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When is the annihilating ideal graph of a zero-dimensional quasisemilocal commutative ring complemented?

S. VISWESWARAN*, HIREN D. PATEL

Department of Mathematics, Saurashtra University, Rajkot, 360 005, India

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Abstract. Let R be a commutative ring with identity. Let $A(R)$ denote the collection of all annihilating ideals of R (that is, $A(R)$ is the collection of all ideals I of R which admits a nonzero annihilator in R). Let $AG(R)$ denote the annihilating ideal graph of R. In this article, necessary and sufficient conditions are determined in order that $AG(R)$ is complemented under the assumption that R is a zero-dimensional quasisemilocal ring which admits at least two nonzero annihilating ideals and as a corollary we determine finite rings R such that $AG(R)$ is complemented under the assumption that $A(R)$ contains at least two nonzero ideals.

Keywords: Annihilating ideal graph of a commutative ring; Complemented graph; Zero-dimensional quasisemilocal ring; Special principal ideal ring

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1. INTRODUCTION

The rings considered in this article are nonzero commutative rings with identity. Recall from [5] that an ideal I of a ring R is an annihilating ideal if there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. As in [5], we denote by $A(R)$, the set of all annihilating ideals of R and by $A(R)^*$, the set of all nonzero annihilating ideals of R. In [5], the authors introduced the concept of annihilating ideal graph of R, denoted by $AG(R)$, which is defined as follows: $AG(R)$ is an undirected graph whose vertex set is $A(R)^*$ and two distinct vertices I and J are adjacent in this graph if and only if $IJ = (0)$. Several graph theoretic properties of the annihilating ideal graph of any commutative ring with identity and their interplay with

* Corresponding author.
E-mail address: s_visweswaran2006@yahoo.co.in (S. Visweswaran).

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the ring theoretic properties have been investigated in [5,6]. Moreover, the annihilating ideal graph of a commutative ring is also studied in [1,7]. In this article we determine necessary and sufficient conditions in order that $AG(R)$ is complemented under the assumption that R is a zero-dimensional quasisemilocal ring such that $A(R)^*$ contains at least two elements. As any finite ring is zero-dimensional and has only finitely many prime ideals, we answer the question of when $AG(R)$ is complemented for any finite ring R which admits at least two nonzero annihilating ideals as a corollary to the results proved in this article.

This article is motivated by the interesting theorems proved on the annihilating ideal graph of a commutative ring in [1,5–7], and moreover, we are very much inspired by the research article [2] in which the authors among other results determined necessary and sufficient conditions in order that $\Gamma(R)$ is complemented, where $\Gamma(R)$ is the zero-divisor graph of R.

It is useful to recall the following definitions from [2,11]. Let $G = (V, E)$ be a simple undirected graph. Let $a, b \in V$. We define $a \leq b$ if a and b are not adjacent and each vertex of G adjacent to b is also adjacent to a. We define $a \sim b$ if $a \leq b$ and $b \leq a$. Thus $a \sim b$ if and only if $\{c \in V | c \text{ is adjacent to } a \text{ in } G\} = \{d \in V | d \text{ is adjacent to } b \text{ in } G\}$. Let $a, b \in V$, $a \neq b$. We say that a and b are orthogonal, written $a \perp b$, if a and b are adjacent and there is no vertex c of G which is adjacent to both a and b. We say that G is complemented, if for each vertex a of G, there is a vertex b of G (called a complement of a) such that $a \perp b$. We say that G is uniquely complemented if G is complemented and whenever $a \perp b$ and $a \perp c$, then $b \sim c$ [2,11]. By dimension of a ring, we mean its Krull dimension and we use the abbreviation $dim\, R$ to denote the dimension of a ring R. A ring R is said to be quasilocal (respectively. quasisemilocal) if R has a unique maximal ideal (respectively, R has only finitely many maximal ideals). By a local (respectively, a semilocal) ring, we mean a Noetherian quasilocal (respectively, a Noetherian quasisemilocal) ring. Recall that a local ring (R, M) is said to be a special principal ideal ring (SPIR), if R is a principal ideal ring and M is nilpotent. Whenever a set A is a subset of a set B and $A \neq B$, we denote it symbolically by $A \subset B$.

It is also useful to recall the following definitions and results from commutative ring theory. Let R be a ring. Let M be a unitary R-module. By the set of zero-divisors of M as an R-module denoted by $Z_R(M)$, we mean $Z_R(M) = \{r \in R | rm = 0 \text{ for some } m \in M, m \neq 0\}$. We denote $Z_R(R)$ simply by $Z(R)$. Recall from [8] that a prime ideal P of R is said to be a maximal N-prime of an ideal I of R, if P is maximal with respect to the property of being contained in $Z_R(R/I)$. It follows from [10, Theorem 1] that maximal N-primes of (0) always exist and if $\{P_\alpha\}_{\alpha \in \Lambda}$ is the set of all maximal N-primes of (0) in R, then $Z(R) = \cup_{\alpha \in \lambda} P_\alpha$.

In Section 2, it is shown that $AG(R)$ is complemented for any reduced ring R which is not an integral domain. Let R be a ring which is not reduced. In Section 3, we state and prove several necessary conditions in order that $AG(R)$ is complemented. The main theorem proved in Section 4 is Theorem 4.8 which determines necessary and sufficient conditions in order that $AG(R)$ is complemented, where R is a zero-dimensional quasilocal ring which admits at least two nonzero annihilating ideals. In Section 5, we consider zero-dimensional quasisemilocal rings R with at least two nonzero annihilating ideals and in Theorem 5.6, necessary and sufficient conditions are determined in order that $AG(R)$ is complemented. In Section 6, we consider rings R which are not reduced and which admit only a finite number of maximal N-primes of (0). We denote the finite set of maximal N-primes of (0) in R by $\{P_1, \ldots, P_n\}$. We determine necessary and sufficient conditions in order that $AG(R)$ is complemented under the additional hypothesis that $\cap_{i=1}^{n} P_i = nil(R)$, where $nil(R)$ denotes the nilradical of R (see Theorems 6.3 and 6.9).

2. A SUFFICIENT CONDITION UNDER WHICH $AG(R)$ IS COMPLEMENTED

The purpose of this section is to prove that if R is any reduced ring which is not an integral domain, then $AG(R)$ is complemented. We begin with the following lemma. This is an analogue to [2, Lemma 3.3]. Again we emphasize that all the rings considered in this article are commutative with identity.

Lemma 2.1. *Let R be a ring. Let $I, J \in A(R)^*$. The following statements are equivalent:*
(i) *$I \perp J$, $I^2 \neq (0)$, and $J^2 \neq (0)$.*
(ii) *$IJ = (0)$ and $I + J \notin A(R)$.*

Proof. (i) $\Rightarrow$ (ii) Since $I \perp J$, it is clear that $IJ = (0)$. Suppose that $I + J \in A(R)$. Then there exists $a \in R\backslash\{0\}$ such that $a(I + J) = (0)$. Hence $aI = (0)$ and $aJ = (0)$. Since $I^2 \neq (0)$ and $J^2 \neq (0)$, it follows that $Ra \neq I$ and $Ra \neq J$. Observe that the ideal $Ra \in A(R)^*$ is such that $I(Ra) = (0)$ and $J(Ra) = (0)$. This is in contradiction to the hypothesis that $I \perp J$. Hence we obtain that $I + J \notin A(R)$.

(ii) $\Rightarrow$ (i) If $I^2 = (0)$, then from $IJ = (0)$, it follows that $(I + J)I = (0)$. This contradicts the assumption that $I + J \notin A(R)$. Hence we obtain that $I^2 \neq (0)$. Similarly, it follows that $J^2 \neq (0)$. Now it is clear that $I \neq J$. Let K be an ideal of R such that $IK = (0)$ and $JK = (0)$. Then $(I + J)K = (0)$. Since $I + J \notin A(R)$, it follows that $K = (0)$. This proves that $I \perp J$. □

Proposition 2.2. *Let R be a reduced ring which is not an integral domain. Then $AG(R)$ is complemented. Moreover, $AG(R)$ is uniquely complemented.*

Proof. Since R is not an integral domain, there exist $a, b \in R\backslash\{0\}$ such that $ab = 0$. Note that $Ra, Rb \in A(R)^*$. Since R is reduced it follows from $ab = 0$ with $a, b \in R\backslash\{0\}$ that $Ra \neq Rb$. Hence $|A(R)^*| \geq 2$.

Let $I \in A(R)^*$. Hence there exists $x \in R\backslash\{0\}$ such that $Ix = (0)$. Let $J = ((0) :_R I)$. As any nonzero element of I annihilates J, it is clear that $J \in A(R)^*$. We assert that $I \perp J$. It is clear that $IJ = (0)$. Hence in view of (ii) $\Rightarrow$ (i) of Lemma 2.1, it is enough to show that $I + J \notin A(R)$. Let $r \in R$ be such that $(I + J)r = (0)$. Then $Ir = (0)$ and $Jr = (0)$. Hence $r \in J$ and from $Jr = (0)$, it follows that $r^2 = 0$. Since R is reduced, we obtain that $r = 0$. This proves that $I \perp J$. Thus each $I \in A(R)^*$ admits a complement in $AG(R)$. This shows that $AG(R)$ is complemented.

We next verify that $AG(R)$ is uniquely complemented. Let $I \in A(R)^*$. Let $J_1, J_2 \in A(R)^*$ be such that $I \perp J_1$ and $I \perp J_2$. Since R is reduced, it follows that $A^2 \neq (0)$ for any nonzero ideal A of R. As $I \perp J_1$ and $I \perp J_2$, we know from (i) $\Rightarrow$ (ii) of Lemma 2.1 that $I + J_i \notin A(R)$ for $i = 1, 2$. Hence $(I + J_1)J_2 \neq (0)$. This implies that $J_1J_2 \neq (0)$ since $IJ_2 = (0)$. Let $K \in A(R)^*$ be such that K is adjacent to J_2. Then $KJ_2 = (0)$. From $IJ_1 = (0)$, it follows that $(I + J_2)KJ_1 = (0)$. As $I + J_2 \notin A(R)$, it follows that $KJ_1 = (0)$. This proves that $J_1 \leq J_2$. Similarly, using the facts that $IJ_2 = (0)$ and $I + J_1 \notin A(R)$, it follows that $J_2 \leq J_1$. Hence we obtain that $J_1 \sim J_2$. This proves that $AG(R)$ is uniquely complemented. □

3. Some necessary conditions in order that $AG(R)$ is complemented, where R is not a reduced ring

In this section we consider rings R such that the nilradical of R is nonzero. We use $nil(R)$ to denote the nilradical of a ring R. The aim of this section is to determine some necessary conditions in order that $AG(R)$ is complemented. We begin with the following lemma.

Lemma 3.1. *Let R be a ring. If $a \in R\backslash\{0\}$, then for any $b \in nil(R)$, $Ra \neq Rab$.*

Proof. If $Ra = Rab$, then $a = rab$ for some $r \in R$. This implies that $a(1 - rb) = 0$. Since $b \in nil(R)$, $1 - rb$ is a unit in R. Hence from $a(1 - rb) = 0$, it follows that $a = 0$. This is a contradiction. Hence $Ra \neq Rab$. $\square$

The following lemma is obvious.

Lemma 3.2. *Let I be a nonzero nilpotent ideal of a ring R. Let n be the least integer $p \geq 2$ with the property that $I^p = (0)$. Then $I^i \neq I^j$ for all distinct $i, j \in \{1, 2, \ldots, n\}$.*

We next have the following lemma which shows that if $AG(R)$ is complemented, then $nil(R)$ must be nilpotent.

Lemma 3.3. *Let R be a ring. If $AG(R)$ is complemented, then $(nil(R))^4 = (0)$.*

Proof. First we show that for any $a \in nil(R)$, $a^4 = 0$. Suppose that $a^4 \neq 0$. Let n be the least integer $p \geq 5$ with the property that $a^p = 0$. Since $AG(R)$ is complemented, there exists $I \in A(R)^*$ such that $Ra^{n-3} \perp I$. It follows from Lemma 3.2 that $Ra^i \neq Ra^j$ for all distinct $i, j \in \{1, 2, \ldots, n\}$. Hence in particular $Ra^{n-1} \neq Ra^{n-2}$. Thus there exists $j \in \{n-2, n-1\}$ such that $I \neq Ra^j$. From $(Ra^{n-3})I = (0)$, it follows that $(Ra^j)I = (0)$. Since $n \geq 5$, it is clear that $Ra^j Ra^{n-3} = (0)$. Hence the ideal Ra^j is adjacent to both Ra^{n-3} and I. This is impossible since $Ra^{n-3} \perp I$. Therefore, for any $a \in nil(R)$, $a^4 = 0$.

Let $a, b, c \in nil(R)$. We assert that $a^2bc = 0$. Suppose that $a^2bc \neq 0$. As $AG(R)$ is complemented, there exists $I \in A(R)^*$ such that $Ra^2 \perp I$. From $(Ra^2)I = (0)$, it follows that $(Ra^2b)I = (Ra^2bc)I = (0)$. It follows from Lemma 3.1 that the ideals Ra^2, Ra^2b, and Ra^2bc are distinct. Hence either $I \neq Ra^2b$ or $I \neq Ra^2bc$. If $I \neq Ra^2b$, then it follows from $a^4 = 0$ that Ra^2b is adjacent to both Ra^2 and I. This is impossible since $Ra^2 \perp I$. Similarly, if $I \neq Ra^2bc$, then we obtain that Ra^2bc is adjacent to both Ra^2 and I. This is not possible since $Ra^2 \perp I$. Hence for any $a, b, c \in nil(R)$, $a^2bc = 0$.

Let $a, b, c, d \in nil(R)$. We claim that $abcd = 0$. Suppose that $abcd \neq 0$. It follows from Lemma 3.1 that the ideals Ra, $Rabc$, and $Rabcd$ are distinct. Since $AG(R)$ is complemented, there exists $I \in A(R)^*$ such that $Ra \perp I$. It follows from $(Ra)I = (0)$ that $(Rabc)I = (0)$ and $(Rabcd)I = (0)$. Observe that either $I \neq Rabc$ or $I \neq Rabcd$. Since $a^2bc = 0$, $(Ra)(Rabc) = (0)$ and $(Ra)(Rabcd) = (0)$. If $I \neq Rabc$, then we obtain that $Rabc$ is adjacent to both Ra and I. This is impossible since $Ra \perp I$. Similarly $I \neq Rabcd$ is also impossible. This proves that for any $a, b, c, d \in nil(R)$, $abcd = 0$.

This shows that $(nil(R))^4 = (0)$. $\square$

The following proposition provides some more necessary conditions on R if $(nil(R))^3 \neq (0)$ and $AG(R)$ is complemented.

Proposition 3.4. *Let R be a ring such that $(nil(R))^3 \neq (0)$. If $AG(R)$ is complemented, then the following hold:*
(i) $(nil(R))^i \perp (nil(R))^3$ *for* $i = 1, 2$.
(ii) $(nil(R))^3 = Rx$ *for any* $x \in (nil(R))^3 \backslash \{0\}$.
(iii) $(nil(R))^2 = Ry$ *for any* $y \in (nil(R))^2 \backslash (nil(R))^3$.
(iv) $nil(R) = Rz$ *for any* $z \in nil(R) \backslash (nil(R))^2$.
(v) *If I is any ideal of R such that $I \subseteq nil(R)$, then $I \in \{(0), nil(R), (nil(R))^2, (nil(R))^3\}$ and so $I \in \{(0), Rz, Rz^2, Rz^3\}$ for any $z \in nil(R) \backslash (nil(R))^2$.*

Proof. Since $AG(R)$ is complemented, we know from Lemma 3.3 that $(nil(R))^4 = (0)$.

(i) Let $i \in \{1, 2\}$. Thus $(nil(R))^i (nil(R))^3 = (0)$. By hypothesis, $(nil(R))^3 \neq (0)$. Hence $(nil(R))^i$ has a nonzero annihilator and so $(nil(R))^i \in A(R)^*$. As $AG(R)$ is complemented, there exists $I \in A(R)^*$ such that $(nil(R))^i \perp I$. Hence $(nil(R))^i I = (0)$ and so it follows that $(nil(R))^3 I = (0)$. It is already noted that $(nil(R))^3 (nil(R))^i = (0)$. Moreover, since $(nil(R))^3 \neq (0)$, we obtain from Lemma 3.2 that $(nil(R))^3 \neq (nil(R))^i$. As $(nil(R))^i \perp I$, the above arguments imply that $I = (nil(R))^3$. This proves that $(nil(R))^i \perp (nil(R))^3$ for $i = 1, 2$.

(ii) Let $x \in (nil(R))^3 \backslash \{0\}$. As $(nil(R))^4 = (0)$, it follows that $Rx(nil(R)) = (0)$ and $Rx(nil(R))^3 = (0)$. We know from (i) that $nil(R) \perp (nil(R))^3$.

Since $Rx \neq (0)$ and $Rx \neq nil(R)$, we obtain that $Rx = (nil(R))^3$.

(iii) Let $y \in (nil(R))^2 \backslash (nil(R))^3$. Since $(nil(R))^4 = (0)$, we obtain that $Ry(nil(R))^2 = (0)$ and $Ry(nil(R))^3 = (0)$. We know from (i) that $(nil(R))^2 \perp (nil(R))^3$. As $Ry \notin \{(0), (nil(R))^3\}$, it follows that $Ry = (nil(R))^2$.

(iv) Let $y \in (nil(R))^2 \backslash (nil(R))^3$. Let $\phi : nil(R) \to (nil(R))^3$ be the mapping given by $\phi(a) = ay$ for any $a \in nil(R)$. It is clear that ϕ is a homomorphism of R-modules. We assert that ϕ is onto. Let $b \in (nil(R))^3$. Note that $(nil(R))^3 = nil(R)(nil(R))^2$. Since $(nil(R))^2 = Ry$ by (iii), we obtain that $(nil(R))^3 = (nil(R))Ry$. Hence $b = ay$ for some $a \in nil(R)$. Hence $\phi(a) = ay = b$. This shows that ϕ is onto. We know from the fundamental theorem of homomorphism of modules that $nil(R)/ker\phi \cong (nil(R))^3$ as R-modules. We know from (ii) that for any nonzero $x \in (nil(R))^3$, $Rx = (nil(R))^3$. Hence it follows that for any $z \in nil(R) \backslash ker\phi$, $nil(R)/ker\phi = R(z + ker\phi)$. We claim that $ker\phi = (nil(R))^2$. As $(nil(R))^4 = (0)$ and $y \in (nil(R))^2$, it is clear that $(nil(R))^2 \subseteq ker\phi$. Let $a \in ker\phi$. Hence $(Ra)Ry = Ra(nil(R))^2 = (0)$ and $Ra(nil(R))^3 = (0)$. By (i), $(nil(R))^2 \perp (nil(R))^3$. Hence we obtain that $Ra \in \{(0), (nil(R))^2, (nil(R))^3\}$. This implies that $a \in (nil(R))^2$. Hence $ker\phi \subseteq (nil(R))^2$ and so $ker\phi = (nil(R))^2$.

Let $z \in nil(R) \backslash (nil(R))^2$. Hence $z \notin ker\phi$. As is remarked in the previous paragraph, $nil(R)/ker\phi = R(z + ker\phi)$. Hence it follows that $nil(R) = Rz + ker\phi = Rz + (nil(R))^2$. Therefore, we obtain that $nil(R) = Rz + (Rz + (nil(R))^2)^2$. Now it is clear that $nil(R) = Rz$ since $(nil(R))^4 = (0)$. Thus for any $z \in nil(R) \backslash (nil(R))^2$, $nil(R) = Rz$.

(v) Let I be any nonzero ideal of R such that $I \subseteq nil(R)$. Since $(nil(R))^4 = (0)$, there exists $i \in \{1, 2, 3\}$ such that $I \subseteq (nil(R))^i$ but $I \nsubseteq (nil(R))^{i+1}$. Let $a \in I \backslash (nil(R))^{i+1}$. Hence $a \in (nil(R))^i \backslash (nil(R))^{i+1}$. It follows from (ii), (iii), or (iv) that $(nil(R))^i = Ra$. As $a \in I$, we obtain that $(nil(R))^i \subseteq I$ and so $I = (nil(R))^i$. This shows that $\{(0), nil(R), (nil(R))^2, (nil(R))^3\}$ is the set of all ideals of R which are contained in $nil(R)$. Let $z \in nil(R) \backslash (nil(R))^2$. Then from (iv), it follows that $nil(R) = Rz$ and so $\{(0), Rz, Rz^2, Rz^3\}$ is the set of all ideals of R which are contained in $nil(R)$. □

We next consider rings R such that $(nil(R))^3 = (0)$ but $(nil(R))^2 \neq (0)$ and determine some necessary conditions in order that $AG(R)$ is complemented.

Proposition 3.5. *Let R be a ring such that $(nil(R))^3 = (0)$ but $(nil(R))^2 \neq (0)$. If $AG(R)$ is complemented, then the following hold:*
(i) $nil(R) \perp (nil(R))^2$.
(ii) $(nil(R))^2 = Rx$ *for any* $x \in (nil(R))^2 \backslash \{0\}$.
(iii) *If* $ab \neq 0$ *for some* $a, b \in nil(R)$*, then* $Ra \perp (nil(R))^2$ *and* $Rb \perp (nil(R))^2$.
(iv) *If* $z^2 \neq 0$ *for some* $z \in nil(R)$*, then* $nil(R) = Ra$ *for any* $a \in nil(R) \backslash (nil(R))^2$.
(v) *If* $z^2 = 0$ *for each* $z \in nil(R)$*, then* $nil(R)$ *is not principal but there exist* $a, b \in nil(R)$ *such that* $nil(R) = Ra + Rb$.

Proof. Since $(nil(R))^2 \neq (0)$ and $(nil(R))^3 = (0)$, it is clear that $nil(R) \in A(R)^*$.

(i) By hypothesis, $AG(R)$ is complemented. Hence there exists $I \in A(R)^*$ such that $nil(R) \perp I$. Hence $I(nil(R)) = (0)$. Therefore, $I(nil(R))^2 = (0)$. Since $(nil(R))^3 = (0)$, it follows that $(nil(R))^2 nil(R) = (0)$. As $nil(R) \perp I$ and $(nil(R))^2 \notin \{(0), nil(R)\}$, we obtain that $(nil(R))^2 = I$. This proves that $nil(R) \perp (nil(R))^2$.

(ii) Let $x \in (nil(R))^2 \backslash \{0\}$. Since $(nil(R))^3 = (0)$, it follows that $Rx(nil(R)) = (0)$ and $Rx(nil(R))^2 = (0)$. We know from (i) that $nil(R) \perp (nil(R))^2$. As $Rx \notin \{(0), nil(R)\}$, we obtain that $Rx = (nil(R))^2$.

(iii) Let $a, b \in nil(R)$ be such that $ab \neq 0$. Since $AG(R)$ is complemented, there exists $I \in A(R)^*$ such that $Ra \perp I$. Hence $(Ra)I = (0)$ and so $(Rab)I = (0)$. As $(nil(R))^3 = (0)$, we obtain that $(Ra)(Rab) = (0)$. Thus the nonzero ideal Rab is such that $(Ra)(Rab) = (0)$ and $(Rab)I = (0)$. Since $Ra \perp I$, $Rab \in \{I, Ra\}$. We know from Lemma 3.1 that $Ra \neq Rab$. Hence $Rab = I$. Therefore, $Ra \perp Rab$. By (ii), $(nil(R))^2 = Rab$. This proves that $Ra \perp (nil(R))^2$. Similarly, it follows that $Rb \perp (nil(R))^2$.

(iv) Suppose that there exists $z \in nil(R)$ such that $z^2 \neq 0$. Consider the mapping $\psi : nil(R) \rightarrow (nil(R))^2$ given by $\psi(y) = yz$ for any $y \in nil(R)$. It is clear that ψ is a homomorphism of R-modules. Since $(nil(R))^2 = Rz^2$ by (ii), it follows that ψ is onto. Hence we obtain from the fundamental theorem of homomorphism of modules that $nil(R)/ker\psi \cong (nil(R))^2$ as R-modules. We know from (ii) that for any nonzero $x \in (nil(R))^2$, $Rx = (nil(R))^2$. Thus for any $a \in nil(R) \backslash ker\psi$, $nil(R)/ker\psi = R(a + ker\psi)$. We claim that $ker\psi = (nil(R))^2$. Since $ker\psi \subseteq nil(R)$ and $(nil(R))^3 = (0)$, it follows that $(ker\psi)(nil(R))^2 = (0)$. As $ker\psi \subseteq ((0) :_R z)$, we obtain that $Rz(ker\psi) = (0)$. We know from (iii) that $Rz \perp (nil(R))^2$. Hence $ker\psi \in \{(0), Rz, (nil(R))^2\}$. As $z^3 = 0$, $z^2 \in ker\psi$. Hence $ker\psi \neq (0)$. Since $z \notin ker\psi$, it is clear that $ker\psi \neq Rz$. Now it follows that $ker\psi = (nil(R))^2$. Let $a \in nil(R) \backslash (nil(R))^2$. Hence as is already observed in this proof we obtain that $nil(R)/(nil(R))^2 = R(a + (nil(R))^2)$. This shows that $nil(R) = Ra + (nil(R))^2$. Hence $nil(R) = Ra + (Ra + (nil(R))^2)^2$. This implies that $nil(R) = Ra$ since $(nil(R))^3 = (0)$.

(v) Since $(nil(R))^2 \neq (0)$ and by assumption $z^2 = 0$ for each $z \in nil(R)$, it is clear that $nil(R)$ is not principal. As $(nil(R))^2 \neq (0)$, there exist $a, b \in nil(R)$ such that $ab \neq (0)$. Consider the homomorphism of R-modules $f : nil(R) \rightarrow (nil(R))^2$ given by $f(z) = zb$ for any $z \in nil(R)$. By (ii), $(nil(R))^2 = Rab$. Hence it follows that f is onto. We assert that $ker\, f = Rb$. Since $ker\, f \subseteq nil(R)$ and $(nil(R))^3 = (0)$, it follows that $(ker\, f)(nil(R))^2 = (0)$. As $ker\, f \subseteq ((0) :_R b)$, it follows that $Rb(ker\, f) = (0)$. We know from (iii) that $Rb \perp (nil(R))^2$. Hence $ker\, f \in \{(0), Rb, (nil(R))^2\}$. Since $b^2 = 0$, $b \in$

$ker\, f$ and so $ker\, f \neq (0)$. We know from Lemma 3.1 that $Rb \neq Rab$. Hence $b \notin (nil(R))^2$. Hence we obtain that $ker\, f \neq (nil(R))^2$. Therefore, it follows that $ker\, f = Rb$. Now f is a homomorphism of R-modules from $nil(R)$ onto $(nil(R))^2$. Hence by the fundamental theorem of homomorphism of modules, it follows that $nil(R)/ker\, f \cong (nil(R))^2$ as R-modules. We know from (ii) that $(nil(R))^2 = Rx$ for any nonzero $x \in (nil(R))^2$ and as $a \notin ker\, f$, it follows that $nil(R)/ker\, f = R(a + ker\, f)$. This implies that $nil(R) = Ra + ker\, f = Ra + Rb$. □

4. Zero-dimensional quasilocal rings R such that $AG(R)$ is complemented

The aim of this section is to determine all zero-dimensional quasilocal rings R such that $AG(R)$ is complemented. We begin with the following lemma.

Lemma 4.1. *Let R be a ring such that $dim\, R = 0$ and R is quasilocal with M as its unique maximal ideal. Suppose that $M^3 \neq (0)$. Then the following statements are equivalent:*
(i) *$AG(R)$ is complemented.*
(ii) *$M^4 = (0)$ and R is a SPIR.*

Proof. (i) ⇒ (ii) By hypothesis, it is clear that M is the only prime ideal of R. Hence $M = nil(R)$. Since $AG(R)$ is complemented by assumption, it follows from Lemma 3.3 that $M^4 = (0)$. By hypothesis, $M^3 \neq (0)$. Now it follows from Proposition 3.4(iv) that $M = Rm$ for any $m \in M \backslash M^2$. As $M^4 = (0)$, $M^3 \neq (0)$, and M is principal, it follows from the proof of (iii) ⇒ (i) of [3, Proposition 8.8] that $\{M = Rm, M^2 = Rm^2, M^3 = Rm^3\}$ is the set of all proper nonzero ideals of R. Hence we obtain that R is a SPIR.

(ii) ⇒ (i) Now R is a SPIR with $M^4 = (0)$ but $M^3 \neq (0)$. Note that $\{M, M^2, M^3\}$ is the set of all nonzero proper ideals of R. Now it is clear that $AG(R)$ is a graph on three vertices $\{M, M^2, M^3\}$, $M \perp M^3$, and $M^2 \perp M^3$. This proves that $AG(R)$ is complemented. □

We next have the following lemma.

Lemma 4.2. *Let R be a quasilocal ring with M as its unique maximal ideal. Suppose that $M^3 = (0)$ but $M^2 \neq (0)$. Then the following statements are equivalent:*
(i) *$AG(R)$ is complemented.*
(ii) *If $z^2 \neq 0$ for some $z \in M$, then M is principal. If $z^2 = 0$ for each $z \in M$, then M is not principal but there exist $a, b \in M$ such that $M = Ra + Rb$.*
(iii) *$I \perp M^2$ for each nonzero proper ideal I of R with $I \neq M^2$.*

Proof. (i) ⇒ (ii) It is clear from the hypothesis that M is the only prime ideal of R. Hence $M = nil(R)$. If $z^2 \neq 0$ for some $z \in M$, then it follows from Proposition 3.5(iv) that M is principal. If $z^2 = (0)$ for each $z \in M$, then it follows from Proposition 3.5(v) that M is not principal but there exist $a, b \in M$ such that $M = Ra + Rb$.

(ii) ⇒ (iii) Suppose that $z^2 \neq 0$ for some $z \in M$. Then M is principal. As $M^3 = (0)$, it follows from the proof of (iii) ⇒ (i) of [3, Proposition 8.8] that M and M^2 are the only proper nonzero ideals of R. Hence in this case, $AG(R)$ is a graph with vertex set $\{M, M^2\}$ and $M \perp M^2$.

Suppose that $z^2 = 0$ for each $z \in M$. Then M is not principal but there exist $a, b \in M$ such that $M = Ra + Rb$. In such a case, $M^2 = Rab$. Let $x \in M^2, x \neq 0$. Then $x = rab$

for some $r \in R$. As $M^3 = (0)$, it follows that r is a unit in R and so $M^2 = Rab = Rx$. Since $M^3 = (0)$ but $M^2 \neq (0)$, it is clear that each nonzero proper ideal is in $A(R)^*$. Let I be any nonzero proper ideal of R. If $I \subseteq M^2$, then as $M^2 = Rx$ for any $x \in I \backslash \{0\}$, it follows that $I = M^2$. Suppose that $I \not\subseteq M^2$. Let $z \in I \backslash M^2$. Since M is not principal but is generated by two elements, it is clear that $dim_{R/M}(M/M^2) = 2$. As $z + M^2 \in M/M^2$ is nonzero, there exists $w \in M$ such that $\{z + M^2, w + M^2\}$ forms a basis of M/M^2 as a vector space over R/M. In such a case, it follows that $M = Rz + Rw + M^2$. This implies that $M = Rz + Rw$ since $M^3 = (0)$. As $z^2 = w^2 = 0$, it follows that $M^2 = Rzw \subseteq I$ since $z \in I$. Note that $dim_{R/M}(I/M^2)$ is either 1 or 2. If $dim_{R/M}(I/M^2) = 1$, then $\{z + M^2\}$ forms a basis of I/M^2. This implies that $I = Rz + M^2$ and so $I = Rz$ since $M^2 = Rzw$. If $dim_{R/M}(I/M^2) = 2$, then it follows that $I/M^2 = M/M^2$ and so $I = M$.

From the above discussion it is clear that if $z^2 = 0$ for each $z \in M$ and if I is any nonzero proper ideal of R, then either $I \in \{M, M^2\}$ or $I = Rz$ for some $z \in I \backslash M^2$. Since $M^3 = (0)$, it follows that $IM^2 = (0)$ for each proper ideal I of R. Let $I \in A(R)^*$ be such that $I \neq M^2$. We verify that $I \perp M^2$. Since $IM^2 = (0)$, I is adjacent to M^2. Let $J \in A(R)^*$ be such that $J \notin \{I, M^2\}$. Suppose that $J = M$. Then $I \neq M$ and so $I = Rz$ for some $z \in I \backslash M^2$. Moreover, it is noted in the previous paragraph that $M^2 = Rzw$ for some $w \in M$. Hence we obtain that $M^2 \subseteq IM = IJ$. Similarly, if $I = M$, then $J \neq M$ and so $J = Rz'$ for some $z' \in J \backslash M^2$ and $M^2 = Rz'w'$ for some $w' \in M$. Therefore, $M^2 = Rw'z' \subseteq MJ = IJ$. Suppose that $I \neq M$ and $J \neq M$. Then there exist $z \in I \backslash M^2$ and $z' \in J \backslash M^2$ such that $I = Rz$ and $J = Rz'$. We claim that $I + J = M$. Indeed, if $I + J \neq M$, then $I + J = Ry$ for some $y \in I + J$ with $y \notin M^2$. Now as $z, z' \in M \backslash M^2$, $I = Rz \subseteq Ry$, and $J = Rz' \subseteq Ry$, we obtain that $I = Rz = Ry = Rz' = J$. But this contradicts the assumption that $I \neq J$. Hence $Rz + Rz' = I + J = M$. Therefore, $M^2 = Rzz' \subseteq IJ$. This shows that if $J \in A(R)^* \backslash \{M^2, I\}$, then $M^2 \subseteq IJ$ and so $IJ \neq (0)$. This proves that $I \perp M^2$ for each $I \in A(R)^*$ with $I \neq M^2$.

(iii) $\Rightarrow$ (i). Since $M^3 = (0)$ but $M^2 \neq (0)$, it is clear that $M \neq M^2$. Hence R admits at least one nonzero proper ideal which is different from M^2. Note that from the preceding observation (iii) $\Rightarrow$ (i) follows immediately. $\square$

We next have the following lemma. We denote the characteristic of a ring R by $char(R)$.

Lemma 4.3. *Let R be a quasilocal ring with M as its unique maximal ideal such that $M^3 = (0)$ but $M^2 \neq (0)$. If $AG(R)$ is complemented and M is not principal, then $char(R/M) = 2$ and moreover, $char(R) \in \{2, 4\}$.*

Proof. Assume that $AG(R)$ is complemented and M is not principal. It follows from the proof of (i) $\Rightarrow$ (ii) of Lemma 4.2 that $z^2 = 0$ for each $z \in M$ and $dim_{R/M}(M/M^2) = 2$. Moreover, it is noted in the proof of (ii) $\Rightarrow$ (iii) of Lemma 4.2 that for any nonzero $x \in M^2$, $M^2 = Rx$ and $I \perp M^2$ for each nonzero proper ideal I of R with $I \neq M^2$. We first verify that $char(R/M) = 2$. Suppose that $char(R/M) \neq 2$. Then $2 \notin M$ and so 2 is a unit in R. Let $a, b \in M$ be such that $\{a + M^2, b + M^2\}$ forms a basis of M/M^2 as a vector space over R/M. Consider the ideals $I_1 = R(a + b)$ and $I_2 = R(a - b)$ of R. From the choice of a, b, it is clear that I_1 and I_2 are nonzero proper ideals of R with $I_i \neq M^2$ for each $i \in \{1, 2\}$. Note that $I_1 \neq I_2$. For if $I_1 = I_2$, then $2b = (a + b) - (a - b) \in I_1$. This implies that $b \in I_1$ since 2 is a unit in R. Hence $b = r(a + b)$ for some $r \in R$. Therefore, $ra + (r - 1)b = 0$. This implies by the choice of a, b that $r \in M$ and $1 - r \in M$. Hence $1 = r + 1 - r \in M$. This is impossible. Thus $I_1 \neq I_2$. Now as $a^2 = b^2 = 0$, it is clear that $I_1I_2 = R(a^2 - b^2) = (0)$. Moreover, as $M^3 = (0)$, it is clear that $I_2M^2 = (0)$. Thus $I_1I_2 = I_2M^2 = (0)$. This is impossible since $I_1 \perp M^2$. Hence $char(R/M) = 2$. Now $2 \in M$ and as $z^2 = 0$ for each $z \in M$, it follows that $4 = 0$ in R. Therefore, $char(R) \in \{2, 4\}$. $\square$

We next provide some examples to illustrate Lemma 4.2.

Example 4.4. Let K be a field with $char(K) = 2$. Let $T = K[x, y]$ be the polynomial ring in two variables over K. Let $I = x^2T + y^2T$ and $R = T/I$. Then $AG(R)$ is complemented.

Proof. Let $N = xT + yT$. Note that $R = T/I$ is a local ring with $M = N/I$ as its unique maximal ideal. For an element $t \in T$, we denote $t + I$ by $\overline{t}$. Observe that $M = \overline{x}R + \overline{y}R, z^2 = \overline{0}$ for each $z \in M$, $M^2 = \overline{xy}R \neq (\overline{0})$, and $M^3 = (\overline{0})$. Now it follows, from (ii) $\Rightarrow$ (iii) of Lemma 4.2, that $J \perp M^2$ for each nonzero proper ideal J of R with $J \neq M^2$. This shows that $AG(R)$ is complemented. $\square$

For any $n \geq 2$, we denote the ring of integers modulo n by $\mathbf{Z}_n$.

Example 4.5. Let $T = \mathbf{Z}_4[x, y]$ be the polynomial ring in two variables over $\mathbf{Z}_4$. Let $I = x^2T + (xy - 2)T + y^2T$ and $R = T/I$. Then $AG(R)$ is complemented.

Proof. Let $N = 2T + xT + yT$. Observe that $R = T/I$ is a local ring with $M = N/I$ as its unique maximal ideal. For any $t \in T$, we denote $t + I$ by $\overline{t}$. Note that $M = \overline{x}R + \overline{y}R, z^2 = \overline{0}$ for each $z \in M$, $M^2 = \overline{2}R$, and $M^3 = (\overline{0})$. Now it follows, from (ii) $\Rightarrow$ (iii) of Lemma 4.2, that $J \perp M^2$ for each nonzero proper ideal J of R with $J \neq M^2$ and hence we obtain that $AG(R)$ is complemented. $\square$

Example 4.6. Let $T = \mathbf{Z}_4[x]$ be the polynomial ring in one variable over $\mathbf{Z}_4$. Let $I = x^2T$. Let $R = T/I$. Then $AG(R)$ is complemented.

Proof. Let $N = 2T + xT$. Note that the ring $R = T/I$ is local with $M = N/I$ as its unique maximal ideal. For any $t \in T$, let us denote $t + I$ by $\overline{t}$. Observe that $M = \overline{2}R + \overline{x}R, z^2 = 0$ for each $z \in M$, $M^2 = \overline{2x}R \neq (\overline{0})$, and $M^3 = (\overline{0})$. It now follows, from (ii) $\Rightarrow$ (iii) of Lemma 4.2, that $J \perp M^2$ for each nonzero proper ideal J of R with $J \neq M^2$ and therefore, we obtain that $AG(R)$ is complemented. $\square$

We make use of the following remark in the proof of Theorem 4.8.

Remark 4.7. Let R be a quasilocal ring with M as its unique maximal ideal. If $M^2 = (0)$ but $M \neq (0)$, then $AG(R)$ is not complemented and indeed one of the following holds:
(i) M is the only element of $A(R)^*$ and hence $AG(R)$ is a graph on a single vertex.
(ii) $A(R)^*$ contains at least three elements and $AG(R)$ is a complete graph.

Proof. Suppose that M is principal. As $M^2 = (0)$, it is clear that M is the only nonzero proper ideal of R. Since the nonzero ideal M annihilates M, it follows that $M \in A(R)^*$. Hence (i) holds. Note that as $AG(R)$ is a graph on a single vertex, it is not complemented.

Suppose that M is not principal. Since $M^2 = (0)$, it is clear that M annihilates any proper nonzero ideal of R and hence $A(R)^*$ is the set of all proper nonzero ideals of R and moreover,

M can be regarded as a vector space over the field R/M. As M is not principal, it follows that $dim_{R/M}M \geq 2$. Let $\{x, y\} \subseteq M$ be such that $\{x, y\}$ is linearly independent over R/M. Note that $Rx, Ry, R(x + y)$ are distinct elements of $A(R)^*$. Since $M^2 = (0)$, we obtain that $IJ = (0)$ for any $I, J \in A(R)^*$. Hence it follows that $AG(R)$ is a complete graph with at least three vertices and hence it is not complemented. □

The following theorem characterizes when $AG(R)$ is complemented, where R is any zero-dimensional quasilocal ring with $AG(R)$ admitting at least two vertices.

Theorem 4.8. *Let R be a zero-dimensional quasilocal ring with M as its unique maximal ideal. Suppose that $AG(R)$ admits at least two vertices. Then $AG(R)$ is complemented if and only if* (a) *and* (b) *hold and moreover, either* (c) *or* (d) *holds, where* (a), (b), (c), *and* (d) *are given below:*

(a) $M^2 \neq (0)$.
(b) $M^4 = (0)$.
(c) *R is a SPIR.*
(d) *$z^2 = 0$ for each $z \in M$, M is not principal but there exist $a, b \in M$ such that $M = Ra + Rb$.*

Proof. We are assuming that $dim\, R = 0$ and R is quasilocal with M as its unique maximal ideal. Hence we obtain that $nil(R) = M$.

Assume that $AG(R)$ admits at least two vertices and is complemented. It follows from Remark 4.7 that $M^2 \neq (0)$. We obtain from Lemma 3.3 that $M^4 = (0)$. If $M^3 \neq (0)$, then it follows from (i) ⇒ (ii) of Lemma 4.1 that R is a SPIR. Suppose that $M^3 = (0)$. If M is principal, then it follows from the proof of (iii) ⇒ (i) of [3, Proposition 8.8] that R is a principal ideal ring and hence R is a SPIR. If M is not principal, then we obtain from (i) ⇒ (ii) of Lemma 4.2 that $z^2 = 0$ for each $z \in M$ and there exist $a, b \in M$ such that $M = Ra + Rb$. Thus if $AG(R)$ is complemented, then (a) and (b) hold. Moreover, either (c) or (d) holds.

Conversely, assume that (a) and (b) hold and moreover, either (c) or (d) holds. Suppose that (c) holds. If $M^3 \neq (0)$, then it follows from (ii) ⇒ (i) of Lemma 4.1 that $AG(R)$ is complemented. If $M^3 = (0)$, then $AG(R)$ is a graph with vertex set $\{M, M^2\}$ and $M \perp M^2$. Hence $AG(R)$ is complemented. Suppose that (d) holds. Then it follows from (ii) ⇒ (i) of Lemma 4.2 that $AG(R)$ is complemented. □

As an immediate consequence of Theorem 4.8, we have the following result.

Corollary 4.9. *Let (R, M) be a finite local ring with $AG(R)$ admitting at least two vertices. Then $AG(R)$ is complemented if and only if* (a), (b) *of Theorem 4.8 hold and either R is a finite SPIR or* (d) *of Theorem 4.8 hold.*

5. Zero-dimensional quasisemilocal rings R such that $AG(R)$ is complemented

The aim of this section is to determine zero-dimensional quasisemilocal rings R such that $AG(R)$ is complemented. We begin with the following lemma.

Lemma 5.1. *Let R be a quasisemilocal ring with $dim\,R = 0$. Let $\{P_1, \dots, P_n\}$ be the set of all maximal ideals of R. If $AG(R)$ is complemented, then there exist quasilocal rings $(R_1, M_1), \dots, (R_n, M_n)$ with $M_i^4 = (0)$ for each $i \in \{1, \dots, n\}$ and $R \cong R_1 \times \cdots \times R_n$ as rings.*

Proof. Since $dim\,R = 0$ and R is quasisemilocal with $\{P_1, \dots, P_n\}$ as its set of all maximal ideals, it is clear that $\{P_1, \dots, P_n\}$ is the set of all prime ideals of R. Hence we obtain that $nil(R) = \cap_{i=1}^n P_i$. Moreover, as $P_i + P_j = R$ for all distinct $i, j \in \{1, \dots, n\}$, it follows from [3, Proposition 1.10(i)] that $nil(R) = \cap_{i=1}^n P_i = \prod_{i=1}^n P_i$.

Suppose that $AG(R)$ is complemented. Then we obtain from Lemma 3.3 that $(nil(R))^4 = (0)$. Hence we obtain that $\prod_{i=1}^n P_i^4 = (0)$. Since $P_i^4 + P_j^4 = R$ for all distinct $i, j \in \{1, \dots, n\}$, it follows from the Chinese remainder theorem [3, Proposition 1.10(ii) and (iii)] that the mapping $f : R \to R/P_1^4 \times \cdots \times R/P_n^4$ given by $f(r) = (r + P_1^4, \dots, r + P_n^4)$ is an isomorphism of rings. Let $i \in \{1, \dots, n\}$ and $R_i = R/P_i^4$. It is clear that R_i is quasilocal with $M_i = P_i/P_i^4$ as its unique maximal ideal and $R \cong R_1 \times \cdots \times R_n$ as rings. Moreover, it is obvious that M_i^4 is the zero ideal of R_i for each $i \in \{1, \dots, n\}$. $\square$

In view of Lemma 5.1, in the rest of this section, we assume that $R = R_1 \times \cdots \times R_n$, where R_i is a quasilocal ring with unique maximal ideal M_i such that $M_i^4 = (0)$ for each $i \in \{1, \dots, n\}$. We proceed to determine when $AG(R)$ is complemented. As Theorem 4.8 determines when $AG(R)$ is complemented in the case where R is a zero-dimensional quasilocal ring, we assume that R is not quasilocal. Hence $n \geq 2$.

Lemma 5.2. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ $(n \geq 2)$, where (R_i, M_i) is a quasilocal ring with $M_i^4 = (0)$ for each $i \in \{1, 2, \dots, n\}$. If $AG(R)$ is complemented, then $M_i^2 = (0)$ and M_i is principal for $i \in \{1, 2, \dots, n\}$; in the case where $M_i \neq (0), M_i = R_i x_i$ for any nonzero element x_i of M_i. Moreover, R_i has at most one proper nonzero ideal for each $i \in \{1, 2, \dots, n\}$.*

Proof. Assume that $AG(R)$ is complemented. Suppose that $M_i^2 \neq (0)$ for some $i \in \{1, 2, \dots, n\}$. Consider the ideal $I = I_1 \times I_2 \times \cdots \times I_n$ of R defined by $I_i = M_i^2$ and $I_j = R_j$ for all $j \in \{1, 2, \dots, n\} \backslash \{i\}$. Since $M_i^4 = (0)$, the ideal $J = J_1 \times J_2 \times \cdots \times J_n$ of R given by $J_i = M_i^2$ and $J_j = (0)$ for all $j \in \{1, 2, \dots, n\} \backslash \{i\}$ is such that $IJ = (0) \times (0) \times \cdots \times (0)$. Hence $I \in A(R)^*$. As $AG(R)$ is complemented, there exists $K \in A(R)^*$ such that $I \perp K$. Now it follows from $IK = (0) \times (0) \times \cdots \times (0)$ and $I_j = R_j$ for all $j \in \{1, 2, \dots, n\} \backslash \{i\}$ that $K_j = (0)$ for all $j \in \{1, 2, \dots, n\} \backslash \{i\}$. Note that $K_i M_i^2 = (0)$. Observe that $JK = IK = (0) \times (0) \times \cdots \times (0)$. Since $I \perp K$ and $J \notin \{(0) \times (0) \times \cdots \times (0), I\}$, it follows that $J = K$. Hence we obtain that $I \perp J$. We next claim that $M_i^3 = (0)$. Indeed, if $M_i^3 \neq (0)$, then the ideal $A = A_1 \times A_2 \times \cdots \times A_n$ of R given by $A_i = M_i^3$ and $A_j = (0)$ for all $j \in \{1, 2, \dots, n\} \backslash \{i\}$ is such that $AI = AJ = (0) \times (0) \times \cdots \times (0)$ and $A \notin \{(0) \times (0) \times \cdots \times (0), I, J\}$. This is impossible since $I \perp J$. Thus $M_i^3 = (0)$. Note that the ideal $B = B_1 \times B_2 \times \cdots \times B_n$ given by $B_i = M_i$ and $B_j = (0)$ for all $j \in \{1, 2, \dots, n\} \backslash \{i\}$ of R is such that $IB = JB = (0) \times (0) \times \cdots \times (0)$ and $B \notin \{(0) \times (0) \times \cdots \times (0), I, J\}$. This cannot happen since $I \perp J$. Hence we obtain that $M_i^2 = (0)$ for each $i \in \{1, 2, \dots, n\}$.

Let $i \in \{1, 2, \dots, n\}$. We next show that M_i is a principal ideal of R_i. If $M_i = (0)$, then it is clear that M_i is principal. Suppose that $M_i \neq (0)$. We show that $M_i = R_i x_i$ for any nonzero $x_i \in M_i$. Consider the ideal $I = I_1 \times I_2 \times \cdots \times I_n$ defined by $I_i = R_i x_i$ and $I_j = R_j$ for all $j \in \{1, 2, \ldots, n\} \backslash \{i\}$. Since $M_i^2 = (0)$, the ideal $J = J_1 \times J_2 \times \cdots \times J_n$ of R given by $J_i = M_i$ and $J_j = (0)$ for all $j \in \{1, 2, \ldots, n\} \backslash \{i\}$ is such that $IJ = (0) \times (0) \times \cdots \times (0)$. Hence $I \in A(R)^*$. Since $AG(R)$ is complemented, there exists $K \in A(R)^*$ such that $I \perp K$. From $IK = (0) \times (0) \times \cdots \times (0)$ and $I_j = R_j$ for all $j \in \{1, 2, \ldots, n\} \backslash \{i\}$, it is clear that $K_j = (0)$. Note that $K_i R_i x_i = (0)$. Hence $K_i \subseteq M_i$. As $M_i^2 = (0)$, it is clear that $JK = (0) \times (0) \times \cdots \times (0)$. Thus $IK = JK = (0) \times (0) \times \cdots \times (0)$. Since $I \perp K$ and $J \notin \{(0) \times (0) \times \cdots \times (0), I\}$, it follows that $J = K$. Thus $I \perp J$. The ideal $A = A_1 \times A_2 \times \cdots \times A_n$ of R given by $A_j = (0)$ for all $j \in \{1, 2, \ldots, n\} \backslash \{i\}$ and $A_i = R_i x_i$ is such that $IA = JA = (0) \times (0) \times \cdots \times (0)$. Since $I \perp J$ and $A \notin \{(0) \times (0) \times \cdots \times (0), I\}$, it follows that $A = J$. Hence we obtain that $M_i = R_i x_i$.

Let $i \in \{1, 2, \ldots, n\}$. If $M_i = (0)$, then R_i is a field and it has no proper nonzero ideal. If $M_i \neq (0)$, then it is noted in the previous paragraph that $M_i = R_i x_i$ for each nonzero $x_i \in M_i$. Hence we obtain that M_i is the only proper nonzero ideal of R_i. This proves that R_i has at most one nonzero proper ideal. □

With R as in the statement of Lemma 5.2, the following lemma provides another necessary condition in order that $AG(R)$ is complemented.

Lemma 5.3. *Let $n \geq 2$ and let $R = R_1 \times R_2 \times \cdots \times R_n$, where (R_i, M_i) is a quasilocal ring with $M_i^4 = (0)$ for each $i \in \{1, 2, \ldots, n\}$. If $AG(R)$ is complemented, then R_j is a field for some $j \in \{1, 2, \ldots, n\}$.*

Proof. Suppose that $AG(R)$ is complemented and R_i is not a field for each $i \in \{1, 2, \ldots, n\}$. Hence $M_i \neq (0)$ for each $i \in \{1, 2, \ldots, n\}$. Let $I = M_1 \times M_2 \times \cdots \times M_n$. We know from Lemma 5.2 that $M_i^2 = (0)$ for each $i \in \{1, 2, \ldots, n\}$. Hence it follows that $I \in A(R)^*$. Since $AG(R)$ is complemented, there exists an ideal $J = J_1 \times J_2 \times \cdots \times J_n$ of R such that $I \perp J$. From $IJ = (0) \times (0) \times \cdots \times (0)$, it follows that $I_i J_i = (0)$ for any $i \in \{1, 2, \ldots, n\}$. Hence $J_i \subseteq M_i$ and moreover, it follows from Lemma 5.2 that $J_i \in \{(0), M_i\}$ for each $i \in \{1, 2, \ldots, n\}$. Since $I \neq J$ and $J \neq (0) \times (0) \times \cdots \times (0)$, it is clear that there exist distinct $r, s \in \{1, 2, \ldots, n\}$ such that $J_r = M_r$ and $J_s = (0)$. Consider the ideal $K = K_1 \times K_2 \times \cdots \times K_n$ of R given by $K_i = (0)$ for all $i \in \{1, 2, \ldots, n\} \backslash \{s\}$ and $K_s = M_s$. Note that the ideal K is such that $K \notin \{(0) \times (0) \times \cdots \times (0), I, J\}$ and $IK = JK = (0) \times (0) \times \cdots \times (0)$. This is impossible as $I \perp J$. Thus if $AG(R)$ is complemented, then R_j is a field for some $j \in \{1, 2, \ldots, n\}$. □

Let R be as in the statement of Lemma 5.2. The following lemma provides another necessary condition in order that $AG(R)$ is complemented.

Lemma 5.4. *Let $n \geq 2$ and let $R = R_1 \times R_2 \times \cdots \times R_n$, where (R_i, M_i) is a quasilocal ring with $M_i^4 = (0)$ for each $i \in \{1, 2, \ldots, n\}$. If $AG(R)$ is complemented, then there exists at most one $i \in \{1, 2, \ldots, n\}$ such that R_i is not a field.*

Proof. Suppose that $AG(R)$ is complemented and there exist distinct $s, t \in \{1, 2, \ldots, n\}$ such that R_s and R_t are not fields. Hence $M_s \neq (0)$ and $M_t \neq (0)$. We know from Lemma 5.3 that there exists $j \in \{1, 2, \ldots, n\}$ such that R_j is a field. It is clear that $j \notin \{s, t\}$. Consider the ideal $I = I_1 \times I_2 \times \cdots \times I_n$ of R given by $I_i = R_i$ for all $i \in \{1, 2, \ldots, n\} \backslash \{s, t\}$,

$I_s = M_s$, and $I_t = M_t$. We know from Lemma 5.2 that $M_s^2 = (0)$ and $M_t^2 = (0)$. Moreover, $R_i(0) = (0)$ for all $i \in \{1, 2, \ldots, n\} \backslash \{s, t\}$. Hence we obtain that $I \in A(R)^*$. Since $AG(R)$ is complemented, there exists an ideal $J = J_1 \times J_2 \times \cdots \times J_n$ of R such that $I \perp J$. Thus $IJ = (0) \times (0) \times \cdots \times (0)$. Therefore, $I_i J_i = (0)$ for each $i \in \{1, 2, \ldots, n\}$. By the choice of I, it is clear that $J_i = (0)$ for all $i \in \{1, 2, \ldots, n\} \backslash \{s, t\}$. Moreover, as we know from Lemma 5.2 that M_s (respectively M_t) is the only proper nonzero ideal of R_s (respectively R_t), it follows that $J_s \in \{(0), M_s\}$ and $J_t \in \{(0), M_t\}$. Since J is a nonzero ideal of R, we must have either $J_s = M_s$ or $J_t = M_t$. Without loss of generality, we may assume that $J_s = M_s$. Note that the ideal $K = K_1 \times K_2 \times \cdots \times K_n$ of R given by $K_i = (0)$ for all $i \in \{1, 2, \ldots, n\} \backslash \{t\}$ and $K_t = M_t$ is such that $K \notin \{(0) \times (0) \times \cdots \times (0), I, J\}$ and $IK = JK = (0) \times (0) \times \cdots \times (0)$. This is impossible since $I \perp J$. Thus if $AG(R)$ is complemented, then there exists at most one $i \in \{1, 2, \ldots, n\}$ such that R_i is not a field. □

With R as in the statement of Lemma 5.2, the following lemma gives another necessary condition in order that $AG(R)$ is complemented under the additional assumption that R is not reduced.

Lemma 5.5. *Let $n \geq 2$ and let $R = R_1 \times R_2 \times \cdots \times R_n$, where (R_i, M_i) is a quasilocal ring with $M_i^4 = (0)$ for each $i \in \{1, 2, \ldots, n\}$. Suppose that R is not reduced. If $AG(R)$ is complemented, then $n = 2$.*

Proof. Suppose that $AG(R)$ is complemented and $n \geq 3$. We are assuming that R is not reduced (that is, R has nonzero nilpotent elements). Hence there exists at least one $i \in \{1, 2, \ldots, n\}$ such that R_i is not a field. Note that by Lemma 5.4, such an i is necessarily unique. Fix $j \in \{1, 2, \ldots, n\}$ with $j \neq i$. Consider the ideal $I = I_1 \times I_2 \times \cdots \times I_n$ of R given by $I_i = M_i$, $I_j = R_j$, and $I_k = (0)$ for all $k \in \{1, 2, \ldots, n\} \backslash \{i, j\}$. Note that $M_i \neq (0)$ and by Lemma 5.2, $M_i^2 = (0)$. Hence it is clear that $I \in A(R)^*$. Since $AG(R)$ is complemented, there exists $J = J_1 \times J_2 \times \cdots \times J_n$ of R such that $I \perp J$. From $IJ = (0) \times (0) \times \cdots \times (0)$, it follows that $I_s J_s = (0)$ for all $s \in \{1, 2, \ldots, n\}$. Since $I_i = M_i$, it follows that $J_i \subseteq M_i$ and as $I_j = R_j$, we obtain that $J_j = (0)$. Since $n \geq 3$, there exists $k \in \{1, 2, \ldots, n\} \backslash \{i, j\}$. As R_k is a field, it follows that $J_k \in \{(0), R_k\}$. Consider the ideal $A = A_1 \times A_2 \times \cdots \times A_n$ of R given by $A_i = M_i, A_j = (0)$, $A_k \in \{(0), R_k\} \backslash \{J_k\}$, and $A_s = (0)$ for all $s \in \{1, 2, \ldots, n\} \backslash \{i, j, k\}$. Note that $AI = AJ = (0) \times (0) \times \cdots \times (0)$ but $A \notin \{(0) \times (0) \times \cdots \times (0), I, J\}$. This is impossible since $I \perp J$. Thus if R is not reduced and $AG(R)$ is complemented, then $n = 2$. □

Let R be a zero-dimensional quasisemilocal ring admitting more than one maximal ideal. The following theorem determines necessary and sufficient conditions in order that $AG(R)$ is complemented.

Theorem 5.6. *Let R be a quasisemilocal ring which is not quasilocal and let $\dim R = 0$. Then the following statements are equivalent:*

(i) *$AG(R)$ is complemented.*

(ii) *Either $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings, where $n \geq 2$ and F_i is a field for all $i \in \{1, 2, \ldots, n\}$, or $R \cong S \times F$ as rings, where (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$ and F is a field.*

Proof. (i) $\Rightarrow$ (ii) Let n be the number of maximal ideals of R. Since R is not quasilocal, it follows that $n \geq 2$. We know from Lemma 5.1 that $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings, where (R_i, M_i) is a quasilocal ring with $M_i^4 = (0)$ for each $i \in \{1, 2, \ldots, n\}$. If $M_i = (0)$ for each $i \in \{1, 2, \ldots, n\}$, then R_i is a field for each i and hence with $F_i = R_i$, we obtain that $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings. Suppose that R_i is not a field for at least one $i \in \{1, 2, \ldots, n\}$. We know from Lemma 5.4 that such an i is necessarily unique. Now R is not reduced. Hence we obtain from Lemma 5.5 that $n = 2$. Thus $R \cong R_1 \times R_2$ as rings, where we may assume that R_1 is not a field and R_2 is a field. We know from Lemma 5.2 that $M_1^2 = (0)$ and $M_1 = R_1 x_1$ for any nonzero $x_1 \in M_1$. Hence M_1 is the only nonzero proper ideal of R_1. Thus (R_1, M_1) is a SPIR with $M_1 \neq (0)$ but $M_1^2 = (0)$. Hence with $S = R_1, M = M_1$, and $F = R_2$, we obtain that $R \cong S \times F$ as rings, where (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$ and F is a field.

(ii) $\Rightarrow$ (i) Suppose that $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings with $n \geq 2$ and F_i is a field for all $i \in \{1, 2, \ldots, n\}$. Note that R is reduced and hence we obtain from Proposition 2.2 that $AG(R)$ is complemented. Indeed $AG(R)$ is uniquely complemented.

Suppose that $R \cong S \times F$ as rings, where (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$ and F is a field. Let $T = S \times F$. Note that M is the only nonzero proper ideal of S. Hence $A(T)^* = \{(0) \times F, M \times (0), M \times F, S \times (0)\}$. It is easy to verify that $(0) \times F \perp M \times (0), M \times F \perp M \times (0)$, and $S \times (0) \perp (0) \times F$. This shows that $AG(T)$ is complemented. As $R \cong T$ as rings, we obtain that $AG(R)$ is complemented. Observe that $(0) \times F \perp M \times (0)$ and $(0) \times F \perp S \times (0)$. As $M \times F$ is adjacent to $M \times (0)$ but $M \times F$ is not adjacent to $S \times (0)$, it follows that $AG(T)$ is not uniquely complemented. Hence we obtain that $AG(R)$ is not uniquely complemented. $\square$

The following corollary determines when $AG(R)$ is complemented, where R is a finite semilocal ring which is not local.

Corollary 5.7. *Let R be a finite semilocal ring which is not local. The following statements are equivalent:*

(i) *$AG(R)$ is complemented.*

(ii) *Either $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings for some $n \geq 2$, where F_i is a finite field for $i = 1, 2, \ldots, n$, or $R \cong S \times F$ as rings, where (S, M) is a finite SPIR with $M \neq (0)$ but $M^2 = (0)$ and F is a finite field.*

Proof. The proof of this corollary follows immediately from Theorem 5.6. Note that the finiteness assertion of F_i for $i = 1, 2, \ldots, n$, S, and F in (ii) follow since R is a finite ring. $\square$

6. Rings R with only finitely many maximal N-primes of (0) such that $AG(R)$ is complemented

Let R be a commutative ring with identity which is not reduced (that is, $nil(R) \neq (0)$). Suppose that R admits only a finite number of maximal N-primes of (0). Let $\{P_1, \ldots, P_n\}$ be the set of all maximal N-primes of (0) in R. Moreover, we assume that $\cap_{i=1}^n P_i = nil(R)$ and $A(R)^*$ contains at least two elements. The purpose of this section is to determine necessary and sufficient conditions in order that $AG(R)$ is complemented. We begin with the following lemma.

Lemma 6.1. *Let R be a ring which is not reduced. Suppose that R admits only one maximal N-prime of (0). Let P be the unique maximal N-prime of (0) in R. If $AG(R)$ is complemented, then P is a maximal ideal of R. Moreover, if $P = nil(R)$, then P is the only prime ideal of R.*

Proof. Suppose that $AG(R)$ is complemented. We prove that P is a maximal ideal of R. Let M be a maximal ideal of R such that $P \subseteq M$. We assert that $P = M$. Suppose that $P \neq M$. Let $a \in M \backslash P$. Since P is the only maximal N-prime of (0) in R, it follows that $P = Z(R)$. Thus $a \notin Z(R)$. As $nil(R) \neq (0)$, there exists $x \in nil(R)\backslash\{0\}$ such that $x^2 = 0$. Let $J = Rx$. Note that $J \in A(R)^*$. Hence there exists $K \in A(R)^*$ such that $J \perp K$. Let $A = Rax$. From $(Rx)K = (0)$, it is clear that $AK = (Rax)K = (0)$. Moreover, as $x^2 = 0$, it follows that $AJ = (Rax)(Rx) = (0)$. Thus the ideal A of R satisfies $AJ = AK = (0)$. Since $J \perp K$, it follows that $A \in \{(0), J, K\}$. Since $x \neq 0$ and $a \notin Z(R)$, $A = Rax \neq (0)$. Observe that $A \neq J$. Indeed, for any $y \in M$, $J = Rx \neq Ryx$. For if $Rx = Ryx$, then $x = ryx$ for some $r \in R$. This implies that $x(1 - ry) = 0$ and so $1 - ry \in Z(R) = P \subseteq M$. As $y \in M$, we obtain that $1 = 1 - ry + ry \in M$. This is impossible since $M \neq R$. This shows that $A \neq J$. Therefore, $A = K$. Hence we obtain that $J \perp A$. Let $B = Ra^2x$. Since $x^2 = 0$, it is clear that $BJ = (Ra^2x)(Rx) = (0)$ and $BA = (Ra^2x)(Rax) = (0)$. As $a \notin Z(R)$ but $a \in M$, it is clear that $Rx \neq Ra^2x$ and $Rax \neq Ra^2x$. Thus the ideal $B = Ra^2x \in A(R)^*$ is adjacent to both J and A. This is impossible since $J \perp A$. Therefore, $P = M$ and this proves that P is a maximal ideal of R.

Suppose that $P = nil(R)$. We next verify that P is the only prime ideal of R. Let Q be any prime ideal of R. Then $Q \supseteq nil(R) = P$ and as P is a maximal ideal of R, it follows that $Q = P$. This shows that P is the only prime ideal of R. □

The following example illustrates that the moreover assertion of Lemma 6.1 may fail to hold if the hypothesis that $P = nil(R)$ is omitted.

Example 6.2. Let $T = \mathbf{Z}[x]$ be the polynomial ring in one variable over $\mathbf{Z}$. Let $I = x^2T + 2xT$. Let $R = T/I$. For any $t \in T$, we denote $t + I$ by $\bar{t}$. Since $\mathbf{Z} \cap I = (0)$, we identify $\overline{n}$ with n for any $n \in \mathbf{Z}$. This example appeared in [2, Example 3.6(a)], where it was noted that $nil(R) = \{0, \overline{x}\}$ and moreover, it was shown that $\Gamma(R)$ is an infinite star graph with center $\overline{x}$, where $\Gamma(R)$ is the zero-divisor graph of R.

Note that $I = x^2T + 2xT = xT \cap (x^2T + 2T)$ is an irredundant primary decomposition of I in T with xT is $P_1 = xT$-primary and $x^2T + 2T$ is $P_2 = xT + 2T$-primary. Observe that xT/I is a P_1/I-primary ideal of R and $(x^2T + 2T)/I$ is a P_2/I-primary ideal of R. Hence it follows that $xT/I \cap (x^2T + 2T)/I$ is an irredundant primary decomposition of the zero ideal of R. We know from [3, Proposition 4.7] that $Z(R) = P_1/I \cup P_2/I$ and as $P_1 \subseteq P_2$, it follows that $Z(R) = P_2/I$. This shows that R admits P_2/I as its only maximal N-prime of (0). Note that $nil(R) = P_1/I \neq P_2/I$.

We now verify that $AG(R)$ is complemented. Indeed, we show that $AG(R)$ is an infinite star graph with center $nil(R)$. Let $J \in A(R)^*$. Then $J \subseteq Z(R) = P_2/I$. Observe that $P_2/I = ((0) :_R \overline{x})$. Hence we obtain that $Jnil(R) = (0)$. Let J_1, J_2 be distinct nonzero ideals of R which are different from $nil(R)$. As $nil(R) = \{0, \overline{x}\}$, it follows that $J_1 \not\subseteq nil(R)$ and $J_2 \not\subseteq nil(R)$. Since $nil(R)$ is a prime ideal of R, we obtain that $J_1J_2 \not\subseteq nil(R)$. Hence we obtain that $J_1J_2 \neq (0)$. It is clear that for any positive integer n, $2^nR \in A(R)^*$ and moreover, for any distinct positive integers n, m, $2^nR \neq 2^mR$. The above arguments show that $AG(R)$ is an infinite star graph with center $nil(R)$. Hence $AG(R)$ is complemented. However, R has an infinite number of prime ideals. □

The following theorem is an immediate consequence of Lemma 6.1 and Theorem 4.8.

Theorem 6.3. *Let R be a ring which is not reduced, admitting only one maximal N-prime P of (0) such that $P = nil(R)$ and $AG(R)$ admits at least two vertices. Then $AG(R)$ is complemented if and only if (a) and (b) hold and moreover, either (c) or (d) holds where (a)–(d) are given below:*

(a) $P^2 \neq (0)$.

(b) $P^4 = (0)$.

(c) *R is a SPIR.*

(d) (d_1) *$z^2 = 0$ for each $z \in P$, P is not principal but there exist $a, b \in P$ such that $P = Ra + Rb$; and (d_2) $P^2 = Rx$ for any nonzero $x \in P^2$.*

Proof. Suppose that $P = nil(R)$ and $AG(R)$ is complemented. Now it follows, from Lemma 6.1, that P is the only prime ideal of R. Hence R is a zero-dimensional quasilocal ring with P as its unique maximal ideal. Applying Theorem 4.8, we obtain that (a) and (b) hold and moreover, either (c) or (d_1) holds. We now verify that when (d_1) holds, then (d_2) holds. From (d_1), $P = Ra + Rb$. As $z^2 = 0$ for each $z \in P$, it follows that $P^2 = Rab$, and $P^3 = (0)$. Let $x \in P^2, x \neq 0$. Hence $x = rab$ for some $r \in R$. Since R is quasilocal with P as its unique maximal ideal and $P^3 = (0)$, it follows that r is a unit in R and so $ab = r^{-1}x$. Hence we obtain that $P^2 = Rab = Rx$.

Conversely, assume that (a) and (b) hold and moreover, either (c) or (d) holds. If (c) holds, then it is clear that P is the unique maximal ideal of R and it follows that either $AG(R)$ is a graph on the vertex set $\{P, P^2, P^3\}$ with $P \perp P^3$ and $P^2 \perp P^3$ or $AG(R)$ is a graph on the vertex set $\{P, P^2\}$ and $P \perp P^2$. Thus if (c) holds, then $AG(R)$ is complemented. Suppose that (d) holds. Let $r \in R \backslash P$. Now $P = Ra + Rb$, $P^2 = Rab$, and $P^3 = (0)$. Since P is the only maximal N-prime of (0) in R, it follows that $Z(R) = P$. As $ab \neq 0$ and $r \in R \backslash Z(R)$, we obtain that $rab \neq 0$. Hence $P^2 = R(rab)$. So there exists $s \in R$ such that $ab = srab$. This implies that $(1 - sr)ab = 0$. Hence we obtain that $1 - sr \in Z(R) = P$. Therefore, $P + Rr = R$. This is true for any $r \in R \backslash P$. Hence it follows that P is a maximal ideal of R. By hypothesis, $P = nil(R)$. So, R must be quasilocal with P as its unique maximal ideal. Now we obtain from (ii) ⇒ (i) of Lemma 4.2 that $AG(R)$ is complemented. □

Let R and $\{P_1, \ldots, P_n\}$ be as in the beginning of this section. We assume that $n \geq 2$ and attempt to determine necessary and sufficient conditions in order that $AG(R)$ is complemented. We next state and prove Lemma 6.4. It is useful to recall the following. Let I be an ideal of a commutative ring T with identity. A prime ideal P of T is said to be a B-prime of I if there exists $t \in T$ such that $P = (I :_T t)$ [9].

Lemma 6.4. *Let R be a ring which is not reduced. Let $n \geq 2$ and let $\{P_1, P_2 \ldots, P_n\}$ be the set of all maximal N-primes of (0) in R. Suppose that $nil(R) = \cap_{i=1}^{n} P_i$. If $AG(R)$ is complemented, then the following hold:*

(i) *For each $i \in \{1, 2, \ldots, n\}$, there exists $x_i \in R$ such that $P_i = ((0) :_R x_i)$ (that is, P_i is a B-prime of (0) in R for each $i \in \{1, 2, \ldots, n\}$). Moreover, for each $i \in \{1, 2, \ldots, n\}$, $x_i \in P_j$ for all $j \in \{1, 2, \ldots, n\} \backslash \{i\}$.*

(ii) *With* $x_1, x_2, \ldots, x_n$ *as in* (i), $x_i \in nil(R)$ *for some* $i \in \{1, 2, \ldots, n\}$ *and moreover, for that i,* P_i *is a maximal ideal of* R.

Proof. (i) As $\{P_1, P_2 \ldots, P_n\}$ is the set of all maximal N-primes of (0) in R, it follows that $Z(R) = \cup_{i=1}^n P_i$. Suppose that $AG(R)$ is complemented. We know, from Lemma 3.3, that $(nil(R))^4 = (0)$. Hence $(\cap_{i=1}^n P_i)^4 = (0)$. Therefore, $\prod_{i=1}^n P_i^4 = (0)$. Let $i \in \{1, 2, \ldots, n\}$. Since $n \geq 2$ and P_k is a maximal N-prime of (0) in R for each $k \in \{1, 2, \ldots, n\}$, it follows that $\prod_{j \in A_i} P_j^4 \neq (0)$, where $A_i = \{1, 2, \ldots, n\} \backslash \{i\}$. Let $y_i \in \prod_{j \in A_i} P_j^4$, $y_i \neq 0$. It now follows that $P_i^4 y_i = (0)$. Let $0 \leq s < 4$ be such that $P_i^s y_i \neq (0)$ but $P_i^{s+1} y_i = (0)$. Let $x_i \in P_i^s y_i \backslash \{0\}$. Observe that $P_i x_i = (0)$. Hence we obtain that $P_i \subseteq ((0) :_R x_i) \subseteq Z(R) = \cup_{k=1}^n P_k$. It now follows that $P_i = ((0) :_R x_i)$. This proves that P_i is a B-prime of (0) in R for each $i \in \{1, 2, \ldots, n\}$. We now prove the moreover assertion. We obtain from [4, Lemma 3.6] that $x_i x_j = 0$ for all distinct $i, j \in \{1, 2, \ldots, n\}$. Hence for each $i \in \{1, 2, \ldots, n\}$, $x_i \in ((0) :_R x_j) = P_j$ for all $j \in \{1, 2, \ldots, n\} \backslash \{i\}$.

(ii) Let $z \in nil(R)$ with $z \neq 0$. Note that $(x_1 + x_2 + \cdots + x_n)z = 0$. Therefore, $x_1 + x_2 + \cdots + x_n \in Z(R) = \cup_{i=1}^n P_i$. Hence we obtain that $x_1 + x_2 + \cdots + x_n \in P_i$ for some $i \in \{1, 2, \ldots, n\}$. We know from (i) that for each $j \in \{1, 2, \ldots, n\} \backslash \{i\}$, $x_j \in P_i$. It follows from $x_1 + x_2 + \cdots + x_n \in P_i$ that $x_i \in P_i = ((0) :_R x_i)$. Therefore, we obtain that $x_i^2 = 0$ and so $x_i \in nil(R)$. We now prove that P_i is a maximal ideal of R. Let M be a maximal ideal of R such that $P_i \subseteq M$. We claim that $M \subseteq Z(R)$. Suppose that $M \not\subseteq Z(R)$. Let $w \in M \backslash Z(R)$. Let $I = Rx_i$. Since $x_i \neq 0$ but $x_i^2 = 0$, it is clear that $I \in A(R)^*$. As $AG(R)$ is complemented, there exists $J \in A(R)^*$ such that $I \perp J$. Let $A = Rwx_i$. Since $x_i^2 = 0$, $IJ = (Rx_i)J = (0)$, it is clear that $AJ = AI = (0)$. It follows from $I \perp J$ that $A \in \{(0), I, J\}$. Since $w \notin Z(R)$, we obtain that $A = Rwx_i \neq (0)$. Observe that $A \neq I$. For if $A = I$, then $x_i \in A$ and so $x_i = rwx_i$ for some $r \in R$. This implies that $(1 - rw)x_i = 0$. Hence $1 - rw \in ((0) :_R x_i) = P_i \subseteq M$. This is impossible since $w \in M$ and M is a proper ideal of R. Hence $A \neq I$ and so $A = J$. Thus we arrive at $I = Rx_i \perp A = Rwx_i$. Note that $B = Rw^2 x_i$ is such that $B \notin \{(0), I, A\}$, but $BI = BA = (0)$. This is in contradiction to the fact that $I \perp A$. Hence we must have $M \subseteq Z(R)$. As M is a maximal ideal of R and $M \subseteq Z(R)$, M is necessarily a maximal N-prime of (0) in R. Since P_i is also a maximal N-prime of (0) in R, it follows from $P_i \subseteq M$ that $P_i = M$. This proves that P_i is a maximal ideal of R. □

With the same hypotheses as in the statement of Lemma 6.4, the following lemma provides another necessary condition in order that $AG(R)$ is complemented.

Lemma 6.5. *Let* R *be a ring which is not reduced. Let* $n \geq 2$ *and let* $\{P_1, P_2, \ldots, P_n\}$ *be the set of all maximal N-primes of* (0) *in* R. *Suppose that* $nil(R) = \cap_{i=1}^n P_i$. *If* $AG(R)$ *is complemented, then* $(nil(R))^2 = (0)$.

Proof. Suppose that $AG(R)$ is complemented. We know from Lemma 3.3 that $(nil(R))^4 = (0)$. We first prove that $(nil(R))^3 = (0)$. Suppose that $(nil(R))^3 \neq (0)$. We know from Proposition 3.4(i) that $nil(R) \perp (nil(R))^3$. Moreover, we know from Lemma 6.4 that there exist elements $x_i \in R$ such that $P_i = ((0) :_R x_i)$ for each $i \in \{1, 2, \ldots, n\}$ and so $(nil(R))(Rx_i) = (0)$ and $(nil(R))^3(Rx_i) = (0)$. Since $nil(R) \perp (nil(R))^3$, it follows that $Rx_1 \in \{nil(R), (nil(R))^3\}$ and $Rx_2 \in \{nil(R), (nil(R))^3\}$. As $(nil(R))^3 \subseteq nil(R)$, it follows that either $Rx_1 \subseteq Rx_2$ or $Rx_2 \subseteq Rx_1$. We may assume without loss of generality that $Rx_1 \subseteq Rx_2$. This implies that $x_1 = rx_2$ for some $r \in R$. Let $a \in P_2$. Then $ax_2 = 0$ and so $ax_1 = a(rx_2) = 0$. This implies that $a \in P_1$. Hence we arrive at $P_2 \subseteq P_1$. This is impossible since P_1 and P_2 are distinct maximal N-primes of (0) in R. Hence we obtain that $(nil(R))^3 = (0)$. We now show that $(nil(R))^2 = (0)$. Suppose that $(nil(R))^2 \neq (0)$. We know from Proposition 3.5(i) that $nil(R) \perp (nil(R))^2$. As $(nil(R))Rx_i = (nil(R))^2Rx_i = (0)$ for each $i \in \{1, 2, \ldots, n\}$, we obtain that $Rx_1 \in \{nil(R), (nil(R))^2\}$ and $Rx_2 \in \{nil(R), (nil(R))^2\}$. Since $(nil(R))^2 \subseteq nil(R)$, proceeding as in the previous paragraph, we obtain a similar contradiction.

This proves that $(nil(R))^2 = (0)$. □

Let $R, \{P_1, P_2, \ldots, P_n\}$ be as in the statement of Lemma 6.4. With the assumption that $nil(R) = \cap_{i=1}^n P_i$, we determine in Theorem 6.9 when $AG(R)$ is complemented. We make use of the following lemmas in the proof of Theorem 6.9. We denote by $Tot(R)$, the total quotient ring of R.

Lemma 6.6. *Let R be a ring which is not reduced. Let $n \geq 2$ and let $\{P_1, P_2, \ldots, P_n\}$ be the set of all maximal N-primes of (0) in R. Suppose that $nil(R) = \cap_{i=1}^n P_i$. If $AG(R)$ is complemented, then $n = 2$.*

Proof. As $\{P_1, P_2, \ldots, P_n\}$ is the set of all maximal N-primes of (0) in R, it is clear that $Z(R) = \cup_{i=1}^n P_i$. Let $S = R\backslash Z(R) = R\backslash(\cup_{i=1}^n P_i)$. Observe that $S^{-1}R = Tot(R)$ is a zero-dimensional quasisemilocal ring and moreover, $\{S^{-1}P_1, S^{-1}P_2, \ldots, S^{-1}P_n\}$ is the set of all its maximal ideals. Furthermore, as R is not reduced, it follows that $Tot(R)$ is not reduced. Since $n \geq 2$, it is clear that $Tot(R)$ is not quasilocal. We want to show that $n = 2$. In view of (i) $\Rightarrow$ (ii) of Theorem 5.6, it is enough to show that $AG(Tot(R))$ is complemented. This is clear if $R = Tot(R)$. Hence we may assume that $R \neq Tot(R)$. Therefore, P_i is not a maximal ideal of R for at least one $i \in \{1, 2, \ldots, n\}$. Without loss of generality we may assume that P_1 is not a maximal ideal of R. We know from Lemma 6.4(i) that there exist elements $x_i \in R$ such that $P_i = ((0) :_R x_i)$ for $i = 1, 2, \ldots, n$. Since P_1 is not a maximal ideal of R, it follows from Lemma 6.4(ii) that $x_1 \notin nil(R)$.

Let $A \in A(Tot(R))^*$. Note that $A = S^{-1}I$ for some ideal $I \in A(R)^*$. Since we are assuming that $AG(R)$ is complemented, there exists $J \in A(R)^*$ such that $I \perp J$ in $AG(R)$. We claim that $A = S^{-1}I \perp S^{-1}J$ in $AG(Tot(R))$. From $IJ = (0)$, it follows that $S^{-1}IS^{-1}J = (0)$. If $B = S^{-1}K$ is any element of $A(Tot(R))^*$ such that $S^{-1}IS^{-1}K = S^{-1}JS^{-1}K = (0)$, it follows that $IK = JK = (0)$. Since $I \perp J$ in $AG(R)$, it follows that $K \in \{I, J\}$ and hence we obtain that either $S^{-1}K = S^{-1}I$ or $S^{-1}K = S^{-1}J$. Now to show $S^{-1}I \perp S^{-1}J$ in $AG(Tot(R))$, we need only to verify that $S^{-1}I \neq S^{-1}J$. Suppose that $S^{-1}I = S^{-1}J$. Then it follows from $S^{-1}IS^{-1}J = (0)$ that $(S^{-1}I)^2 = (S^{-1}J)^2 = (0)$. Therefore, we obtain that $I^2 = J^2 = (0)$. Hence it follows that $I \subseteq nil(R)$ and $J \subseteq nil(R)$. Note that $I(Rx_1) = J(Rx_1) = (0)$. As $x_1 \notin nil(R)$, it is clear that $Rx_1 \notin \{(0), I, J\}$. Thus we obtain that the ideal Rx_1 is adjacent to I and J in $AG(R)$. This is impossible since $I \perp J$ in $AG(R)$. This proves that $S^{-1}I \neq S^{-1}J$ and so as is noted already, we obtain that $S^{-1}I \perp S^{-1}J$ in $AG(Tot(R))$. This shows that $AG(Tot(R))$ is complemented and so as is remarked earlier in this proof, it follows that $n = 2$. □

Lemma 6.7. *Let T_1, T_2 be commutative rings with identity. Suppose that N_i is the unique maximal N-prime of (0) in T_i for each $i \in \{1, 2\}$ with $nil(T_i) = N_i$. Let $T = T_1 \times T_2$. Suppose that $AG(T)$ is complemented. If $N_2 \neq (0)$, then N_2 is a maximal ideal of T_2.*

Proof. Since $nil(T_2) = N_2 \neq (0)$, there exists $t_2 \in N_2$ such that $t_2 \neq 0$ but $t_2^2 = 0$. By contradiction, suppose that N_2 is not a maximal ideal of T_2. Let M be a maximal ideal of T_2 such that $N_2 \subset M$. Consider the ideal $I = T_1 \times T_2t_2$. Note that $I \in A(T)^*$. As $AG(T)$ is complemented, there exists $J \in A(T)^*$ such that $I \perp J$. Observe that $J = J_1 \times J_2$ for some ideal J_1 of T_1 and an ideal J_2 of T_2. From $IJ = (0) \times (0)$, it follows that $J_1 = (0)$ and $(T_2t_2)J_2 = (0)$. Let $y \in M \backslash N_2$. Since $Z(T_2) = N_2$, we obtain that $y \notin Z(T_2)$. As $t_2 \neq 0$, it follows that $yt_2 \neq 0$. Note that the nonzero ideal $K = (0) \times T_2(yt_2)$ is such that $IK = JK = (0) \times (0)$. Since $I \perp J$, we obtain that $K \in \{I, J\}$. It is clear that $K \neq I$. Hence $K = J$. Therefore, we obtain that $T_2t_2 = T_2(yt_2)$. So there exists $s_2 \in T_2$ such that $t_2 = s_2yt_2$. This implies that $t_2(1 - s_2y) = 0$. Thus $1 - s_2y \in Z(T_2) = N_2 \subset M$. As $y \in M$, it follows that $1 = 1 - s_2y + s_2y \in M$. This is impossible. Therefore, N_2 must be a maximal ideal of T_2. □

We also make use of the following lemma in the proof of Theorem 6.9.

Lemma 6.8. *Let (S, M) be a SPIR with $M \neq (0)$ but $M^2 = (0)$ and D be an integral domain. Let $R = S \times D$. Then $AG(R)$ is complemented.*

Proof. If D is a field, then it is already verified in the proof of (ii) ⇒ (i) of Theorem 5.6 that $AG(R)$ is complemented. Suppose that D is not a field. Observe that $A(R)^* = \{(0) \times I | I$ varies over all nonzero ideals of $D\} \cup \{M \times J | J$ varies over all ideals of $D\} \cup \{S \times (0)\}$. It is easy to verify that for any nonzero ideal I of D, $(0) \times I \perp M \times (0)$, for any nonzero ideal J of D, $M \times J \perp M \times (0)$, and $S \times (0) \perp (0) \times D$. This proves that each element of $A(R)^*$ admits a complement in $AG(R)$ and hence we obtain that $AG(R)$ is complemented. □

With the help of Lemmas 6.4–6.8, we prove the following theorem.

Theorem 6.9. *Let R be a ring which is not reduced. Let $n \geq 2$ and let $\{P_1, P_2, \ldots, P_n\}$ be the set of all maximal N-primes of (0) in R. Suppose that $nil(R) = \cap_{i=1}^n P_i$. Then $AG(R)$ is complemented if and only if either R is isomorphic to $F \times S$ as rings, where F is a field and (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$, or is isomorphic to $S \times D$ as rings, where (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$ and D is an integral domain which is not a field.*

Proof. Suppose that $AG(R)$ is complemented. We know from Lemma 6.6 that $n = 2$. Thus $\{P_1, P_2\}$ is the set of all maximal N-primes of (0) in R. We know from Lemma 6.4(i) that there exist $x_1, x_2 \in R$ such that $P_1 = ((0) :_R x_1), P_2 = ((0) :_R x_2)$ and moreover, $x_1 \in P_2$ and $x_2 \in P_1$. Furthermore, we know from Lemma 6.4(ii) that either $x_1 \in nil(R)$ or $x_2 \in nil(R)$. We may assume without loss of generality that $x_1 \in nil(R)$. In such a case, it follows from Lemma 6.4(ii) that P_1 is a maximal ideal of R. As P_1, P_2 are distinct maximal N-primes of (0) in R, we obtain that $P_1 + P_2 = R$. We know from Lemma 6.5 that $(nil(R))^2 = (0)$ and so $(P_i \cap P_2)^2 = (0)$. Hence $P_1^2P_2^2 = (0)$. As $P_1^2 + P_2^2 = R$, we obtain from the Chinese remainder theorem [3, Proposition 1.10(ii) and (iii)] that the mapping $f : R \to R/P_1^2 \times R/P_2^2$ given by $f(r) = (r + P_1^2, r + P_2^2)$ is an isomorphism of rings. Let us denote R/P_1^2 by T_1 and R/P_2^2 by T_2. Moreover, let us denote $P_1/P_1^2 = N_1$ and P_2/P_2^2 by N_2. Note that $f(Z(R)) = f(P_1 \cup P_2) = (N_1 \times T_2) \cup (T_1 \times N_2)$. As f is an isomorphism of rings, it follows that $f(Z(R)) = Z(T_1 \times T_2) = (Z(T_1) \times T_2) \cup (T_1 \times Z(T_2))$. Hence we obtain that $Z(T_1) = N_1$ and $Z(T_2) = N_2$. Therefore, N_i is the unique maximal N-prime of the zero ideal of T_i for each $i \in \{1, 2\}$. Moreover, $f(nil(R)) = f(P_1 \cap P_2) = P_1/P_1^2 \times P_2/P_2^2 = nil(T_1) \times nil(T_2)$. Hence it follows that $nil(T_1) = P_1/P_1^2 = N_1$ and $nil(T_2) = P_2/P_2^2 = N_2$.

We consider two cases.

Case (i). P_2 is a maximal ideal of R.

As P_1 is already a maximal ideal of R and $nil(R) = P_1 \cap P_2$, it follows that R is a zero-dimensional quasisemilocal ring with $\{P_1, P_2\}$ as its set of all prime ideals of R. Now $AG(R)$ is complemented and R is not reduced. Hence it follows from (i) $\Rightarrow$ (ii) of Theorem 5.6 that R must be isomorphic to $S \times F$ as rings, where (S, M) is a SPIR with $M \neq (0)$ but $M^2 = (0)$ and F is a field.

Case (ii). P_2 is not a maximal ideal of R.

Note that $N_2 = P_2/P_2^2$ is not a maximal ideal of T_2. Since $AG(R)$ is complemented and R is isomorphic to $T_1 \times T_2$ as rings, we obtain that $AG(T_1 \times T_2)$ is complemented. Now it follows from Lemma 6.7 that $nil(T_2)$ is the zero ideal of T_2. Hence we obtain that $P_2 = P_2^2$ and so $T_2 = R/P_2^2 = R/P_2$ is an integral domain. By assumption, P_2 is not a maximal ideal of R and so T_2 is not a field. Let us denote $T_1 \times T_2$ by T. Since T is not reduced, it follows that $nil(T_1)$ is a nonzero ideal of T_1. Hence $P_1 \neq P_1^2$. We assert that any $x \in P_1 \backslash P_1^2$, $P_1/P_1^2 = T_1(x + P_1^2)$. Observe that $I = T_1(x + P_1^2) \times T_2 \in A(T)^*$. As $AG(T)$ is complemented, there exists an ideal J_1 of T_1 and an ideal J_2 of T_2 such that $I = T_1(x + P_1^2) \times T_2 \perp J = J_1 \times J_2$. Hence $J_2 = (0 + P_2^2)$ and from $T_1(x + P_1^2)J_1 = (0 + P_1^2)$, it follows that $J_1 \subseteq P_1/P_1^2$. Note that the ideal $K = P_1/P_1^2 \times (0 + P_2^2)$ is such that $IK = JK = (0 + P_1^2) \times (0 + P_2^2)$. Since $I \perp J$ and as $K \notin \{(0 + P_1^2) \times (0 + P_2^2), I\}$, it follows that $K = J$. Hence we obtain that $I \perp K$. Now the ideal $B = T_1(x + P_1^2) \times (0 + P_2^2)$ is such that $BI = BK = (0 + P_1^2) \times (0 + P_2^2)$. Since $I \perp K$ and $B \notin \{(0 + P_1^2) \times (0 + P_2^2), I\}$, we obtain that $B = K$. Hence $P_1/P_1^2 = T_1(x + P_1^2)$. As P_1/P_1^2 is a maximal ideal of T_1, it is clear that (T_1, N_1) is a SPIR with N_1 is a nonzero ideal of T_1 but N_1^2 is the zero ideal of T_1. Let $S = T_1$, $M = N_1$, and $D = T_2$. Note that $(S.M)$ is a SPIR with $M \neq (0)$ but $M^2 = (0)$, D is an integral domain which is not a field and moreover, $R \cong S \times D$ as rings.

The converse follows immediately from Lemma 6.8. $\square$

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