be the signed set

\[
X \setminus A(e) = \begin{cases} 
  X(e) & \text{if } e \notin A \\
  0 & \text{if } e \in A.
\end{cases}
\]

**Definition 1.6.** Let \( M \) be an oriented matroid on \( E \), and let \( A \) be a subset of \( E \). The contraction \( M \setminus A \) of \( M \) by \( A \) is the oriented matroid on \( E \) with covectors \( \{X \in V^*(M) \mid X(A) = 0\} \). The deletion \( M \setminus A \) of \( A \) from \( M \) is the oriented matroid on \( E \) with covectors \( \{X \setminus A \mid X \in V^*(M)\} \).

(Note this definition of deletion is a slight departure from that in, for instance, [5]. We do not actually delete elements, we just make them loops.)

2. **Construction of \( g \) and \( g^* \)**

2.1. **Covectors and \( g^* \).** This section will prove the covector statements of Theorems 0.1 and 0.2.

**Proposition 2.1.** Let \( M_1 \leadsto M_2 \) be a weak map of oriented matroids.

1. For any \( X \in V^*(M_1) \), the set \( \{Y \in V^*(M_2) \mid Y \leq X\} \) has a unique maximal element, which we will denote \( g^*(X) \).
2. The map \( g^* : V^*(M_1) \to V^*(M_2) \) described above is a surjective poset map.
3. If \( M_1 \) and \( M_2 \) have the same rank, then \( (g^*)^{-1}(0) = \{0\} \).

As alluded to in the introduction, a closely related result appears in [3].

**Proof.**

1. First note that \( X \geq 0 \in V^*(M_2) \) for any \( X \in V^*(M_1) \). Now assume \( Y_1 \) and \( Y_2 \) are two covectors of \( M_2 \) such that \( X \geq Y_1 \) and \( X \geq Y_2 \). Then for any element \( e \) of \( M_2 \), we know \( Y_1(e)Y_2(e) \geq 0 \), and so \( Y_1 \circ Y_2 = Y_2 \circ Y_1 \). Thus the composition \( Y \) of all \( Y_i \in V^*(M_2) \) such that \( X \geq Y_i \) is independent of the order of the \( Y_i \), and hence is the unique maximal covector such that \( X \geq Y \).
2. Let \( X, Y \) be elements of \( V^*(M_1) \) such that \( X \geq Y \). Then \( Y \geq g^*(Y) \) implies that \( X \geq g^*(Y) \). Thus by unique maximality of \( g^*(X) \), we have that \( g^*(X) \geq g^*(Y) \). Thus \( g^* \) is a poset map.
To see $g^*$ is surjective, we induct on rank($M_2$). If rank($M_2$) = 1, then $M_2$ has three covectors, the 0 covector and two maximal covectors. Certainly 0 $\in$ $(g^*)^{-1}(0)$, and by definition the preimage of a maximal covector is non-empty.

For $M_2$ of larger rank, let $Z = A^+B^{-1}C^0$ be a covector of $M_2$. We have two cases:

- If $C$ is the set of loops of $M_2$, then $Z$ is maximal in $V^*(M_2)$, and again clearly $(g^*)^{-1}(Z)$ $\neq \emptyset$.
- Otherwise, let $e$ be an element of $C$ which is a non-loop in $M_2$. Then by Lemma A.15 in [2] $M_1/e \rightsquigarrow M_2/e$, and so by the induction hypothesis there is some $X$ $\in$ $V^*(M_1/e)$ $\subset$ $V^*(M_1)$ such that $Z$ is the maximal covector in $M_2/e$ such that $X \geq Z$.

But since $X(e) = 0$, we know $g^*(X) \in V^*(M_2/e)$. Thus $g^*(X) = Z$.

Finally, we show that if rank($M_1$) = rank($M_2$) and $X$ $\in$ $V^*(M_1)\setminus\{0\}$, then $g^*(X)$ $\neq 0$. It suffices to show that for any minimal non-zero element $X$ of $V^*(M_1)$, there is some non-zero $Y$ in $V^*(M_2)$ such that $X \geq Y$. We induct on rank, and within a given rank we induct on the number of non-loops of $M_1$. In the base case of either induction the map $g^*$ is clearly a poset isomorphism.

Above the base case, if $X$ is minimal, then there is some non-loop $e$ of $M_1$ such that $X(e) = 0$. We have two cases:

- If $e$ is a loop in $M_2$, then $M_1\setminus e \rightsquigarrow M_2\setminus e$ and $X\setminus e \in V^*(M_1\setminus e)$, and so by induction on the number of elements we get a non-zero $Y$ $\in$ $V^*(M_2\setminus e)$ such that $Y \leq X\setminus e$. But since $e$ is a loop of $M_2$, $Y$ is also an element of $V^*(M_2)$.
- If $e$ is a non-loop in $M_2$, then by Lemma A.15 in [2], $M_1/e \rightsquigarrow M_2/e$, and $X$ $\in$ $V^*(M_1/e)$. Thus by induction on rank we get a non-zero $Y$ $\in$ $V^*(M_2/e)$ $\subset$ $V^*(M_2)$.

This proves one direction of Theorem 0.1. The other direction is clear. The particular $g^*$ given by the above proposition is the map of covectors representing the weak map.

The proof of the covector statement of Theorem 0.2 will uses a result of Quillen:

**QUILLEN’S FIBER THEOREM (cf. [4]).** Let $f : P \to Q$ be a poset map. If for all $q$ $\in$ $Q$, $\Delta f^{-1}(q_{\geq q})$ is contractible, then $\Delta f$ is a homotopy equivalence.

**PROOF OF THEOREM 0.2 (COVECTOR STATEMENT).** We saw above that $(g^*)^{-1}(0) = \{0\}$. Thus if $M_1$ and $M_2$ have the same rank, then $g^*$ restricts to a map $V^*(M_1)\setminus\{0\} \to V^*(M_2)\setminus\{0\}$. Taking order complexes, we get a simplicial map $h = \Delta g^*[V^*(M_1)\setminus\{0\}] : \Delta V^*(M_1)\setminus\{0\} \to \Delta V^*(M_2)\setminus\{0\}$ of spheres. To see this is a homotopy equivalence, note that for any $Y$ $\in$ $V^*(M_2)\setminus\{0\}$, the set $h^{-1}(V^*(M_2)_{\geq Y})$ is just $\Delta V^*(M_1)_{\geq Y}$. This is easily seen to be contractible: $Y$ is a maximal cell in a pseudosphere arrangement corresponding to $M_1\setminus 0$, and $\Delta V^*(M_1)_{\geq Y}$ is the barycentric subdivision of a subdivision of the interior of this cell. Thus $h$ is a homotopy equivalence by Quillen’s Theorem A.

2.2. **Vectors and g.** The statements about vectors in Theorems 0.1 and 0.2 now follow easily.

**LEMMA 2.2.** Let $M_1 \rightsquigarrow M_2$ be a weak map of oriented matroids.

1. For every $X$ $\in$ $V(M_1)$ there is a unique maximal $g(X)$ $\in$ $V(M_2)$ such that $g(X) \leq X$.
   Further, if $X \neq 0$ then $g(X) \neq 0$.
2. The resulting map $g : V(M_1) \to V(M_2)$ is a poset map.
The map $g$ given by the preceding lemma is the map of vectors representing the weak map.

Note that $g$ may not be surjective. For instance, let $M_1$ be the unique oriented matroid on $E$ of rank $|E|$ and let $M_2$ be any other oriented matroid on $E$. Then $M_1 \sim M_2$, but $|V(M_1)| = 1$ and $V(M_2) > 1$. However, if $M_1$ and $M_2$ have the same rank, then by Proposition 1.4.2 the map $g$ can be viewed as the map of covectors induced by $M_1^e \sim M_2^e$. Thus in this case $g$ is surjective and induces a homotopy equivalence of pseudospheres, by the results of the previous section. This completes the proofs of Theorems 0.1 and 0.2.

3. A Not-Quite Functorial Relationship

The above results raise hopes for a functorial relationship between the MacPhersonian and the category of simplicial spheres and simplicial maps. The aim of this section is to dash these hopes. (Proposition 3.2 will then partially resurrect them, by showing that our maps $g^*$ and $g$ give something homotopically `close' to functors.)

As noted in the introduction, the map $M \to V^*(M)\setminus\{0\}$ maps objects of the category MacP$_n$ to the objects of $S_n$, and the map $M \to V(M)\setminus\{0\}$ maps objects of MacP$(n,k)$ to objects of $S_{k-n}$. To extend one of these maps $F$ to a functor, one would need to associate to each rank-preserving weak map $M_1 \sim M_2$ a simplicial map $F(M_1 \sim M_2)$ from $F(M_1)$ to $F(M_2)$ in such a way that whenever $M_1 \sim M_2 \sim M_3$ the maps $F$ satisfy $F(M_1 \sim M_3) = F(M_2 \sim M_3) \circ F(M_1 \sim M_2)$. The simplicial maps $g$ and $g^*$ do not give us such functors. Indeed, there is no good functor between these categories. We will show this for the map $M \to V^*(M)$. Then by Proposition 1.4(2), a similar result holds for vectors.

By a `good' functor, we mean a functor $F$ that satisfies the following three conditions.

**Object Axiom:** For every oriented matroid $M$, $F(M) = \Delta V^*(M)$.

**Contraction Axiom:** If $M_1 \sim M_2$ and $e$ is non-zero in $M_2$, then $F(M_1 \sim M_2)|_{\Delta V^*(M_1/e)} = F(M_1/e \sim M_2/e)$. (Note: as mentioned before, Lemma A.15 in [2] says that $M_1/e \sim M_2/e$.)

**Deletion Axiom:** For every oriented matroid $M$, non-coloop $e$ of $M$, and covector $X$ of $M$, $F(M \sim M\setminus e)(X) = X\setminus e$.

Put informally, the latter two axioms require $F$ to `respect contraction and deletion'. One easily checks that the covector maps given by $g^*$ satisfy all three axioms.

**Theorem 3.1.** For every $n > 1$ and $k > n$, there is no functor from MacP$(n,k)$ to $S_n$ satisfying all three axioms.

**Proof.** We first show there is no such functor for $n = 2$ and $k > n$. Assume there were such a functor $F$. Consider the three rank 2 realizable oriented matroids given by the hyperplane arrangements in Figure 1. (Here 3 is a loop of $M_3$ and $\{4, 5, \ldots, k\}$ are zero elements of all three oriented matroids.) Let $M_1 \sim M_2 \sim M_3$.

Let $X = \{1,2\}^{-\{3, \ldots, k\}} \in V^*(M_1)$. Then

$$F(M_1 \sim M_3)(X) = F(M_2 \sim M_3) \circ F(M_1 \sim M_2)(X)$$

by functoriality of $F$

$$= F(M_2 \sim M_3) \circ F(M_1/3 \sim M_2/3)(X)$$

by Contraction Axiom

$$= F(M_2 \sim M_3)(Y),$$

where $Y$ is some element of $V^*(M_2/3)$. 


Thus \( F(M_1 \leadsto M_2)(X) \) is either \( 1^{-\{2, \ldots, k\}} \) or \( 1^{+\{2, \ldots, k\}} \). In particular, \( F(M_1 \leadsto M_2)(X) \in V^+(M_2/[2]) \). Applying the Contraction Axiom, we see \( F(M_1 \leadsto M_3)(X) = F(M_2/2 \leadsto M_3/2)(Y) \), and so \( F(M_1 \leadsto M_3)(2) = 0 \). On the other hand, applying the Deletion Axiom to the weak map \( M_1 \leadsto M_3 \), we see \( F(M_1 \leadsto M_3)(X) = X^{1^3} = [1, 2]^{-\{3, \ldots, k\}} \), a contradiction.

Using this we can show there is no such functor \( \text{MacP}(n, k) \to S_n \) for all \( n > 2 \) and \( k > n \). For \( i \in \{1, 2, 3\} \), let \( M_i \) be the rank two oriented matroid \( M_i \) arising in the previous argument in the case \( k = 3 \). Let \( M_0 \) be the unique rank \( n - 2 \) oriented matroid with non-loops \( \{4, 5, \ldots, n + 1\} \) and loops \( \{n + 2, \ldots, k\} \). For \( i \in \{1, 2, 3\} \), let \( M'_i = M_i \oplus M_0 \in \text{MacP}(n, k) \). Then \( M_i \leadsto M'_i \leadsto M'_i \). By the Contraction Axiom, for each \( i < j \) we have \( F(M'_i \leadsto M'_j)(X) = F(M'_i/[4, \ldots, k] \leadsto M'_j/[4, \ldots, k])(X) \). But by the above argument we know \( F(M'_i/[4, \ldots, k] \leadsto M'_j/[4, \ldots, k])(X) \neq F(M'_i/[4, \ldots, k] \leadsto M'_j/[4, \ldots, k]) \circ F(M'_i/[4, \ldots, k] \leadsto M'_j/[4, \ldots, k])(X) \).

Finally, the following shows that the correspondences given by \( g \) and \( g^* \) are, at least, ‘functors up to homotopy’.

**Proposition 3.2.** Let \( M_1, M_2, M_3 \) be oriented matroids such that \( M_1 \leadsto M_2 \) and \( M_2 \leadsto M_3 \) (and hence \( M_1 \leadsto M_3 \)). For each \( i > j \), let \( g_{ij} : V(M_1) \to V(M_2) \) and \( g^*_{ij} : V^*(M_1) \to V^*(M_2) \) be the poset maps representing the weak map. Then for every \( X \in V(M_1) \) and \( Y \in V^*(M_1) \), we have \( g_{31}(X) \geq g_{32} \circ g_{21}(X) \) and \( g_{31}^*(Y) \geq g_{32}^* \circ g_{21}^*(Y) \).

It is the failure of these inequalities to be equalities that shows \( g \) and \( g^* \) do not give functors. An immediate corollary is that \( g_{31} \) is homotopic to \( g_{32} \circ g_{21} \) and \( g_{31}^* \) is homotopic to \( g_{32}^* \circ g_{21}^* \) (cf. Theorem 10.11 in [4]).

**Proof.** Let \( X \) be an element of \( V(M) \) and \( e \) be an element of \( E \). If \( g_{13}(X)(e) = 0 \), then there is no \( Z \in V(M_3) \) such that \( Z(e) \neq 0 \) and \( Z \leq X \). Since \( g_{21}(X) \leq X \), this implies there is no \( Z \in V(M_3) \) such that \( Z(e) \neq 0 \) and \( Z \leq g_{21}(X) \). Thus \( g_{32} \circ g_{21}(X)(e) = 0 \).

A similar proof shows the result for \( g^* \) and \( Y \).
ACKNOWLEDGEMENTS

The author thanks James Davis and Eric Babson for numerous observations. This work was partially supported by NSF grant DMS 9803615.

REFERENCES


Received 1 February 1999 in revised form 1 November 1999

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