Twice Q-Polynomial Distance-regular Graphs are Thin

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Let $\Gamma$ be a distance-regular graph of diameter $d \geq 3$. For each vertex $x$ of $\Gamma$, let $T(x)$ denote the subconstituent algebra for $\Gamma$ with respect to $x$. An irreducible $T(x)$-module $W$ is said to be thin if $\dim E^i(x)W < 1$ for $0 \leq i \leq d$, where $E^i(x)$ is the projection onto the $i$th subconstituent for $\Gamma$ with respect to $x$. The graph $\Gamma$ is said to be thin if, for each vertex $x$ of $\Gamma$, every irreducible $T(x)$-module is thin. Our main result is the following Theorem: If $\Gamma$ has two Q-polynomial structures, then $\Gamma$ is thin.

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1. INTRODUCTION

If $\Gamma$ is a distance-regular graph which is Q-polynomial with respect to some eigenvalue $\theta$, and the eigenvalues of $\Gamma$ are not all integers, then $\Gamma$ is also Q-polynomial with respect to an eigenvalue $\psi \neq \theta$ [1, p. 360]. Since the existence of a single Q-polynomial structure gives us regularity conditions on $\Gamma$ [5–7], we expect that the interaction of two Q-polynomial structures will give us extra regularity conditions. We found in our earlier paper [3] that such a graph $\Gamma$ has vertex neighborhood graphs which are strongly regular, and calculated the parameters of these neighborhood graphs.

In the present paper, we find in Theorem 2.1 that a distance-regular graph of diameter at least 3 which has two Q-polynomial structures is thin in the sense of Terwilliger [5].

In Sections 1.1–1.6 we introduce notation and definitions used in the remainder of the paper; the thin property is defined in Section 1.6. See [1], [2], and [5] for further introduction to this material.

1.1. Preliminaries. Let $\Gamma = (X, \mathcal{R})$ be a connected, undirected graph with vertex set $X$ and edge set $\mathcal{R}$. Let $\partial$ denote the usual path-length distance on $\Gamma$, and let $d$ denote the maximum of $\partial(x, y)$ over $x, y \in X$. We refer to $d$ as the diameter of $\Gamma$.

Let $\text{Mat}_X$ denote the algebra of matrices over $\mathbb{R}$ with rows and columns indexed by $X$. The distance matrices $A_0, \ldots, A_d \in \text{Mat}_X$ for $\Gamma$ are defined by

$$
(A_i)_{xy} = \begin{cases} 
1 & \text{if } \partial(x, y) = i, \\
0 & \text{otherwise}; 
\end{cases} \quad x, y \in X.
$$

Observe that $A = A_1$ is the usual adjacency matrix for $\Gamma$.

Let $V = \mathbb{R}^X$ denote the column space of $\text{Mat}_X$. Place an inner product on $V$ by letting $\langle v, w \rangle = v^t w$ for vectors $v, w \in V$. We refer to $V$ as the standard module for $\Gamma$. For each vertex $x \in X$, let $\mathcal{E}$ denote the vector in $V$ with a 1 in the $x$ entry and 0's elsewhere. Observe by (1) that

$$
\langle A_i \mathcal{E}, \mathcal{E} \rangle = \begin{cases} 
1 & \text{if } \partial(x, y) = i, \\
0 & \text{otherwise} \end{cases} \quad x, y \in X.
$$

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1.2. Distance-regular graphs. Let $\Gamma = (X, \mathcal{R})$ be a connected, undirected graph of diameter $d$. Recall that $\Gamma$ is distance-regular if there are integers $p_{ij}^h$, $0 \leq h, i, j \leq d$ so that for any integer $h$ with $0 \leq h \leq d$ and any vertices $x, y \in X$ with $\partial(x, y) = h$,

$$p_{ij}^h = |\{z \in X : \partial(x, z) = i, \partial(y, z) = j\}|, \quad 0 \leq i, j \leq d.$$  

(3)

The integers $p_{ij}^h$ are called the intersection numbers for $\Gamma$, and are abbreviated $k = p_{11}^0$, $a_i = p_{ii}^0$, $b_i = p_{i+1,1}^0$, $c_i = p_{i-1,1}^0$. It can be shown using (3) that $b_0, \ldots, b_{d-1}$ and $c_1, \ldots, c_d$ are not zero.

1.3. Eigenvalues and dual eigenvalues. Let $\Gamma = (X, \mathcal{R})$ be a distance-regular graph of diameter $d \geq 3$. By an eigenvalue of $\Gamma$ we mean an eigenvalue of the adjacency matrix $A$. Let $\theta$ be an eigenvalue of $\Gamma$, and let $E$ be the corresponding primitive idempotent. It is known that $E$ is of the form

$$E = \frac{1}{|X|} \sum_{i=0}^{d} \theta_i^* A_i$$

(4)

for some real numbers $\theta_0^*, \ldots, \theta_d^*$. The numbers $\theta_0^*, \ldots, \theta_d^*$ are referred to as the dual eigenvalues to $\theta$.

It is known that $\theta$ is not the trivial eigenvalue $k$ exactly when $\theta_0^* \neq \theta_d^*$. In this case, the dual eigenvalues satisfy

$$\theta = b_1 \frac{\theta_1^* - \theta_d^*}{\theta_0^* - \theta_d^*} - 1$$

(5)

[3]. Combining (2) and (4), we have

$$|X| \langle E \hat{\tau}, \hat{y} \rangle = \theta_j^*$$

(6)

for each pair of vertices $x, y \in X$ with $\partial(x, y) = j$. The identities (5) and (6) will be useful in the proof of Lemma 2.4.

1.4. Q-polynomial distance-regular graphs. Let $\Gamma = (X, \mathcal{R})$ be a distance-regular graph of diameter $d \geq 3$. Let $\theta$ be an eigenvalue of $\Gamma$, with dual eigenvalues $\theta_0^*, \ldots, \theta_d^*$. We say that $\Gamma$ is Q-polynomial with respect to $\theta$ if, for each integer $i$ with $0 \leq i \leq d$, there exists a polynomial $q_i$ of degree exactly $i$ so that the matrix

$$E_i = \sum_{j=0}^{d} q_i(\theta_j^*) A_j$$

(7)

is a primitive idempotent of $\Gamma$ [2, p. 135].

If $\Gamma$ is Q-polynomial with respect to $\theta$ then, for $0 \leq i, j \leq d$,

$$\theta_i^* \neq \theta_j^* \quad \text{whenever } i \neq j$$

(8)

[2, p. 135]. In particular, the identity (5) holds.

1.5. The subconstituent algebra. Let $\Gamma = (X, \mathcal{R})$ be a distance-regular graph of diameter $d$, and fix a vertex $x \in X$. For $0 \leq i \leq d$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X$ with

$$(E_i^*)_{yy} = \begin{cases} 
1 & \text{if } \partial(x, y) = i, \\
0 & \text{otherwise},
\end{cases} \quad y \in X.$$ 

(9)

We refer to $E_i^*$ as the $i$th dual idempotent for $\Gamma$ with respect to $x$. Let $T = T(x)$ denote the sub-algebra of $\text{Mat}_X$ generated by the distance matrices $A_0, \ldots, A_d$ and the dual
idempotents $E_0^*, \ldots, E_d^*$. We refer to $T$ as the subconstituent algebra for $\Gamma$ with respect to $x$ [5, Section 1].

There are three matrices in $T$ which are of particular interest to us. These are the lowering matrix $L = L(x)$, the flat matrix $F = F(x)$, and the raising matrix $R = R(x)$, defined by

$$
L = \sum_{i=1}^{d} E_{i-1}^* AE_i^*, \quad F = \sum_{i=0}^{d} E_i^* AE_i^*, \quad R = \sum_{i=0}^{d-1} E_{i+1}^* AE_i^*. 
$$

1.6. Thin distance-regular graphs. Let $\Gamma = (X, \mathcal{R})$ be a distance-regular graph of diameter $d$. Fix a vertex $x \in X$, and write $T = T(x)$, $E_i^* = E_i^*(x)$, $0 \leq i \leq d$. By a $T$-module we mean a subspace of the standard module $V$ which is invariant under $T$-multiplication. A $T$-module is said to be irreducible if it properly contains no $T$-modules other than 0. An irreducible $T$-module $W$ is said to be thin if

$$
\dim E_i^* W \leq 1, \quad 0 \leq i \leq d. 
$$

The graph $\Gamma$ is said to be thin if, for each $x \in X$, every irreducible $T(x)$-module is thin. See [7, Section 6] for many examples of distance-regular graphs which are thin.

2. Results

Our main result is the following theorem.

**Theorem 2.1.** Let $\Gamma = (X, \mathcal{R})$ be a distance-regular graph of diameter $d \geq 3$. If $\Gamma$ is $Q$-polynomial with respect to distinct eigenvalues $\theta$ and $\psi$, then $\Gamma$ is thin.

We break the proof of Theorem 2.1 into cases according to the value of the intersection number $a_2$. Lemma 2.3 shows that $\Gamma$ is thin in the case $a_2 = 0$, and Lemma 2.5 shows that $\Gamma$ is thin in the case $a_2 \neq 0$. The following lemma is used to prove both results. (Recall the matrix commutator $[B, C] = BC - CB$.)

**Lemma 2.2.** (Terwilliger). Let $\Gamma = (X, \mathcal{R})$ be a distance-regular graph of diameter $d \geq 3$. Suppose that $\Gamma$ is Q-polynomial with respect to an eigenvalue $\theta$, and let

$$
E = \frac{1}{|X|} \sum_{h=0}^{d} \theta_h^* A_h 
$$

be the primitive idempotent corresponding to $\theta$. Fix a vertex $x \in X$, and write $L = L(x)$, $F = F(x)$, $R = R(x)$ and $E_i^* = E_i^*(x)$, $0 \leq i \leq d$. Then:

(i) [8, Lemma 5.5(i)]

$$
[F, LR]E_i^* = (\mu_i(\theta) - 1)[F, RL]E_i^*, \quad 1 \leq i \leq d - 1, 
$$

where

$$
\mu_i(\theta) = \frac{\theta_{i-1}^* - \theta_{i+1}^*}{\theta_i^* - \theta_{i+1}^*}, \quad 1 \leq i \leq d - 1; 
$$

(ii) [8, Theorem 5.1(v)] $\Gamma$ is thin iff

$$
[F, RL]E_i^* = 0, \quad 2 \leq i \leq d - 1. 
$$
LEMMA 2.3. Let $\Gamma = (X, \mathcal{R})$ be a distance-regular graph of diameter $d \geq 3$. Suppose that $\Gamma$ is $Q$-polynomial with respect to an eigenvalue $\theta$, and suppose that the intersection number $a_2$ is zero. Then:

(i) there exist real numbers $\gamma(\theta), g^-_i(\theta)$ and $g^+_i(\theta)$ such that

\[
g^-_i(\theta)a_{i-2} + a_{i-1} + g^+_i(\theta)a_i = \gamma(\theta), \quad 2 \leq i \leq d, \quad (18)
g^+_i(\theta) \neq 0, \quad 2 \leq i \leq d-1; \quad (19)
\]

(ii) the intersection numbers $a_1, \ldots, a_{d-1}$ are all zero;

(iii) the graph $\Gamma$ is thin.

PROOF OF (i). Fix a vertex $x \in X$, and write $L = L(x)$, $F = F(x)$, $R = R(x)$ and $E_i^* = E_i^*(x)$, $0 \leq i \leq d$. Then, by Terwilliger [7, Lemma 5.5(iii), Lemma 5.6] there exist real numbers $\gamma(\theta), g^-_i(\theta)$ and $g^+_i(\theta)$ satisfying (19), so that

\[
(g^-_i(\theta)FL^2 + LFL + g^+_i(\theta)L^2F - \gamma(\theta)L^2)E_i^* = 0, \quad 2 \leq i \leq d. \quad (20)
\]

Let $\delta_i$ denote the vector

\[
\delta_i = \sum_{y \in X} \delta_{(x,y) = i}, \quad 0 \leq i \leq d \quad (21)
\]

and observe by (10) and (11) that

\[
L\delta_i = b_{i-1}\delta_{i-1}, \quad 1 \leq i \leq d, \quad (22)

F\delta_i = a_i\delta_i, \quad 0 \leq i \leq d. \quad (23)
\]

Applying (20) to the vector $\delta_i$ and using (22) and (23), we obtain

\[
b_{i-2}b_{i-1}(g^-_i(\theta)a_{i-2} + a_{i-1} + g^+_i(\theta)a_i - \gamma(\theta))\delta_{i-2} = 0, \quad 2 \leq i \leq d. \quad (24)
\]

and (18) follows immediately. \hfill \Box

PROOF OF (ii). If $a_1 \neq 0$ then $a_2, \ldots, a_{d-1} \neq 0$ [2, Proposition 5.5.1(i)]. Since $a_2 = 0$, we have $a_1 = 0$; trivially, $a_0 = 0$. Setting $i = 2$ in (18), we find that $\gamma(\theta) = 0$, and (18) becomes

\[
g^-_i(\theta)a_{i-2} + a_{i-1} + g^+_i(\theta)a_i = 0, \quad 2 \leq i \leq d. \quad (25)
\]

Recalling from (19) that $g^+_i(\theta)$ is not zero for $2 \leq i \leq d - 1$, a simple induction shows that $a_1, \ldots, a_{d-1}$ are zero, as desired. \hfill \Box

PROOF OF (iii). Observe by part (ii) and (11) that $FE_i^* = 0$ for $1 \leq i \leq d - 1$. Also, observe by (10) and (12) that the matrices $RL$ and $E_i^*$ commute. Now

\[
[F, RL]E_i^* = FRLE_i^* - RLFE_i^* \quad (26)
\]

\[
= FE_i^*RL - RLFE_i^* \quad (27)
\]

\[
= 0 \quad (28)
\]

for $1 \leq i \leq d - 1$. Applying Lemma 2.2(ii), we see that $\Gamma$ is thin as desired. \hfill \Box

For the proof of Theorem 2.1 in the case $a_2 \neq 0$, we introduce integers $\sigma$ and $\sigma'$
which count certain four-vertex configurations in \( \Gamma \). We show in Lemma 2.4 that \( \theta \) can be computed from \( \mu_i(\theta) \), the intersection numbers of \( \Gamma \), \( \sigma \) and \( \sigma' \). This will lead to our main result, Lemma 2.5.

**Lemma 2.4.** Let \( \Gamma = (X, \mathcal{R}) \) be a distance-regular graph of diameter \( d \geq 3 \). Suppose that \( \Gamma \) is \( Q \)-polynomial with respect to an eigenvalue \( \theta \), and suppose that the intersection number \( b_2 \) is not zero.

Fix an integer \( i \) with \( 2 \leq i \leq d - 1 \). Fix vertices \( x, y, z \in X \) with \( d(x, y) = 2 \), \( d(y, z) = i - 1 \) and \( d(x, z) = i + 1 \), and let \( \sigma \) and \( \sigma' \) denote the integers

\[
\sigma = |\{u \in X | d(x, u) = 2, \ d(y, u) = 1, \ d(z, u) = i - 1\}|, \\
\sigma' = |\{v \in X | d(x, v) = 1, \ d(y, v) = 2, \ d(z, v) = i + 1\}|
\]

Then

\[
b_1 = (\theta + b_1 + 1)

\left(1 - \frac{\sigma' + \sigma(\mu_i(\theta) - 1)}{a_2\mu_i(\theta)}\right)
\]

where \( \mu_i(\theta) \) is as in (16). In particular, the eigenvalue \( \theta \) may be calculated from \( \mu_i(\theta) \), the integers \( \sigma \) and \( \sigma' \), and the intersection numbers \( b_1 \) and \( a_2 \).

**Proof.** Let \( \theta \) have corresponding primitive idempotent \( E \) and dual eigenvalues \( \theta_0^*, \ldots, \theta_d^* \). For \( 0 \leq i, j \leq d \), let \( \delta_{ij} \) denote the vector

\[
\delta_{ij} = \sum_{w \in X} \alpha(x, w) \alpha(y, w) \delta_{\left\{ w \right\}}
\]

Then the vector

\[
\delta_{21} - \delta_{12} - a_2 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_2^*} (\tilde{y} - \tilde{x})
\]

is in \( EV^\perp \) [4, Theorem 1.1; 8, Theorem 3.3]; in particular,

\[
0 = \left\langle E \tilde{x}, \delta_{21} - \delta_{12} - a_2 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_2^*} (\tilde{y} - \tilde{x}) \right\rangle.
\]

There are exactly \( a_2 \) vertices \( u \in X \) with \( d(x, u) = 2 \) and \( d(y, u) = 1 \). Of these, \( \sigma \) have \( d(z, u) = i - 1 \), leaving \( a_2 - \sigma \) with \( d(z, u) = i \). Applying (6),

\[
|X| \left\langle E \tilde{x}, \delta_{21} \right\rangle = \sigma \theta_{i-1}^* + (a_2 - \sigma) \theta_i^*.
\]

Similarly, of the \( a_2 \) vertices \( v \in X \) with \( d(x, v) = 1 \) and \( d(y, v) = 2 \), exactly \( \sigma' \) have \( d(z, v) = i + 1 \), leaving \( a_2 - \sigma' \) with \( d(z, v) = i \); we have

\[
|X| \left\langle E \tilde{x}, \delta_{12} \right\rangle = \sigma' \theta_{i+1}^* + (a_2 - \sigma') \theta_i^*.
\]

Evaluating (34) using (35), (36) and (6), we obtain

\[
0 = (\sigma \theta_{i-1}^* + (a_2 - \sigma) \theta_i^*) - (\sigma' \theta_{i+1}^* + (a_2 - \sigma') \theta_i^*) - a_2 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_2^*} (\theta_{i+1}^* - \theta_{i-1}^*)
\]

\[
= \sigma (\theta_{i-1}^* - \theta_i^*) + \sigma' (\theta_i^* - \theta_{i+1}^*) - a_2 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_2^*} (\theta_{i+1}^* - \theta_{i-1}^*).
\]
Rearranging (38) and applying (16), we have
\[
\frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_2^*} = \frac{\sigma'(\theta_1^* - \theta_{i+1}^*) + \sigma(\theta_{i-1}^* - \theta_{i+1}^*)}{a_2(\theta_{i-1}^* - \theta_{i+1}^*)} = \frac{\sigma' + \sigma(\mu_i(\theta) - 1)}{a_2\mu_i(\theta)}.
\]  
(39)

(40)

Now, by (5) and (40),
\[
b_1 = (\theta + b_1 + 1) \left( 1 - \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_2^*} \right)
\]  
(41)

\[
= (\theta + b_1 + 1) \left( 1 - \frac{\sigma' + \sigma(\mu_i(\theta) - 1)}{a_2\mu_i(\theta)} \right).
\]  
(42)

as desired.

\[\square\]

**LEMMA 2.5.** Let \( \Gamma = (X, \mathcal{R}) \) be a distance-regular graph of diameter \( d \geq 3 \). Suppose that \( \Gamma \) is \( Q \)-polynomial with respect to distinct eigenvalues \( \theta \) and \( \psi \), and suppose that the intersection number \( a_2 \) is not zero. Then \( \Gamma \) is thin.

**PROOF.** Suppose, for a contradiction, that \( \Gamma \) is not thin. Fix a vertex \( x \in X \), and write \( L = L(x) \), \( F = F(x) \), \( R = R(x) \) and \( E_i^* = E_i^*(x) \), \( 0 \leq i \leq d \). Then, by Lemma 2.2(ii), there exists an integer \( i \) with \( 2 \leq i \leq d - 1 \) so that \( [F, RL]E_i^* \) is not zero. Fix such an integer \( i \).

Since \( [F, RL]E_i^* \) is not zero, we can solve the equation (15) for \( \mu_i(\theta) \), obtaining a constant which does not depend on \( \theta \). In particular, \( \mu_i(\theta) = \mu_i(\psi) \). Now (31) applies to both \( \theta \) and \( \psi \), forcing \( \theta = \psi \). This is a contradiction, and \( \Gamma \) is thin as desired. \[\square\]

**REFERENCES**


Received 24 August 1994 and accepted in revised form 28 January 1995

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