When is the annihilating ideal graph of a zero-dimensional quasisemilocal commutative ring complemented?

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Abstract. Let $R$ be a commutative ring with identity. Let $A(R)$ denote the collection of all annihilating ideals of $R$ (that is, $A(R)$ is the collection of all ideals $I$ of $R$ which admits a nonzero annihilator in $R$). Let $AG(R)$ denote the annihilating ideal graph of $R$. In this article, necessary and sufficient conditions are determined in order that $AG(R)$ is complemented under the assumption that $R$ is a zero-dimensional quasisemilocal ring which admits at least two nonzero annihilating ideals and as a corollary we determine finite rings $R$ such that $AG(R)$ is complemented under the assumption that $A(R)$ contains at least two nonzero ideals.

Keywords: Annihilating ideal graph of a commutative ring; Complemented graph; Zero-dimensional quasisemilocal ring; Special principal ideal ring

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1. INTRODUCTION

The rings considered in this article are nonzero commutative rings with identity. Recall from [5] that an ideal $I$ of a ring $R$ is an annihilating ideal if there exists $r \in R \setminus \{0\}$ such that $Ir = (0)$. As in [5], we denote by $A(R)$, the set of all annihilating ideals of $R$ and by $A(R)^*$, the set of all nonzero annihilating ideals of $R$. In [5], the authors introduced the concept of annihilating ideal graph of $R$, denoted by $AG(R)$, which is defined as follows: $AG(R)$ is an undirected graph whose vertex set is $A(R)^*$ and two distinct vertices $I$ and $J$ are adjacent in this graph if and only if $IJ = (0)$. Several graph theoretic properties of the annihilating ideal graph of any commutative ring with identity and their interplay with
the ring theoretic properties have been investigated in [5,6]. Moreover, the annihilating ideal graph of a commutative ring is also studied in [1,7]. In this article we determine necessary and sufficient conditions in order that $AG(R)$ is complemented under the assumption that $R$ is a zero-dimensional quasisemilocal ring such that $A(R)^*$ contains at least two elements. As any finite ring is zero-dimensional and has only finitely many prime ideals, we answer the question of when $AG(R)$ is complemented for any finite ring $R$ which admits at least two nonzero annihilating ideals as a corollary to the results proved in this article.

This article is motivated by the interesting theorems proved on the annihilating ideal graph of a commutative ring in [1,5–7], and moreover, we are very much inspired by the research article [2] in which the authors among other results determined necessary and sufficient conditions in order that $\Gamma(R)$ is complemented, where $\Gamma(R)$ is the zero-divisor graph of $R$.

It is useful to recall the following definitions from [2,11]. Let $G = (V, E)$ be a simple undirected graph. Let $a, b \in V$. We define $a \leq b$ if $a$ and $b$ are not adjacent and each vertex of $G$ adjacent to $b$ is also adjacent to $a$. We define $a \sim b$ if $a \leq b$ and $b \leq a$. Thus $a \sim b$ if and only if $\{c \in V|c$ is adjacent to $a in G\} = \{d \in V|d$ is adjacent to $b in G\}$. Let $a, b \in V$, $a \neq b$. We say that $a$ and $b$ are orthogonal, written $a \perp b$, if $a$ and $b$ are adjacent and there is no vertex $c$ of $G$ which is adjacent to both $a$ and $b$. We say that $G$ is complemented, if for each vertex $a$ of $G$, there is a vertex $b$ of $G$ (called a complement of $a$) such that $a \perp b$. We say that $G$ is uniquely complemented if $G$ is complemented and whenever $a \perp b$ and $a \perp c$, then $b \sim c$ [2,11]. By dimension of a ring, we mean its Krull dimension and we use the abbreviation $\text{dim} R$ to denote the dimension of a ring $R$. A ring $R$ is said to be quasilocal (respectively, quasisemilocal) if $R$ has a unique maximal ideal (respectively, $R$ has only finitely many maximal ideals). By a local (respectively, a semilocal) ring, we mean a Noetherian quasilocal (respectively, a Noetherian quasisemilocal) ring. Recall that a local ring $(R, M)$ is said to be a special principal ideal ring (SPIR), if $R$ is a principal ideal ring and $M$ is nilpotent. Whenever a set $A$ is a subset of a set $B$ and $A \neq B$, we denote it symbolically by $A \subset B$.

It is also useful to recall the following definitions and results from commutative ring theory. Let $R$ be a ring. Let $M$ be a unitary $R$-module. By the set of zero-divisors of $M$ as an $R$-module denoted by $Z_{R}(M)$, we mean $Z_{R}(M) = \{r \in R|rm = 0 \text{ for some } m \in M, m \neq 0\}$. We denote $Z_{R}(R)$ simply by $Z(R)$. Recall from [8] that a prime ideal $P$ of $R$ is said to be a maximal N-prime of an ideal $I$ of $R$, if $P$ is maximal with respect to the property of being contained in $Z_{R}(R/I)$. It follows from [10, Theorem 1] that maximal N-primes of $(0)$ always exist and if $\{P_{\alpha}\}_{\alpha \in A}$ is the set of all maximal N-primes of $(0)$ in $R$, then $Z(R) = \cup_{\alpha \in A} P_{\alpha}$.

In Section 2, it is shown that $AG(R)$ is complemented for any reduced ring $R$ which is not an integral domain. Let $R$ be a ring which is not reduced. In Section 3, we state and prove several necessary conditions in order that $AG(R)$ is complemented. The main theorem proved in Section 4 is Theorem 4.8 which determines necessary and sufficient conditions in order that $AG(R)$ is complemented, where $R$ is a zero-dimensional quasilocal ring which admits at least two nonzero annihilating ideals. In Section 5, we consider zero-dimensional quasisemilocal rings $R$ with at least two nonzero annihilating ideals and in Theorem 5.6, necessary and sufficient conditions are determined in order that $AG(R)$ is complemented. In Section 6, we consider rings $R$ which are not reduced and which admit only a finite number of maximal N-primes of $(0)$. We denote the finite set of maximal N-primes of $(0)$ in $R$ by $\{P_{1}, \ldots, P_{n}\}$. We determine necessary and sufficient conditions in order that $AG(R)$ is
complemented under the additional hypothesis that $\cap_{i=1}^{n} P_i = \text{nil}(R)$, where $\text{nil}(R)$ denotes the nilradical of $R$ (see Theorems 6.3 and 6.9).

2. A SUFFICIENT CONDITION UNDER WHICH $AG(R)$ IS COMPLEMENTED

The purpose of this section is to prove that if $R$ is any reduced ring which is not an integral domain, then $AG(R)$ is complemented. We begin with the following lemma. This is an analogue to [2, Lemma 3.3]. Again we emphasize that all the rings considered in this article are commutative with identity.

**Lemma 2.1.** Let $R$ be a ring. Let $I, J \in A(R)^*$. The following statements are equivalent:

(i) $I \perp J, I^2 \neq (0)$, and $J^2 \neq (0)$.

(ii) $IJ = (0)$ and $I + J \not\in A(R)$.

**Proof.** (i) ⇒ (ii) Since $I \perp J$, it is clear that $IJ = (0)$. Suppose that $I + J \in A(R)$. Then there exists $a \in R \setminus \{0\}$ such that $a(I + J) = (0)$. Hence $aI = (0)$ and $aJ = (0)$. Since $I^2 \neq (0)$ and $J^2 \neq (0)$, it follows that $Ra \neq I$ and $Ra \neq J$. Observe that the ideal $Ra \in A(R)^*$ is such that $I(Ra) = (0)$ and $J(Ra) = (0)$. This is in contradiction to the hypothesis that $I \perp J$. Hence we obtain that $I + J \not\in A(R)$.

(ii) ⇒ (i) If $I^2 = (0)$, then from $IJ = (0)$, it follows that $(I + J)I = (0)$. This contradicts the assumption that $I + J \not\in A(R)$. Hence we obtain that $I^2 \neq (0)$. Similarly, it follows that $J^2 \neq (0)$. Now it is clear that $I \neq J$. Let $K$ be an ideal of $R$ such that $IK = (0)$ and $JK = (0)$. Then $(I + J)K = (0)$. Since $I + J \not\in A(R)$, it follows that $K = (0)$. This proves that $I \perp J$. □

**Proposition 2.2.** Let $R$ be a reduced ring which is not an integral domain. Then $AG(R)$ is complemented. Moreover, $AG(R)$ is uniquely complemented.

**Proof.** Since $R$ is not an integral domain, there exist $a, b \in R \setminus \{0\}$ such that $ab = 0$. Note that $Ra, Rb \in A(R)^*$. Since $R$ is reduced it follows from $ab = 0$ with $a, b \in R \setminus \{0\}$ that $Ra \neq Rb$. Hence $|A(R)^*| \geq 2$.

Let $I \in A(R)^*$. Hence there exists $x \in R \setminus \{0\}$ such that $Ix = (0)$. Let $J = ((0) :_R I)$. As any nonzero element of $I$ annihilates $J$, it is clear that $J \in A(R)^*$. We assert that $I \perp J$. It is clear that $IJ = (0)$. Hence in view of (ii) ⇒ (i) of Lemma 2.1, it is enough to show that $I + J \not\in A(R)$. Let $r \in R$ be such that $(I + J)r = (0)$. Then $Ir = (0)$ and $Jr = (0)$. Hence $r \in J$ and from $Jr = (0)$, it follows that $r^2 = 0$. Since $R$ is reduced, we obtain that $r = 0$. This proves that $I \perp J$. Thus each $I \in A(R)^*$ admits a complement in $AG(R)$. This shows that $AG(R)$ is complemented.

We next verify that $AG(R)$ is uniquely complemented. Let $I \in A(R)^*$. Let $J_1, J_2 \in A(R)^*$ be such that $I \perp J_1$ and $I \perp J_2$. Since $R$ is reduced, it follows that $A^2 \neq (0)$ for any nonzero ideal $A$ of $R$. As $I \perp J_1$ and $I \perp J_2$, we know from (i) ⇒ (ii) of Lemma 2.1 that $I + J_i \not\in A(R)$ for $i = 1, 2$. Hence $(I + J_1)J_2 \neq (0)$. This implies that $J_1J_2 \neq (0)$ since $IJ_2 = (0)$. Let $K \in A(R)^*$ be such that $K$ is adjacent to $J_2$. Then $KJ_2 = (0)$. From $IJ_1 = (0)$, it follows that $(I + J_2)KJ_1 = (0)$. As $I + J_2 \not\in A(R)$, it follows that $KJ_1 = (0)$. This proves that $J_1 \preceq J_2$. Similarly, using the facts that $IJ_2 = (0)$ and $I + J_1 \not\in A(R)$, it follows that $J_2 \preceq J_1$. Hence we obtain that $J_1 \sim J_2$. This proves that $AG(R)$ is uniquely complemented. □
3. **Some Necessary Conditions in Order that $AG(R)$ is Complemented, Where $R$ is Not a Reduced Ring**

In this section we consider rings $R$ such that the nilradical of $R$ is nonzero. We use $\text{nil}(R)$ to denote the nilradical of a ring $R$. The aim of this section is to determine some necessary conditions in order that $AG(R)$ is complemented. We begin with the following lemma.

**Lemma 3.1.** Let $R$ be a ring. If $a \in R \setminus \{0\}$, then for any $b \in \text{nil}(R)$, $Ra \neq Rab$.

**Proof.** If $Ra = Rab$, then $a = rab$ for some $r \in R$. This implies that $a(1 - rb) = 0$. Since $b \in \text{nil}(R)$, $1 - rb$ is a unit in $R$. Hence from $a(1 - rb) = 0$, it follows that $a = 0$. This is a contradiction. Hence $Ra \neq Rab$. □

The following lemma is obvious.

**Lemma 3.2.** Let $I$ be a nonzero nilpotent ideal of a ring $R$. Let $n$ be the least integer $p \geq 2$ with the property that $I^p = (0)$. Then $I^i \neq I^j$ for all distinct $i, j \in \{1, 2, \ldots, n\}$.

We next have the following lemma which shows that if $AG(R)$ is complemented, then $\text{nil}(R)$ must be nilpotent.

**Lemma 3.3.** Let $R$ be a ring. If $AG(R)$ is complemented, then $(\text{nil}(R))^4 = (0)$.

**Proof.** First we show that for any $a \in \text{nil}(R)$, $a^4 = 0$. Suppose that $a^4 \neq 0$. Let $n$ be the least integer $p \geq 5$ with the property that $a^p = 0$. Since $AG(R)$ is complemented, there exists $I \in A(R)^\ast$ such that $Ra^{n-3} \perp I$. It follows from Lemma 3.2 that $Ra^i \neq Ra^j$ for all distinct $i, j \in \{1, 2, \ldots, n\}$. Hence in particular $Ra^{n-1} \neq Ra^{n-2}$. Thus there exists $j \in \{n-2, n-1\}$ such that $I \neq Ra^j$. From $(Ra^{n-3})I = (0)$, it follows that $(Ra^j)I = (0)$. Since $n \geq 5$, it is clear that $Ra^jRa^{n-3} = (0)$. Hence the ideal $Ra^j$ is adjacent to both $Ra^{n-3} - 4$ and $I$. This is impossible since $Ra^{n-3} \perp I$. Therefore, for any $a \in \text{nil}(R)$, $a^4 = 0$.

Let $a, b, c \in \text{nil}(R)$. We assert that $a^2bc = 0$. Suppose that $a^2bc \neq 0$. As $AG(R)$ is complemented, there exists $I \in A(R)^\ast$ such that $Ra^2 \perp I$. From $(Ra^2)I = (0)$, it follows that $(Ra^2b)I = (Ra^2bc)I = (0)$. It follows from Lemma 3.1 that the ideals $Ra^2, Ra^2b$, and $Ra^2bc$ are distinct. Hence either $I \neq Ra^2b$ or $I \neq Ra^2bc$. If $I \neq Ra^2b$, then it follows from $a^4 = 0$ that $Ra^2b$ is adjacent to both $Ra^2$ and $I$. This is impossible since $Ra^2 \perp I$. Similarly, if $I \neq Ra^2bc$, then we obtain that $Ra^2bc$ is adjacent to both $Ra^2$ and $I$. This is not possible since $Ra^2 \perp I$. Hence for any $a, b, c \in \text{nil}(R)$, $a^2bc = 0$.

Let $a, b, c, d \in \text{nil}(R)$. We claim that $abcd = 0$. Suppose that $abcd \neq 0$. It follows from Lemma 3.1 that the ideals $Ra, Rabc$, and $Rabcd$ are distinct. Since $AG(R)$ is complemented, there exists $I \in A(R)^\ast$ such that $Ra \perp I$. It follows from $(Ra)I = (0)$ that $(Rabc)I = (0)$ and $(Rabcd)I = (0)$. Observe that either $I \neq Rabc$ or $I \neq Rabcd$. Since $a^2bc = 0$, $(Ra)(Rabc) = (0)$ and $(Ra)(Rabcd) = (0)$. If $I \neq Rabc$, then we obtain that $Rabc$ is adjacent to both $Ra$ and $I$. This is impossible since $Ra \perp I$. Similarly $I \neq Rabcd$ is also impossible. This proves that for any $a, b, c, d \in \text{nil}(R)$, $abcd = 0$.

This shows that $(\text{nil}(R))^4 = (0)$. □

The following proposition provides some more necessary conditions on $R$ if $(\text{nil}(R))^3 \neq (0)$ and $AG(R)$ is complemented.
Proposition 3.4. Let $R$ be a ring such that $(\text{nil}(R))^3 \neq (0)$. If $AG(R)$ is complemented, then the following hold:

(i) $(\text{nil}(R))^i \perp (\text{nil}(R))^3$ for $i = 1, 2$.
(ii) $(\text{nil}(R))^3 = Rx$ for any $x \in (\text{nil}(R))^3 \setminus \{0\}$.
(iii) $(\text{nil}(R))^2 = Ry$ for any $y \in (\text{nil}(R))^2 \setminus (\text{nil}(R))^3$.
(iv) $\text{nil}(R) = Rz$ for any $z \in \text{nil}(R) \setminus (\text{nil}(R))^2$.
(v) If $I$ is any ideal of $R$ such that $I \subseteq \text{nil}(R)$, then $I \in \{(0), \text{nil}(R), (\text{nil}(R))^2, (\text{nil}(R))^3\}$ and so $I \subseteq \{(0), Rz, Rz^2, Rz^3\}$ for any $z \in \text{nil}(R) \setminus (\text{nil}(R))^2$.

Proof. Since $AG(R)$ is complemented, we know from Lemma 3.3 that $(\text{nil}(R))^4 = (0)$.

(i) Let $i \in \{1, 2\}$. Thus $(\text{nil}(R))^i(\text{nil}(R))^3 = (0)$. By hypothesis, $(\text{nil}(R))^3 \neq (0)$. Hence $(\text{nil}(R))^i$ has a nonzero annihilator and so $(\text{nil}(R))^i \in A(R)^*$. As $AG(R)$ is complemented, there exists $I \in A(R)^*$ such that $(\text{nil}(R))^i \perp I$. Hence $(\text{nil}(R))^iI = (0)$ and so it follows that $(\text{nil}(R))^3I = (0)$. It is already noted that $(\text{nil}(R))^3(\text{nil}(R))^2 = (0)$.

Moreover, since $(\text{nil}(R))^3 \neq (0)$, we obtain from Lemma 3.2 that $(\text{nil}(R))^3 \neq (\text{nil}(R))^4$. As $(\text{nil}(R))^i \perp I$, the above arguments imply that $I = (\text{nil}(R))^3$. This proves that $(\text{nil}(R))^i \perp (\text{nil}(R))^3$ for $i = 1, 2$.

(ii) Let $x \in (\text{nil}(R))^3 \setminus \{0\}$. As $(\text{nil}(R))^4 = (0)$, it follows that $Rx(\text{nil}(R))^3 = (0)$ and $Rx(\text{nil}(R))^3 = (0)$. We know from (i) that $\text{nil}(R) \perp (\text{nil}(R))^3$.

Since $Rx \neq (0)$ and $Rx \neq \text{nil}(R)$, we obtain that $Rx = (\text{nil}(R))^3$.

(iii) Let $y \in (\text{nil}(R))^2 \setminus (\text{nil}(R))^3$. Since $(\text{nil}(R))^2 = (0)$, we obtain that $Ry(\text{nil}(R))^2 = (0)$ and $Ry(\text{nil}(R))^3 = (0)$. We know from (i) that $(\text{nil}(R))^2 \perp (\text{nil}(R))^3$. As $Ry \not\in \{(0), (\text{nil}(R))^3\}$, it follows that $Ry = (\text{nil}(R))^2$.

(iv) Let $y \in (\text{nil}(R))^2 \setminus (\text{nil}(R))^3$. Let $\phi : \text{nil}(R) \to (\text{nil}(R))^3$ be the mapping given by $\phi(a) = ay$ for any $a \in \text{nil}(R)$. It is clear that $\phi$ is a homomorphism of $R$-modules. We assert that $\phi$ is onto. Let $b \in (\text{nil}(R))^3$. Note that $(\text{nil}(R))^3 = \text{nil}(R)(\text{nil}(R))^2$. Since $(\text{nil}(R))^2 = Ry$ by (iii), we obtain that $(\text{nil}(R))^3 = (\text{nil}(R))^2Ry$. Hence $b = ay$ for some $a \in \text{nil}(R)$. Hence $\phi(a) = ay = b$. This shows that $\phi$ is onto. We know from the fundamental theorem of homomorphism of modules that $\text{nil}(R)/\text{ker}\phi \cong (\text{nil}(R))^3$ as $R$-modules.

We know from (ii) that for any nonzero $x \in (\text{nil}(R))^3$, $Rx = (\text{nil}(R))^3$. Hence it follows that for any $z \in \text{nil}(R)/\text{ker}\phi$, $\text{nil}(R)/\text{ker}\phi = R(z + \text{ker}\phi)$. We claim that $\text{ker}\phi = (\text{nil}(R))^2$. As $(\text{nil}(R))^4 = (0)$ and $y \in (\text{nil}(R))^2$, it is clear that $(\text{nil}(R))^2 \subseteq \text{ker}\phi$. Let $a \in \text{ker}\phi$. Hence $Ra(\text{nil}(R))^2 = (0)$ and $Ra(\text{nil}(R))^3 = (0)$. By (i), $(\text{nil}(R))^2 \perp (\text{nil}(R))^3$. Hence we obtain that $Ra \in \{(0), (\text{nil}(R))^2, (\text{nil}(R))^3\}$. This implies that $a \in (\text{nil}(R))^2$.

Hence $\text{ker}\phi \subseteq (\text{nil}(R))^2$ and so $\text{ker}\phi = (\text{nil}(R))^2$.

Let $z \in \text{nil}(R) \setminus (\text{nil}(R))^2$. Hence $z \not\in \text{ker}\phi$. As is remarked in the previous paragraph, $\text{nil}(R)/\text{ker}\phi = R(z + \text{ker}\phi)$. Hence it follows that $\text{nil}(R) = Rz + \text{ker}\phi = Rz + (\text{nil}(R))^2$. Therefore, we obtain that $\text{nil}(R) = Rz + (Rz + (\text{nil}(R))^2)^2$. Now it is clear that $\text{nil}(R) = Rz$ since $(\text{nil}(R))^4 = (0)$. Thus for any $z \in \text{nil}(R) \setminus (\text{nil}(R))^2$, $\text{nil}(R) = Rz$.

(v) Let $I$ be any nonzero ideal of $R$ such that $I \subseteq \text{nil}(R)$. Since $(\text{nil}(R))^4 = (0)$, there exists $i \in \{1, 2, 3\}$ such that $I \subseteq (\text{nil}(R))^i$ but $I \not\subseteq (\text{nil}(R))^{i+1}$. Let $a \in I \setminus (\text{nil}(R))^{i+1}$. Hence $a \in (\text{nil}(R))^i(\text{nil}(R))^{i+1}$. It follows from (ii), (iii), or (iv) that $(\text{nil}(R))^4 = Ra$.

As $a \in I$, we obtain that $(\text{nil}(R))^i \subseteq I$ and so $I = (\text{nil}(R))^i$. This shows that $\{(0), \text{nil}(R), (\text{nil}(R))^2, (\text{nil}(R))^3\}$ is the set of all ideals of $R$ which are contained in $\text{nil}(R)$. Let $z \in \text{nil}(R) \setminus (\text{nil}(R))^2$. Then from (iv), it follows that $\text{nil}(R) = Rz$ and so $\{(0), Rz, Rz^2, Rz^3\}$ is the set of all ideals of $R$ which are contained in $\text{nil}(R)$. □
We next consider rings $R$ such that $(\text{nil}(R))^3 = (0)$ but $(\text{nil}(R))^2 \neq (0)$ and determine some necessary conditions in order that $AG(R)$ is complemented.

**Proposition 3.5.** Let $R$ be a ring such that $(\text{nil}(R))^3 = (0)$ but $(\text{nil}(R))^2 \neq (0)$. If $AG(R)$ is complemented, then the following hold:

(i) $(\text{nil}(R)) \perp (\text{nil}(R))^2$.

(ii) $(\text{nil}(R))^2 = Rx$ for any $x \in (\text{nil}(R))^2 \setminus \{0\}$.

(iii) If $ab \neq 0$ for some $a, b \in \text{nil}(R)$, then $Ra \perp (\text{nil}(R))^2$ and $Rb \perp (\text{nil}(R))^2$.

(iv) If $z^2 \neq 0$ for some $z \in \text{nil}(R)$, then $\text{nil}(R) = Ra$ for any $a \in \text{nil}(R) \setminus (\text{nil}(R))^2$.

(v) If $z^2 = 0$ for each $z \in \text{nil}(R)$, then $\text{nil}(R)$ is not principal but there exist $a, b \in \text{nil}(R)$ such that $\text{nil}(R) = Ra + Rb$.

**Proof.** Since $(\text{nil}(R))^2 \neq (0)$ and $(\text{nil}(R))^3 = (0)$, it is clear that $\text{nil}(R) \in A(R)^*$.

(i) By hypothesis, $AG(R)$ is complemented. Hence there exists $I \in A(R)^*$ such that $\text{nil}(R) \perp I$. Hence $I(\text{nil}(R)) = (0)$. Therefore, $I((\text{nil}(R))^2) = (0)$. Since $(\text{nil}(R))^3 = (0)$, it follows that $(\text{nil}(R))^2 \cap \text{nil}(R) = (0)$. As $\text{nil}(R) \perp I$ and $(\text{nil}(R))^2 \notin \{(0), \text{nil}(R)\}$, we obtain that $(\text{nil}(R))^2 = I$. This proves that $\text{nil}(R) \perp (\text{nil}(R))^2$.

(ii) Let $x \in (\text{nil}(R))^2 \setminus \{0\}$. Since $(\text{nil}(R))^3 = (0)$, it follows that $Rx(\text{nil}(R)) = (0)$ and $Rx(\text{nil}(R))^2 = (0)$. We know from (i) that $\text{nil}(R) \perp (\text{nil}(R))^2$. As $Rx \notin \{(0), \text{nil}(R)\}$, we obtain that $Rx = (\text{nil}(R))^2$.

(iii) Let $a, b \in \text{nil}(R)$ be such that $ab \neq 0$. Since $AG(R)$ is complemented, there exists $I \in A(R)^*$ such that $Ra \perp I$. Hence $(Ra)I = (0)$ and so $(Ra)I = (0)$. As $(\text{nil}(R))^3 = (0)$, we obtain that $(Ra)(\text{nil}(R)) = (0)$. Thus the nonzero ideal $Ra$ is such that $(Ra)(\text{nil}(R)) = (0)$ and $(Ra)I = (0)$. Since $Ra \perp I$, $Ra \in \{I, Ra\}$. We know from Lemma 3.1 that $Ra \neq Rab$. Hence $Ra \perp Ra$. By (ii), $(\text{nil}(R))^2 = R$. This proves that $Ra \perp (\text{nil}(R))^2$. Similarly, it follows that $Rb \perp (\text{nil}(R))^2$.

(iv) Suppose that there exists $z \in \text{nil}(R)$ such that $z^2 = 0$. Consider the mapping $\psi : \text{nil}(R) \to (\text{nil}(R))^2$ given by $\psi(y) = yz$ for any $y \in \text{nil}(R)$. It is clear that $\psi$ is a homomorphism of $R$-modules. Since $(\text{nil}(R))^2 = Rz^2$ by (ii), it follows that $\psi$ is onto. Hence we obtain from the fundamental theorem of homomorphism of modules that $\text{nil}(R)/\ker \psi \cong (\text{nil}(R))^2$ as $R$-modules. We know from (ii) that for any nonzero $x \in (\text{nil}(R))^2$, $Rx = (\text{nil}(R))^2$. Thus for any $a \in \text{nil}(R) \setminus \ker \psi$, $\text{nil}(R)/\ker \psi = R(a + \ker \psi)$. We claim that $\ker \psi = (\text{nil}(R))^2$. Since $\ker \psi \subseteq \text{nil}(R)$ and $(\text{nil}(R))^3 = (0)$, it follows that $(\ker \psi)(\text{nil}(R)) = (0)$. As $\ker \psi \subseteq (0 : R z)$, we obtain that $Rz(\ker \psi) = (0)$. We know from (iii) that $Rz \perp (\text{nil}(R))^2$. Hence $\ker \psi \in \{(0), Rz, (\text{nil}(R))^2\}$. As $z^2 = 0$, $z^2 \in \ker \psi$. Hence $\ker \psi \neq (0)$. Since $z \notin \ker \psi$, it is clear that $\ker \psi \neq Rz$. Now it follows that $\ker \psi = (\text{nil}(R))^2$. Let $a \in \text{nil}(R) \setminus (\text{nil}(R))^2$. Hence as is already observed in this proof we obtain that $(\text{nil}(R))(\text{nil}(R))^2 = R(a + (\text{nil}(R))^2)$. This shows that $\text{nil}(R) = Ra + (\text{nil}(R))^2$. Hence $\text{nil}(R) = Ra + (Ra + (\text{nil}(R))^2)$. This implies that $\text{nil}(R) = Ra$ since $(\text{nil}(R))^3 = (0)$.

(v) Since $(\text{nil}(R))^2 \neq (0)$ and by assumption $z^2 = 0$ for each $z \in \text{nil}(R)$, it is clear that $\text{nil}(R)$ is not principal. As $(\text{nil}(R))^2 \neq (0)$, there exist $a, b \in \text{nil}(R)$ such that $ab \neq 0$. Consider the homomorphism of $R$-modules $f : \text{nil}(R) \to (\text{nil}(R))^2$ given by $f(z) = zb$ for any $z \in \text{nil}(R)$. By (ii), $(\text{nil}(R))^2 = Rab$. Hence it follows that $f$ is onto. We assert that $\ker f = Rb$. Since $\ker f \subseteq \text{nil}(R)$ and $(\text{nil}(R))^3 = (0)$, it follows that $(\ker f)(\text{nil}(R))^2 = (0)$. As $\ker f \subseteq (0 : R b)$, it follows that $Rb(\ker f) = (0)$. We know from (iii) that $Rb \perp (\text{nil}(R))^2$. Hence $\ker f \in \{(0), Rb, (\text{nil}(R))^2\}$. Since $b^2 = 0, b \in \text{nil}(R)$, we obtain that $\ker f = Rb$.
ker f and so ker f ≠ (0). We know from Lemma 3.1 that Rb ≠ Rab. Hence b ∉ (nil(R))^2. Hence we obtain that ker f ≠ (nil(R))^2. Therefore, it follows that ker f = Rb. Now f is a homomorphism of R-modules from nil(R) onto (nil(R))^2. Hence by the fundamental theorem of homomorphism of modules, it follows that nil(R)/ker f ≅ (nil(R))^2 as R-modules. We know from (ii) that (nil(R))^2 = Rx for any nonzero x ∈ (nil(R))^2 and as a ∉ ker f, it follows that nil(R)/ker f = R(a + ker f). This implies that nil(R) = Ra + ker f = Ra + Rb. □

4. ZERO-DIMENSIONAL QUASILOCAL RINGS R SUCH THAT AG(R) IS COMPLEMENTED

The aim of this section is to determine all zero-dimensional quasilocal rings R such that AG(R) is complemented. We begin with the following lemma.

Lemma 4.1. Let R be a ring such that dim R = 0 and R is quasilocal with M as its unique maximal ideal. Suppose that M^3 ≠ (0). Then the following statements are equivalent:

(i) AG(R) is complemented.
(ii) M^4 = (0) and R is a SPIR.

Proof. (i) ⇒ (ii) By hypothesis, it is clear that M is the only prime ideal of R. Hence M = nil(R). Since AG(R) is complemented by assumption, it follows from Lemma 3.3 that M^4 = (0). By hypothesis, M^3 ≠ (0). Now it follows from Proposition 3.4(iv) that M = Rm, M^2 ≠ (0), and M is principal, it follows from the proof of (iii) ⇒ (i) of [3, Proposition 8.8] that {M = Rm, M^2 = Rm^2, M^3 = Rm^3} is the set of all proper nonzero ideals of R. Hence we obtain that R is a SPIR.

(ii) ⇒ (i) Now R is a SPIR with M^4 = (0) but M^3 ≠ (0). Note that {M, M^2, M^3} is the set of all nonzero proper ideals of R. Now it is clear that AG(R) is a graph on three vertices {M, M^2, M^3}, M ⊥ M^2, and M^2 ⊥ M^3. This proves that AG(R) is complemented. □

We next have the following lemma.

Lemma 4.2. Let R be a quasilocal ring with M as its unique maximal ideal. Suppose that M^3 = (0) but M^2 ≠ (0). Then the following statements are equivalent:

(i) AG(R) is complemented.
(ii) If z^2 ≠ 0 for some z ∈ M, then M is principal. If z^2 = 0 for each z ∈ M, then M is not principal but there exist a, b ∈ M such that M = Ra + Rb.
(iii) I ⊥ M^2 for each nonzero proper ideal I of R with I ≠ M^2.

Proof. (i) ⇒ (ii) It is clear from the hypothesis that M is the only prime ideal of R. Hence M = nil(R). If z^2 ≠ 0 for some z ∈ M, then it follows from Proposition 3.5(iv) that M is principal. If z^2 = (0) for each z ∈ M, then it follows from Proposition 3.5(v) that M is not principal but there exist a, b ∈ M such that M = Ra + Rb.

(ii) ⇒ (iii) Suppose that z^2 ≠ 0 for some z ∈ M. Then M is principal. As M^3 = (0), it follows from the proof of (iii) ⇒ (i) of [3, Proposition 8.8] that M and M^2 are the only proper nonzero ideals of R. Hence in this case, AG(R) is a graph with vertex set {M, M^2} and M ⊥ M^2.

Suppose that z^2 = 0 for each z ∈ M. Then M is not principal but there exist a, b ∈ M such that M = Ra + Rb. In such a case, M^2 = Rab. Let x ∈ M^2, x ≠ 0. Then x = rab
for some \( r \in R \). As \( M^3 = (0) \), it follows that \( r \) is a unit in \( R \) and so \( M^2 = Rab = Rx \). Since \( M^3 = (0) \) but \( M^2 \neq (0) \), it is clear that each nonzero proper ideal is in \( A(R)^* \). Let \( I \) be any nonzero proper ideal of \( R \). If \( I \subseteq M^2 \), then as \( M^2 = Rx \) for any \( x \in I \setminus \{0\} \), it follows that \( I = M^2 \). Suppose that \( I \not\subseteq M^2 \). Let \( z \in I \setminus M^2 \). Since \( M \) is not principal but is generated by two elements, it is clear that \( \dim_{R/M}(M/M^2) = 2 \). As \( z^2 = w^2 = 0 \), it follows that \( M^2 = M^2 + z^2M^2 = \{z + M^2, w + M^2\} \) forms a basis of \( M/M^2 \) as a vector space over \( R/M \). In such a case, it follows that \( M = Rz + Rw + M^2 \). This implies that \( M = Rz + Rw + M^2 \) since \( M^3 = (0) \). As \( z^2 = w^2 = 0 \), it follows that \( M^2 = Rz + M^2 \). From the above discussion it is clear that if \( z^2 = 0 \) for each \( z \in M \) and if \( I \) is any nonzero proper ideal of \( R \), then either \( I \subseteq \{M, M^2\} \) or \( I = Rz \) for some \( z \in I \setminus M^2 \). Since \( M^3 = (0) \), it follows that \( IM^2 = (0) \) for each proper ideal \( I \) of \( R \). Let \( I \in A(R)^* \) be such that \( I \neq M^2 \). We verify that \( I \cap M^2 \). Since \( IM^2 = (0) \), \( I \) is adjacent to \( M^2 \). Let \( J \in A(R)^* \) be such that \( J \not\subseteq \{I, M^2\} \). Suppose that \( J = M \). Then \( I \neq M \) and so \( I = Rz \) for some \( z \in I \setminus M^2 \). Moreover, it is noted in the previous paragraph that \( M^2 = Rzw \) for some \( w \in M \). Hence we obtain that \( M^2 \subseteq IM = IJ \). Similarly, if \( I = M \), then \( J \neq M \) and so \( J = Rz' \) for some \( z' \in J \setminus M^2 \) and \( M^2 = Rz'w' \) for some \( w' \in M \). Therefore, \( M^2 = Rz'w' \subseteq MJ = IJ \). Suppose that \( I \neq M \) and \( J \neq M \). Then there exist \( z \in I \setminus M^2 \) and \( z' \in J \setminus M^2 \) such that \( I = Rz \) and \( J = Rz' \). We claim that \( I + J = M \). Indeed, if \( I + J \neq M \), then \( I + J = Ry \) for some \( y \in I + J \) with \( y \not\in M^2 \). Now as \( z, z' \in M \setminus M^2 \), \( I = Rz \subseteq Ry \), and \( J = Rz' \subseteq Ry \), we obtain that \( I = Rz = Ry = Rz' = J \). But this contradicts the assumption that \( I \neq J \). Hence \( Rz + Rz' = I + J = M \). Therefore, \( M^2 = Rzz' \subseteq IJ \). This shows that if \( J \in A(R)^* \setminus \{M^2, I\} \), then \( M^2 \subseteq IJ \) and so \( IJ \neq 0 \). This proves that \( I \cap M^2 \) for each \( I \in A(R)^* \) with \( I \neq M^2 \).

\( \text{(iii)} \implies \text{(i)} \). Since \( M^3 = (0) \) but \( M^2 \neq (0) \), it is clear that \( M \neq M^2 \). Hence \( R \) admits at least one nonzero proper ideal which is different from \( M^2 \). Note that from the preceding observation \( \text{(iii)} \implies \text{(i)} \) follows immediately. \( \Box \)

We next have the following lemma. We denote the characteristic of a ring \( R \) by \( char(R) \).

**Lemma 4.3.** Let \( R \) be a quasilocal ring with \( M \) as its unique maximal ideal such that \( M^3 = (0) \) but \( M^2 \neq (0) \). If \( AG(R) \) is complemented and \( M \) is not principal, then \( char(R/M) = 2 \) and moreover, \( char(R) \in \{2, 4\} \).

**Proof.** Assume that \( AG(R) \) is complemented and \( M \) is not principal. It follows from the proof of \( \text{(i)} \implies \text{(ii)} \) of **Lemma 4.2** that \( z^2 = 0 \) for each \( z \in M \) and \( \dim_{R/M}(M/M^2) = 2 \). Moreover, it is noted in the proof of \( \text{(ii)} \implies \text{(iii)} \) of **Lemma 4.2** that for any nonzero \( x \in M^2 \), \( M^2 = Rz \) and \( I \cap M^2 \) for each nonzero proper ideal \( I \) of \( R \) with \( I \neq M^2 \). We first verify that \( char(R/M) = 2 \). Suppose that \( char(R/M) \neq 2 \). Then \( 2 \not\in M \) and so \( 2 \) is a unit in \( R \). Let \( a, b \in M \) be such that \( \{a + M^2, b + M^2\} \) forms a basis of \( M/M^2 \) as a vector space over \( R/M \). Consider the ideals \( I_1 = R(a + b) \) and \( I_2 = R(a - b) \) of \( R \). From the choice of \( a, b \), it is clear that \( I_1 \) and \( I_2 \) are nonzero proper ideals of \( R \) with \( I_i \neq M^2 \) for each \( i \in \{1, 2\} \). Note that \( I_1 \neq I_2 \). For if \( I_1 = I_2 \), then \( 2b = (a + b) - (a - b) \in I_1 \). This implies that \( b \in I_1 \) since \( 2 \) is a unit in \( R \). Hence \( b = r(a + b) \) for some \( r \in R \). Therefore, \( ra + (r - 1)b = 0 \). This implies by the choice of \( a, b \) that \( r \in M \) and \( 1 - r \in M \). Hence \( 1 = r + 1 - r \in M \). This
Proof. Let \( I = x^2 = b^2 = 0 \), it is clear that \( I_1I_2 = R(a^2 - b^2) = (0) \). Moreover, as \( M^3 = (0) \), it is clear that \( I_2M^2 = (0) \). Thus \( I_1I_2 = I_2M^2 = (0) \). This is impossible since \( I_1 \perp M^2 \). Hence \( \text{char}(R/M) = 2 \). Now \( 2 \in M \) and as \( z^2 = 0 \) for each \( z \in M \), it follows that \( 4 = 0 \) in \( R \). Therefore, \( \text{char}(R) \in \{2, 4\} \). \( \square \)

We next provide some examples to illustrate Lemma 4.2.

Example 4.4. Let \( K \) be a field with \( \text{char}(K) = 2 \). Let \( T = K[x, y] \) be the polynomial ring in two variables over \( K \). Let \( I = x^2T + y^2T \) and \( R = T/I \). Then \( AG(R) \) is complemented.

Proof. Let \( N = xT + yT \). Note that \( R = T/I \) is a local ring with \( M = N/I \) as its unique maximal ideal. For an element \( t \in T \), we denote \( t + I \) by \( \bar{t} \). Observe that \( M = \bar{R} + \bar{y}R, z^2 = \bar{0} \) for each \( z \in M \), \( M^2 = \bar{R} \neq (0) \), and \( M^3 = (0) \). Now it follows, from (ii) \( \Rightarrow \) (iii) of Lemma 4.2, that \( J \perp M^2 \) for each nonzero proper ideal \( J \) of \( R \) with \( J \neq M^2 \). This shows that \( AG(R) \) is complemented. \( \square \)

For any \( n \geq 2 \), we denote the ring of integers modulo \( n \) by \( \mathbb{Z}_n \).

Example 4.5. Let \( T = \mathbb{Z}_4[x, y] \) be the polynomial ring in two variables over \( \mathbb{Z}_4 \). Let \( I = x^2T + (xy - 2)T + y^2T \) and \( R = T/I \). Then \( AG(R) \) is complemented.

Proof. Let \( N = 2T + xT + yT \). Observe that \( R = T/I \) is a local ring with \( M = N/I \) as its unique maximal ideal. For any \( t \in T \), we denote \( t + I \) by \( \bar{t} \). Note that \( M = \bar{R} + \bar{y}R, z^2 = \bar{0} \) for each \( z \in M \), \( M^2 = \bar{R} \), and \( M^3 = (0) \). Now it follows, from (ii) \( \Rightarrow \) (iii) of Lemma 4.2, that \( J \perp M^2 \) for each nonzero proper ideal \( J \) of \( R \) with \( J \neq M^2 \) and hence we obtain that \( AG(R) \) is complemented. \( \square \)

Example 4.6. Let \( T = \mathbb{Z}_4[x] \) be the polynomial ring in one variable over \( \mathbb{Z}_4 \). Let \( I = x^2T \). Let \( R = T/I \). Then \( AG(R) \) is complemented.

Proof. Let \( N = 2T + xT \). Note that the ring \( R = T/I \) is local with \( M = N/I \) as its unique maximal ideal. For any \( t \in T \), let us denote \( t + I \) by \( \bar{t} \). Observe that \( M = \bar{R} + \bar{xR}, z^2 = \bar{0} \) for each \( z \in M \), \( M^2 = \bar{R} \neq (0) \), and \( M^3 = (0) \). It now follows, from (ii) \( \Rightarrow \) (iii) of Lemma 4.2, that \( J \perp M^2 \) for each nonzero proper ideal \( J \) of \( R \) with \( J \neq M^2 \) and therefore, we obtain that \( AG(R) \) is complemented. \( \square \)

We make use of the following remark in the proof of Theorem 4.8.

Remark 4.7. Let \( R \) be a quasilocal ring with \( M \) as its unique maximal ideal. If \( M^2 = (0) \) but \( M \neq (0) \), then \( AG(R) \) is not complemented and indeed one of the following holds:

(i) \( M \) is the only element of \( A(R)^* \) and hence \( AG(R) \) is a graph on a single vertex.

(ii) \( A(R)^* \) contains at least three elements and \( AG(R) \) is a complete graph.

Proof. Suppose that \( M \) is principal. As \( M^2 = (0) \), it is clear that \( M \) is the only nonzero proper ideal of \( R \). Since the nonzero ideal \( M \) annihilates \( M \), it follows that \( M \in A(R)^* \). Hence (i) holds. Note that as \( AG(R) \) is a graph on a single vertex, it is not complemented.

Suppose that \( M \) is not principal. Since \( M^2 = (0) \), it is clear that \( M \) annihilates any proper nonzero ideal of \( R \) and hence \( A(R)^* \) is the set of all proper nonzero ideals of \( R \) and moreover,
$M$ can be regarded as a vector space over the field $R/M$. As $M$ is not principal, it follows that \( \dim_{R/M} M \geq 2 \). Let \( \{x, y\} \subseteq M \) be such that \( \{x, y\} \) is linearly independent over $R/M$. Note that \( Rx, Ry, R(x+y) \) are distinct elements of $A(R)^*$. Since $M^2 = (0)$, we obtain that $IJ = (0)$ for any $I, J \in A(R)^*$. Hence it follows that $AG(R)$ is a complete graph with at least three vertices and hence it is not complemented. \( \square \)

The following theorem characterizes when $AG(R)$ is complemented, where $R$ is any zero-dimensional quasilocal ring with $AG(R)$ admitting at least two vertices.

**Theorem 4.8.** Let $R$ be a zero-dimensional quasilocal ring with $M$ as its unique maximal ideal. Suppose that $AG(R)$ admits at least two vertices. Then $AG(R)$ is complemented if and only if (a) and (b) hold and moreover, either (c) or (d) holds, where (a), (b), (c), and (d) are given below:

- (a) $M^2 \neq (0)$.
- (b) $M^4 = (0)$.
- (c) $R$ is a SPIR.
- (d) $z^2 = 0$ for each $z \in M$, $M$ is not principal but there exist $a, b \in M$ such that $M = Ra + Rb$.

**Proof.** We are assuming that $\dim R = 0$ and $R$ is quasilocal with $M$ as its unique maximal ideal. Hence we obtain that $nil(R) = M$.

Assume that $AG(R)$ admits at least two vertices and is complemented. It follows from Remark 4.7 that $M^2 \neq (0)$. We obtain from Lemma 3.3 that $M^4 = (0)$. If $M^3 \neq (0)$, then it follows from (i) $\Rightarrow$ (ii) of Lemma 4.1 that $R$ is a SPIR. Suppose that $M^3 = (0)$. If $M$ is principal, then it follows from the proof of (iii) $\Rightarrow$ (i) of [3, Proposition 8.8] that $R$ is a principal ideal ring and hence $R$ is a SPIR. If $M$ is not principal, then we obtain from (i) $\Rightarrow$ (ii) of Lemma 4.2 that $z^2 = 0$ for each $z \in M$ and there exist $a, b \in M$ such that $M = Ra + Rb$. Thus if $AG(R)$ is complemented, then (a) and (b) hold. Moreover, either (c) or (d) holds.

Conversely, assume that (a) and (b) hold and moreover, either (c) or (d) holds. Suppose that (c) holds. If $M^3 \neq (0)$, then it follows from (ii) $\Rightarrow$ (i) of Lemma 4.1 that $AG(R)$ is complemented. If $M^3 = (0)$, then $AG(R)$ is a graph with vertex set $\{M, M^2\}$ and $M \perp M^2$. Hence $AG(R)$ is complemented. Suppose that (d) holds. Then it follows from (ii) $\Rightarrow$ (i) of Lemma 4.2 that $AG(R)$ is complemented. \( \square \)

As an immediate consequence of Theorem 4.8, we have the following result.

**Corollary 4.9.** Let $(R, M)$ be a finite local ring with $AG(R)$ admitting at least two vertices. Then $AG(R)$ is complemented if and only if (a), (b) of Theorem 4.8 hold and either $R$ is a finite SPIR or (d) of Theorem 4.8 hold.

5. **ZERO-DIMENSIONAL QUASISEMILOCAL RINGS $R$ SUCH THAT $AG(R)$ IS COMPLEMENTED**

The aim of this section is to determine zero-dimensional quasisemilocal rings $R$ such that $AG(R)$ is complemented. We begin with the following lemma.
Lemma 5.1. Let \( R \) be a quasilocal ring with \( \dim R = 0 \). Let \( \{P_1, \ldots, P_n\} \) be the set of all maximal ideals of \( R \). If \( AG(R) \) is complemented, then there exist quasilocal rings \((R_1, M_1), \ldots, (R_n, M_n)\) with \( M_i^4 = (0) \) for each \( i \in \{1, \ldots, n\} \) and \( R \cong R_1 \times \cdots \times R_n \) as rings.

Proof. Since \( \dim R = 0 \) and \( R \) is quasilocal with \( \{P_1, \ldots, P_n\} \) as its set of all maximal ideals, it is clear that \( \{P_1, \ldots, P_n\} \) is the set of all prime ideals of \( R \). Hence we obtain that \( \text{nil}(R) = \bigcap_{i=1}^n P_i \). Moreover, as \( P_i + P_j = R \) for all distinct \( i, j \in \{1, \ldots, n\} \), it follows from \([3, \text{Proposition 1.10(i)}]\) that \( \text{nil}(R) = \bigcap_{i=1}^n P_i = \prod_{i=1}^n P_i \).

Suppose that \( AG(R) \) is complemented. Then we obtain from \( \text{Lemma 3.3} \) that \((\text{nil}(R))^4 = (0)\). Hence we obtain that \( \prod_{i=1}^n P_i^4 = (0) \). Since \( P_i^4 + P_j^4 = R \) for all distinct \( i, j \in \{1, \ldots, n\} \), it follows from the Chinese remainder theorem \([3, \text{Proposition 1.10(ii) and (iii)}]\) that the mapping \( f : R \rightarrow R/P_i^4 \times \cdots \times R/P_n^4 \) given by \( f(r) = (r + P_i^4, \ldots, r + P_n^4) \) is an isomorphism of rings. Let \( i \in \{1, \ldots, n\} \) and \( R_i = R/P_i^4 \). It is clear that \( R_i \) is quasilocal with \( M_i = P_i^4 \) as its unique maximal ideal and \( R \cong R_1 \times \cdots \times R_n \) as rings. Moreover, it is obvious that \( M_i^4 \) is the zero ideal of \( R_i \) for each \( i \in \{1, \ldots, n\} \). □

In view of \( \text{Lemma 5.1} \), in the rest of this section, we assume that \( R = R_1 \times \cdots \times R_n \), where \( R_i \) is a quasilocal ring with unique maximal ideal \( M_i \) such that \( M_i^4 = (0) \) for each \( i \in \{1, \ldots, n\} \). We proceed to determine when \( AG(R) \) is complemented. As \( \text{Theorem 4.8} \) determines when \( AG(R) \) is complemented in the case where \( R \) is a zero-dimensional quasilocal ring, we assume that \( R \) is not quasilocal. Hence \( n \geq 2 \).

Lemma 5.2. Let \( R = R_1 \times R_2 \times \cdots \times R_n \) (\( n \geq 2 \)), where \( (R_i, M_i) \) is a quasilocal ring with \( M_i^4 = (0) \) for each \( i \in \{1, 2, \ldots, n\} \). If \( AG(R) \) is complemented, then \( M_i^2 = (0) \) and \( M_i \) is principal for \( i \in \{1, 2, \ldots, n\} \); in the case where \( M_i^2 \neq (0) \), \( M_i = R_i x_i \) for any nonzero element \( x_i \) of \( M_i \). Moreover, \( R_i \) has at most one proper nonzero ideal for each \( i \in \{1, 2, \ldots, n\} \).

Proof. Assume that \( AG(R) \) is complemented. Suppose that \( M_i^2 \neq (0) \) for some \( i \in \{1, 2, \ldots, n\} \). Consider the ideal \( I = I_1 \times I_2 \times \cdots \times I_n \) of \( R \) defined by \( I_i = M_i^2 \) and \( I_j = R_j \) for all \( j \in \{1, 2, \ldots, n\} \setminus \{i\} \). Since \( M_i^4 = (0) \), the ideal \( J_i = J_1 \times J_2 \cdots \times J_n \) of \( R \) given by \( J_i = M_i^2 \) and \( J_j = (0) \) for all \( j \in \{1, 2, \ldots, n\} \setminus \{i\} \) is such that \( IJ = (0) \times (0) \cdots \times (0) \). Hence \( I \in A(R)^\times \). As \( AG(R) \) is complemented, there exists \( K \in A(R)^\times \) such that \( I \perp K \). Now it follows from \( IK = (0) \times (0) \cdots \times (0) \) and \( I_j = R_j \) for all \( j \in \{1, 2, \ldots, n\} \setminus \{i\} \) that \( K_j = (0) \) for all \( j \in \{1, 2, \ldots, n\} \setminus \{i\} \). Note that \( K_i M_i^2 = (0) \). Observe that \( JK = IK = (0) \times (0) \cdots \times (0) \). Since \( I \perp K \) and \( J \not\in \{0 \times (0) \cdots \times (0), I\} \), it follows that \( J = K \). Hence we obtain that \( I \perp J \). We next claim that \( M_i^3 = (0) \). Indeed, if \( M_i^3 \neq (0) \), then the ideal \( A = A_1 \times A_2 \cdots \times A_n \) of \( R \) given by \( A_i = M_i^3 \) and \( A_j = (0) \) for all \( j \in \{1, 2, \ldots, n\} \setminus \{i\} \) is such that \( AI = AJ = (0) \times (0) \cdots \times (0) \) and \( A \not\in \{(0) \times (0) \cdots \times (0), I, J\} \). This is impossible since \( I \perp J \). Thus \( M_i^3 = (0) \). Note that the ideal \( B = B_1 \times B_2 \cdots \times B_n \) of \( R \) is such that \( IB = JB = (0) \times (0) \cdots \times (0) \) and \( B \not\in \{(0) \times (0) \cdots \times (0), I, J\} \). This cannot happen since \( I \perp J \). Hence we obtain that \( M_i^2 = (0) \) for each \( i \in \{1, 2, \ldots, n\} \).

Let \( i \in \{1, 2, \ldots, n\} \). We next show that \( M_i \) is a principal ideal of \( R_i \). If \( M_i = (0) \), then it is clear that \( M_i \) is principal. Suppose that \( M_i \neq (0) \). We show that \( M_i = R_i x_i \) for any
nonzero \( x_i \in M_i \). Consider the ideal \( I = I_1 \times I_2 \times \cdots \times I_n \) defined by \( I_i = R_i x_i \) and \( I_j = R_j \) for all \( j \in \{1, 2, \ldots, n\} \setminus \{i\} \). Since \( M_i^2 = (0) \), the ideal \( J = J_1 \times J_2 \times \cdots \times J_n \) of \( R \) given by \( J_i = M_i \) and \( J_j = (0) \) for all \( j \in \{1, 2, \ldots, n\} \setminus \{i\} \) is such that \( IJ = (0) \times (0) \times \cdots \times (0) \). Hence \( I \subset A(R)^* \). Since \( AG(R) \) is complemented, there exists \( K \subset A(R)^* \) such that \( I \perp K \). From \( IK = (0) \times (0) \times \cdots \times (0) \) and \( I_j = R_j \) for all \( j \in \{1, 2, \ldots, n\} \setminus \{i\} \), it is clear that \( K_i = (0) \). Note that \( K_i R_i x_i = (0) \). Hence \( K_i \subseteq M_i \). As \( M_i^2 = (0) \), it is clear that \( JK = (0) \times (0) \times \cdots \times (0) \). Thus \( IJ = JK = (0) \times (0) \times \cdots \times (0) \). Since \( I \perp K \) and \( J \not\subset \{(0) \times (0) \times \cdots \times (0) \} \), it follows that \( J = K \). Thus \( I \perp J \). The ideal \( A = A_1 \times A_2 \times \cdots \times A_n \) of \( R \) given by \( A_j = (0) \) for all \( j \in \{1, 2, \ldots, n\} \setminus \{i\} \) and \( A_i = R_i x_i \) is such that \( IA = JA = (0) \times (0) \times \cdots \times (0) \). Since \( I \perp J \) and \( A \not\subset \{(0) \times (0) \times \cdots \times (0) \} \), it follows that \( A = J \). Hence we obtain that \( M_i = R_i x_i \).

Let \( i \in \{1, 2, \ldots, n\} \). If \( M_i = (0) \), then \( R_i \) is a field and it has no proper nonzero ideal. If \( M_i \neq (0) \), then it is noted in the previous paragraph that \( M_i = R_i x_i \) for each nonzero \( x_i \in M_i \). Hence we obtain that \( M_i \) is the only proper nonzero ideal of \( R_i \). This proves that \( R_i \) has at most one nonzero proper ideal. □

With \( R \) as in the statement of Lemma 5.2, the following lemma provides another necessary condition in order that \( AG(R) \) is complemented.

**Lemma 5.3.** Let \( n \geq 2 \) and let \( R = R_1 \times R_2 \times \cdots \times R_n \), where \( (R_i, M_i) \) is a quasilocal ring with \( M_i^4 = (0) \) for each \( i \in \{1, 2, \ldots, n\} \). If \( AG(R) \) is complemented, then \( R_j \) is a field for some \( j \in \{1, 2, \ldots, n\} \).

**Proof.** Suppose that \( AG(R) \) is complemented and \( R_i \) is not a field for each \( i \in \{1, 2, \ldots, n\} \). Hence \( M_i \neq (0) \) for each \( i \in \{1, 2, \ldots, n\} \). Let \( I = M_1 \times M_2 \times \cdots \times M_n \). We know from Lemma 5.2 that \( M_i^2 = (0) \) for each \( i \in \{1, 2, \ldots, n\} \). Hence it follows that \( I \subset A(R)^* \). Since \( AG(R) \) is complemented, there exists an ideal \( J = J_1 \times J_2 \times \cdots \times J_n \) of \( R \) such that \( I \perp J \). From \( IJ = (0) \times (0) \times \cdots \times (0) \), it follows that \( I_i J_i = (0) \) for any \( i \in \{1, 2, \ldots, n\} \). Hence \( J_i \subseteq M_i \) and moreover, it follows from Lemma 5.2 that \( J_i \in \{(0), M_i\} \) for each \( i \in \{1, 2, \ldots, n\} \). Since \( I \neq J \) and \( J \neq (0) \times (0) \times \cdots \times (0) \), it is clear that there exist distinct \( r, s \in \{1, 2, \ldots, n\} \) such that \( J_r = M_r \) and \( J_s = (0) \). Consider the ideal \( K = K_1 \times K_2 \times \cdots \times K_n \) of \( R \) given by \( K_i = (0) \) for all \( i \in \{1, 2, \ldots, n\} \setminus \{s\} \) and \( K_s = M_s \). Note that the ideal \( K \) is such that \( K \not\subset \{(0) \times (0) \times \cdots \times (0) \} \) and \( IJ = JK = (0) \times (0) \times \cdots \times (0) \). This is impossible as \( I \perp J \). Thus if \( AG(R) \) is complemented, then \( R_j \) is a field for some \( j \in \{1, 2, \ldots, n\} \). □

Let \( R \) be as in the statement of Lemma 5.2. The following lemma provides another necessary condition in order that \( AG(R) \) is complemented.

**Lemma 5.4.** Let \( n \geq 2 \) and let \( R = R_1 \times R_2 \times \cdots \times R_n \), where \( (R_i, M_i) \) is a quasilocal ring with \( M_i^4 = (0) \) for each \( i \in \{1, 2, \ldots, n\} \). If \( AG(R) \) is complemented, then there exists at most one \( i \in \{1, 2, \ldots, n\} \) such that \( R_i \) is not a field.

**Proof.** Suppose that \( AG(R) \) is complemented and there exist distinct \( s, t \in \{1, 2, \ldots, n\} \) such that \( R_s \) and \( R_t \) are not fields. Hence \( M_s \neq (0) \) and \( M_t \neq (0) \). We know from Lemma 5.3 that there exists \( j \in \{1, 2, \ldots, n\} \) such that \( R_j \) is a field. It is clear that \( j \not\in \{s, t\} \). Consider the ideal \( I = I_1 \times I_2 \times \cdots \times I_n \) of \( R \) given by \( I_i = R_i \) for all \( i \in \{1, 2, \ldots, n\} \setminus \{s, t\} \),
Let $R$ be a zero-dimensional quasisemilocal ring admitting more than one maximal ideal. The following theorem determines necessary and sufficient conditions in order that $AG(R)$ is complemented.

**Theorem 5.6.** Let $R$ be a quasisemilocal ring which is not quasilocal and let $\dim R = 0$. Then the following statements are equivalent:

(i) $AG(R)$ is complemented.

(ii) Either $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings, where $n \geq 2$ and $F_i$ is a field for all $i \in \{1, 2, \ldots, n\}$, or $R \cong S \times F$ as rings, where $(S, M)$ is a SPIR with $M \neq (0)$ but $M^2 = (0)$ and $F$ is a field.
Proof. (i) ⇒ (ii) Let \( n \) be the number of maximal ideals of \( R \). Since \( R \) is not quasilocal, it follows that \( n \geq 2 \). We know from Lemma 5.1 that \( R \cong R_1 \times R_2 \times \cdots \times R_n \) as rings, where \((R_i, M_i)\) is a quasilocal ring with \( M_i = (0) \) for each \( i \in \{1, 2, \ldots, n\} \). If \( M_i = (0) \) for each \( i \in \{1, 2, \ldots, n\} \), then \( R_i \) is a field for each \( i \) and hence with \( F_i = R_i \), we obtain that \( R \cong F_1 \times F_2 \times \cdots \times F_n \) as rings. Suppose that \( R_i \) is not a field for at least one \( i \in \{1, 2, \ldots, n\} \). We know from Lemma 5.4 that such an \( i \) is necessarily unique. Now \( R \) is not reduced. Hence we obtain from Lemma 5.5 that \( n = 2 \). Thus \( R \cong R_1 \times R_2 \) as rings, where we may assume that \( R_1 \) is not a field and \( R_2 \) is a field. We know from Lemma 5.2 that \( M_2^2 = (0) \) and \( M_1 = R_1 x_1 \) for any nonzero \( x_1 \in M_1 \). Hence \( M_1 \) is the only nonzero proper ideal of \( R_1 \). Thus \((R_1, M_1)\) is a SPIR with \( M_1 \neq (0) \) but \( M_1^2 = (0) \). Hence with \( S = R_1, M = M_1, \) and \( F = R_2 \), we obtain that \( R \cong S \times F \) as rings, where \((S, M)\) is a SPIR with \( M \neq (0) \) but \( M^2 = (0) \) and \( F \) is a field.

(ii) ⇒ (i) Suppose that \( R \cong F_1 \times F_2 \times \cdots \times F_n \) as rings with \( n \geq 2 \) and \( F_i \) is a field for all \( i \in \{1, 2, \ldots, n\} \). Note that \( R \) is reduced and hence we obtain from Proposition 2.2 that \( AG(R) \) is complemented. Indeed \( AG(R) \) is uniquely complemented.

Suppose that \( R \cong S \times F \) as rings, where \((S, M)\) is a SPIR with \( M \neq (0) \) but \( M^2 = (0) \) and \( F \) is a field. Let \( T = S \times F \). Note that \( M \) is the only nonzero proper ideal of \( S \). Hence \( A(T)^* = \{(0) \times F, M \times (0), M \times F, S \times (0)\} \). It is easy to verify that \( (0) \times F \perp M \times (0), M \times F \perp M \times (0), \) and \( S \times (0) \perp (0) \times F \). This shows that \( AG(T) \) is complemented. As \( R \cong T \) as rings, we obtain that \( AG(R) \) is complemented. Observe that \( (0) \times F \perp M \times (0) \) and \( (0) \times F \perp S \times (0) \). As \( M \times F \) is adjacent to \( M \times (0) \) but \( M \times F \) is not adjacent to \( S \times (0) \), it follows that \( AG(T) \) is not uniquely complemented. Hence we obtain that \( AG(R) \) is not uniquely complemented. □

The following corollary determines when \( AG(R) \) is complemented, where \( R \) is a finite semilocal ring which is not local.

Corollary 5.7. Let \( R \) be a finite semilocal ring which is not local. The following statements are equivalent:

(i) \( AG(R) \) is complemented.

(ii) Either \( R \cong F_1 \times F_2 \times \cdots \times F_n \) as rings for some \( n \geq 2 \), where \( F_i \) is a finite field for \( i = 1, 2, \ldots, n \), or \( R \cong S \times F \) as rings, where \((S, M)\) is a finite SPIR with \( M \neq (0) \) but \( M^2 = (0) \) and \( F \) is a finite field.

Proof. The proof of this corollary follows immediately from Theorem 5.6. Note that the finiteness assertion of \( F_i \) for \( i = 1, 2, \ldots, n \), \( S \), and \( F \) in (ii) follow since \( R \) is a finite ring. □

6. Rings \( R \) With Only Finitely Many Maximal N-primes of \((0)\) Such That \( AG(R) \) is Complemented

Let \( R \) be a commutative ring with identity which is not reduced (that is, \( nil(R) \neq (0) \)). Suppose that \( R \) admits only a finite number of maximal N-primes of \((0)\). Let \( \{P_1, \ldots, P_n\} \) be the set of all maximal N-primes of \((0)\) in \( R \). Moreover, we assume that \( \cap_{i=1}^{n} P_i = nil(R) \) and \( A(R)^* \) contains at least two elements. The purpose of this section is to determine necessary and sufficient conditions in order that \( AG(R) \) is complemented. We begin with the following lemma.
Lemma 6.1. Let $R$ be a ring which is not reduced. Suppose that $R$ admits only one maximal $N$-prime of $(0)$. Let $P$ be the unique maximal $N$-prime of $(0)$ in $R$. If $AG(R)$ is complemented, then $P$ is a maximal ideal of $R$. Moreover, if $P = \text{nil}(R)$, then $P$ is the only prime ideal of $R$.

Proof. Suppose that $AG(R)$ is complemented. We prove that $P$ is a maximal ideal of $R$. Let $M$ be a maximal ideal of $R$ such that $P \subseteq M$. We assert that $P = M$. Suppose that $P \neq M$. Let $a \in M \setminus P$. Since $P$ is the only maximal $N$-prime of $(0)$ in $R$, it follows that $P = Z(R)$. Thus $a \notin Z(R)$. As $\text{nil}(R) \neq (0)$, there exists $x \in \text{nil}(R) \setminus \{0\}$ such that $x^2 = 0$. Let $J = Rax$. Note that $J \in A(R)^*$. Hence there exists $K \in A(R)^*$ such that $J \perp K$. Let $A = Rax$. From $(Rx)K = (0)$, it is clear that $AK = (Rax)K = (0)$. Moreover, as $x^2 = 0$, it follows that $AJ = (Rax)(Rx) = (0)$. Thus the ideal $A$ of $R$ satisfies $AJ = AK = (0)$. Since $J \perp K$, it follows that $A \subseteq \{(0), J, K\}$. Since $x \neq 0$ and $a \notin Z(R)$, $A = Rax \neq (0)$. Observe that $A \neq J$. Indeed, for any $y \in M$, $J = Rax \neq Ryx$. For if $Rx = Ryx$, then $x = ryx$ for some $r \in R$. This implies that $x(1 - ry) = 0$ and so $1 - ry \in Z(R) = P \subseteq M$. As $y \in M$, we obtain that $1 = 1 - ry + ry \in M$. This is impossible since $M \neq R$. This shows that $A \neq J$. Therefore, $A = K$. Hence we obtain that $J \perp A$. Let $B = Ra^2x$. Since $x^2 = 0$, it is clear that $BJ = (Ra^2x)(Rx) = (0)$ and $BA = (Ra^2x)(Rax) = (0)$. As $a \notin Z(R)$ but $a \in M$, it is clear that $Rx \neq Ra^2x$ and $Rax \neq Ra^2x$. Thus the ideal $B = Ra^2x \in A(R)^*$ is adjacent to both $J$ and $A$. This is impossible since $J \perp A$. Therefore, $P = M$ and this proves that $P$ is a maximal ideal of $R$.

Suppose that $P = \text{nil}(R)$. We next verify that $P$ is the only prime ideal of $R$. Let $Q$ be any prime ideal of $R$. Then $Q \supseteq \text{nil}(R) = P$ and as $P$ is a maximal ideal of $R$, it follows that $Q = P$. This shows that $P$ is the only prime ideal of $R$. □

The following example illustrates that the moreover assertion of Lemma 6.1 may fail if the hypothesis that $P = \text{nil}(R)$ is omitted.

Example 6.2. Let $T = \mathbb{Z}[x]$ be the polynomial ring in one variable over $\mathbb{Z}$. Let $I = x^2T + 2xT$. Let $R = T/I$. For any $t \in T$, we denote $t + I$ by $\bar{t}$. Since $\mathbb{Z} \cap I = (0)$, we identify $\mathbb{Z}$ with $\bar{n}$ for any $n \in \mathbb{Z}$. This example appeared in [2, Example 3.6(a)], where it was noted that $\text{nil}(R) = \{0, \pi\}$ and moreover, it was shown that $\Gamma(R)$ is an infinite star graph with center $\pi$, where $\Gamma(R)$ is the zero-divisor graph of $R$.

Note that $I = x^2T + 2xT = xT \cap (x^2T + 2T)$ is an irredundant primary decomposition of $I$ in $T$ with $xT$ is $P_1 = xT$-primary and $x^2T + 2T$ is $P_2 = xT + 2T$-primary. Observe that $xT/I$ is a $P_1/I$-primary ideal of $R$ and $(x^2T + 2T)/I$ is a $P_2/I$-primary ideal of $R$. Hence it follows that $xT/I \cap (x^2T + 2T)/I$ is an irredundant primary decomposition of the zero ideal of $R$. We know from [3, Proposition 4.7] that $Z(R) = P_1/I \cup P_2/I$ and as $P_1 \subseteq P_2$, it follows that $Z(R) = P_2/I$. This shows that $R$ admits $P_2/I$ as its only maximal $N$-prime of $(0)$. Note that $\text{nil}(R) = P_1/I \neq P_2/I$.

We now verify that $AG(R)$ is complemented. Indeed, we show that $AG(R)$ is an infinite star graph with center $\text{nil}(R)$. Let $J \in A(R)^*$. Then $J \subseteq Z(R) = P_2/I$. Observe that $P_2/I = ((0) :_R \pi)$. Hence we obtain that $J \cap \text{nil}(R) = (0)$. Let $J_1$, $J_2$ be distinct nonzero ideals of $R$ which are different from $\text{nil}(R)$. As $\text{nil}(R) = \{0, \pi\}$, it follows that $J_1 \nsubseteq \text{nil}(R)$ and $J_2 \nsubseteq \text{nil}(R)$. Since $\text{nil}(R)$ is a prime ideal of $R$, we obtain that $J_1J_2 \nsubseteq \text{nil}(R)$. Hence we obtain that $J_1J_2 \neq (0)$. It is clear that for any positive integer $n$, $2^nR \in A(R)^*$ and moreover, for any distinct positive integers $n, m$, $2^nR \neq 2^mR$. The above arguments show
that \(AG(R)\) is an infinite star graph with center \(\text{nil}(R)\). Hence \(AG(R)\) is complemented. However, \(R\) has an infinite number of prime ideals. \(\square\)

The following theorem is an immediate consequence of Lemma 6.1 and Theorem 4.8.

**Theorem 6.3.** Let \(R\) be a ring which is not reduced, admitting only one maximal \(N\)-prime \(P\) of \((0)\) such that \(P = \text{nil}(R)\) and \(AG(R)\) admits at least two vertices. Then \(AG(R)\) is complemented if and only if (a) and (b) hold and moreover, either (c) or (d) holds where (a)–(d) are given below:

(a) \(P^2 \neq (0)\).
(b) \(P^4 = (0)\).
(c) \(R\) is a SPIR.
(d) \((d_1)\) \(z^2 = 0\) for each \(z \in P\), \(P\) is not principal but there exist \(a, b \in P\) such that \(P = Ra + Rb\); and (d_2) \(P^2 = Rx\) for any nonzero \(x \in P^2\).

**Proof.** Suppose that \(P = \text{nil}(R)\) and \(AG(R)\) is complemented. Now it follows, from Lemma 6.1, that \(P\) is the only prime ideal of \(R\). Hence \(R\) is a zero-dimensional quasilocal ring with \(P\) as its unique maximal ideal. Applying Theorem 4.8, we obtain that (a) and (b) hold and moreover, either (c) or (d) holds. We now verify that when (d_1) holds, then (d_2) holds. From (d_1), \(P = Ra + Rb\). As \(z^2 = 0\) for each \(z \in P\), it follows that \(P^2 = Rab\), and \(P^3 = (0)\). Let \(x \in P^2\), \(x \neq 0\). Hence \(x = rab\) for some \(r \in R\). Since \(R\) is quasilocal with \(P\) as its unique maximal ideal and \(P^3 = (0)\), it follows that \(r\) is a unit in \(R\) and so \(ab = r^{-1}x\). Hence we obtain that \(P^2 = Rab = Rx\).

Conversely, assume that (a) and (b) hold and moreover, either (c) or (d) holds. If (c) holds, then it is clear that \(P\) is the unique maximal ideal of \(R\) and it follows that either \(AG(R)\) is a graph on the vertex set \(\{P, P^2, P^3\}\) with \(P \perp P^3\) and \(P^2 \perp P^3\) or \(AG(R)\) is a graph on the vertex set \(\{P, P^2\}\) and \(P \perp P^2\). Thus if (c) holds, then \(AG(R)\) is complemented. Suppose that (d) holds. Let \(r \in R\setminus P\). Now \(P = Ra + Rb\), \(P^2 = Rab\), and \(P^3 = (0)\). Since \(P\) is the only maximal \(N\)-prime of \((0)\) in \(R\), it follows that \(Z(R) = P\). As \(ab \neq 0\) and \(r \in R\setminus Z(R)\), we obtain that \(rab \neq 0\). Hence \(P^2 = R(ab)\). So there exists \(s \in R\) such that \(ab = srab\). This implies that \((1 - sr)ab = 0\). Hence we obtain that \(1 - sr \in Z(R) = P\). Therefore, \(P + Rr = R\). This is true for any \(r \in R\setminus P\). Hence it follows that \(P\) is a maximal ideal of \(R\). By hypothesis, \(P = \text{nil}(R)\). So, \(R\) must be quasilocal with \(P\) as its unique maximal ideal. Now we obtain from (ii) \(\Rightarrow\) (i) of Lemma 4.2 that \(AG(R)\) is complemented. \(\square\)

Let \(R\) and \(\{P_1, \ldots, P_n\}\) be as in the beginning of this section. We assume that \(n \geq 2\) and attempt to determine necessary and sufficient conditions in order that \(AG(R)\) is complemented. We next state and prove Lemma 6.4. It is useful to recall the following. Let \(I\) be an ideal of a commutative ring \(T\) with identity. A prime ideal \(P\) of \(T\) is said to be a \(B\)-prime of \(I\) if there exists \(t \in T\) such that \(P = (I :_T t)\) [9].

**Lemma 6.4.** Let \(R\) be a ring which is not reduced. Let \(n \geq 2\) and let \(\{P_1, P_2, \ldots, P_n\}\) be the set of all maximal \(N\)-primes of \((0)\) in \(R\). Suppose that \(\text{nil}(R) = \cap_{i=1}^n P_i\). If \(AG(R)\) is complemented, then the following hold:

(i) For each \(i \in \{1, 2, \ldots, n\}\), there exists \(x_i \in R\) such that \(P_i = ((0) :_R x_i)\) (that is, \(P_i\) is a \(B\)-prime of \((0)\) in \(R\) for each \(i \in \{1, 2, \ldots, n\}\)). Moreover, for each \(i \in \{1, 2, \ldots, n\}\), \(x_i \in P_j\) for all \(j \in \{1, 2, \ldots, n\}\) \(\setminus \{i\}\).
(ii) With $x_1, x_2, \ldots, x_n$ as in (i), $x_i \in \text{nil}(R)$ for some $i \in \{1,2,\ldots, n\}$ and moreover, for that $i$, $P_i$ is a maximal ideal of $R$.

**Proof.** (i) As $\{P_1, P_2, \ldots, P_n\}$ is the set of all maximal N-primes of $(0)$ in $R$, it follows that $Z(R) = \bigcup_{i=1}^n P_i$. Suppose that $AG(R)$ is complemented. We know, from Lemma 3.3, that $(\text{nil}(R))^4 = (0)$. Hence $(\cap_{i=1}^n P_i)^4 = (0)$. Therefore, $\prod_{i=1}^n P_i^4 = (0)$. Let $i \in \{1,2,\ldots, n\}$. Since $n \geq 2$ and $P_k$ is a maximal N-prime of $(0)$ in $R$ for each $k \in \{1,2,\ldots, n\}$, it follows that $\prod_{j \in A_i} P_j^4 \neq (0)$, where $A_i = \{1,2,\ldots, n\}\{i\}$. Let $y_i \in \prod_{j \in A_i} P_j^4$, $y_i \neq 0$. It now follows that $P_i^4 y_i = (0)$. Let $0 \leq s < 4$ be such that $P_i^s y_i \neq (0)$ but $P_i^{s+1} y_i = (0)$. Let $x_i \in P_i^s y_i \{0\}$. Observe that $P_i x_i = (0)$. Hence we obtain that $P_i \subseteq ((0):_R x_i) \subseteq Z(R) = \cap_{k=1}^n P_k$. It now follows that $P_i = ((0):_R x_i)$. This proves that $P_i$ is a B-prime of $(0)$ in $R$ for each $i \in \{1,2,\ldots, n\}$. We now prove that assertion. We obtain from [4, Lemma 3.6] that $x_i x_j = 0$ for all distinct $i, j \in \{1,2,\ldots, n\}$. Hence for each $i \in \{1,2,\ldots, n\}$, $x_i \in ((0):_R x_j) = P_j$ for all $j \in \{1,2,\ldots, n\}\{i\}$.

(ii) Let $z \in \text{nil}(R)$ with $z \neq 0$. Note that $(x_1 + x_2 + \cdots + x_n)z = 0$. Therefore, $x_1 + x_2 + \cdots + x_n \in Z(R) = \cap_{i=1}^n P_i$. Hence we obtain that $x_1 + x_2 + \cdots + x_n \in P_i$ for some $i \in \{1,2,\ldots, n\}$. We know from (i) that for each $j \in \{1,2,\ldots, n\}\{i\}$, $x_j \in P_i$. It follows from $x_1 + x_2 + \cdots + x_n \in P_i$ that $x_i \in P_i = ((0):_R x_i)$. Therefore, we obtain that $x_i^2 = 0$ and so $x_i \in \text{nil}(R)$. We now prove that $P_i$ is a maximal ideal of $R$. Let $M$ be a maximal ideal of $R$ such that $P_i \subseteq M$. We claim that $M \subseteq Z(R)$. Suppose that $M \not\subseteq Z(R)$. Let $I = Rx_i$. Since $x_i \neq 0$ but $x_i^2 = 0$, it is clear that $I \subseteq A(R)$. As $AG(R)$ is complemented, there exists $J \in A(R)^*$ such that $I \perp J$. Let $A = Rw x_i$. Since $x_i^2 = 0$, $I = (Rx_i)J = (0)$, it is clear that $AJ = IA = (0)$. It follows from $I \perp J$ that $A \subseteq \{(0), I, J\}$. Since $w \not\in Z(R)$, we obtain that $A = Rw x_i \neq (0)$. Observe that $A \neq I$. For if $A = I$, then $x_i \in A$ and so $x_i = rw x_i$ for some $r \in R$. This implies that $(1 - rw)x_i = 0$. Hence $1 - rw \in ((0):_R x_i) = P_i \subseteq M$. This is impossible since $w \in M$ and $M$ is a proper ideal of $R$. Hence $A \neq I$ and so $A = J$. Thus we arrive at $I = Rx_i \perp A = Rw x_i$. Note that $B = Rw^2 x_i$ is such that $B \not\subseteq \{(0), I, A\}$, but $BI = BA = (0)$. This is in contradiction to the fact that $I \perp A$. Hence we must have $M \subseteq Z(R)$. As $M$ is a maximal ideal of $R$ and $M \subseteq Z(R)$, $M$ is necessarily a maximal N-prime of $(0)$ in $R$. Since $P_i$ is also a maximal N-prime of $(0)$ in $R$, it follows from $P_i \subseteq M$ that $P_i = M$. This proves that $P_i$ is a maximal ideal of $R$. \(\square\)

With the same hypotheses as in the statement of Lemma 6.4, the following lemma provides another necessary condition in order that $AG(R)$ is complemented.

**Lemma 6.5.** Let $R$ be a ring which is not reduced. Let $n \geq 2$ and let $\{P_1, P_2, \ldots, P_n\}$ be the set of all maximal N-primes of $(0)$ in $R$. Suppose that $\text{nil}(R) = \cap_{i=1}^n P_i$. If $AG(R)$ is complemented, then $(\text{nil}(R))^2 = (0)$.

**Proof.** Suppose that $AG(R)$ is complemented. We know from Lemma 3.3 that $(\text{nil}(R))^4 = (0)$. We first prove that $(\text{nil}(R))^3 = (0)$. Suppose that $(\text{nil}(R))^3 \neq (0)$. We know from Proposition 3.4(i) that $\text{nil}(R) \perp (\text{nil}(R))^3$. Moreover, we know from Lemma 6.4 that there exist elements $x_i \in R$ such that $P_i = ((0):_R x_i)$ for each $i \in \{1,2,\ldots, n\}$ and so $(\text{nil}(R))(Rx_i) = (0)$ and $(\text{nil}(R))^3(Rx_i) = (0)$. Since $\text{nil}(R) \perp (\text{nil}(R))^3$, it follows that $Rx_1 \in \{\text{nil}(R), (\text{nil}(R))^3\}$ and $Rx_2 \in \{\text{nil}(R), (\text{nil}(R))^3\}$. As $(\text{nil}(R))^3 \subseteq \text{nil}(R)$, it follows that either $Rx_1 \subseteq Rx_2$ or $Rx_2 \subseteq Rx_1$. We may assume without loss of
generality that $Rx_1 \subseteq Rx_2$. This implies that $x_1 = rx_2$ for some $r \in R$. Let $a \in P_2$. Then $ax_2 = 0$ and so $ax_1 = a(rx_2) = 0$. This implies that $a \in P_1$. Hence we arrive at $P_2 \subseteq P_1$. This is impossible since $P_1$ and $P_2$ are distinct maximal N-primes of $(0)$ in $R$. Hence we obtain that $(\operatorname{nil}(R))^3 = (0)$. We now show that $(\operatorname{nil}(R))^2 = (0)$. Suppose that $(\operatorname{nil}(R))^2 \neq (0)$. We know from Proposition 3.5(i) that $\operatorname{nil}(R) \perp (\operatorname{nil}(R))^2$. As $(\operatorname{nil}(R))Rx_i = (\operatorname{nil}(R))^2Rx_i = (0)$ for each $i \in \{1, 2, \ldots, n\}$, we obtain that $Rx_1 \in \{\operatorname{nil}(R), (\operatorname{nil}(R))^2\}$ and $Rx_2 \in \{\operatorname{nil}(R), (\operatorname{nil}(R))^2\}$. Since $(\operatorname{nil}(R))^2 \subseteq \operatorname{nil}(R)$, proceeding as in the previous paragraph, we obtain a similar contradiction.

This proves that $(\operatorname{nil}(R))^2 = (0)$. □

Let $R, \{P_1, P_2, \ldots, P_n\}$ be as in the statement of Lemma 6.4. With the assumption that $\operatorname{nil}(R) = \bigcap_{i=1}^n P_i$, we determine in Theorem 6.9 when $\operatorname{AG}(R)$ is complemented. We make use of the following lemmas in the proof of Theorem 6.9. We denote by $\operatorname{Tot}(R)$, the total quotient ring of $R$.

**Lemma 6.6.** Let $R$ be a ring which is not reduced. Let $n \geq 2$ and let $\{P_1, P_2, \ldots, P_n\}$ be the set of all maximal N-primes of $(0)$ in $R$. Suppose that $\operatorname{nil}(R) = \bigcap_{i=1}^n P_i$. If $\operatorname{AG}(R)$ is complemented, then $n = 2$.

**Proof.** As $\{P_1, P_2, \ldots, P_n\}$ is the set of all maximal N-primes of $(0)$ in $R$, it is clear that $Z(R) = \bigcup_{i=1}^n P_i$. Let $S = R \setminus Z(R) = R \setminus (\bigcup_{i=1}^n P_i)$. Observe that $S^{-1}R = \operatorname{Tot}(R)$ is a zero-dimensional quasisemilocal ring and moreover, $\{S^{-1}P_1, S^{-1}P_2, \ldots, S^{-1}P_n\}$ is the set of all its maximal ideals. Furthermore, as $R$ is not reduced, it follows that $\operatorname{Tot}(R)$ is not reduced. Since $n \geq 2$, it is clear that $\operatorname{Tot}(R)$ is not quasilocal. We want to show that $n = 2$. In view of (i) ⇒ (ii) of Theorem 5.6, it is enough to show that $\operatorname{AG}(\operatorname{Tot}(R))$ is complemented. This is clear if $R = \operatorname{Tot}(R)$. Hence we may assume that $R \neq \operatorname{Tot}(R)$. Therefore, $P_i$ is not a maximal ideal of $R$ for at least one $i \in \{1, 2, \ldots, n\}$. Without loss of generality we may assume that $P_1$ is not a maximal ideal of $R$. We know from Lemma 6.4(i) that there exist elements $x_i \in R$ such that $P_i = ((0) :_R x_i)$ for $i = 1, 2, \ldots, n$. Since $P_1$ is not a maximal ideal of $R$, it follows from Lemma 6.4(ii) that $x_1 \notin \operatorname{nil}(R)$.

Let $A \in A(\operatorname{Tot}(R))^*$. Note that $A = S^{-1}I$ for some ideal $I \in A(R)^*$. Since we are assuming that $\operatorname{AG}(R)$ is complemented, there exists $J \in A(R)^*$ such that $I \perp J$ in $\operatorname{AG}(R)$. We claim that $A = S^{-1}I \perp S^{-1}J$ in $\operatorname{AG}(\operatorname{Tot}(R))$. From $IJ = (0)$, it follows that $S^{-1}IS^{-1}J = (0)$. If $B = S^{-1}K$ is any element of $A(\operatorname{Tot}(R))^*$ such that $S^{-1}IS^{-1}K = S^{-1}JS^{-1}K = (0)$, it follows that $IK = JK = (0)$. Since $I \perp J$ in $\operatorname{AG}(R)$, it follows that $K \in \{I, J\}$ and hence we obtain that either $S^{-1}K = S^{-1}I$ or $S^{-1}K = S^{-1}J$. Now to show $S^{-1}I \perp S^{-1}J$ in $\operatorname{AG}(\operatorname{Tot}(R))$, we need only to verify that $S^{-1}I \neq S^{-1}J$. Suppose that $S^{-1}I = S^{-1}J$. Then it follows from $S^{-1}IS^{-1}J = (0)$ that $(S^{-1}I)^2 = (S^{-1}J)^2 = (0)$. Therefore, we obtain that $I^2 = J^2 = (0)$. Hence it follows that $I \subseteq \operatorname{nil}(R)$ and $J \subseteq \operatorname{nil}(R)$. Note that $I(Rx_1) = J(Rx_1) = (0)$. As $x_1 \notin \operatorname{nil}(R)$, it is clear that $Rx_1 \notin ((0), I, J)$. Thus we obtain that the ideal $Rx_1$ is adjacent to $I$ and $J$ in $\operatorname{AG}(R)$. This is impossible since $I \perp J$ in $\operatorname{AG}(R)$. This proves that $S^{-1}I \neq S^{-1}J$ and so as is noted already, we obtain that $S^{-1}I \perp S^{-1}J$ in $\operatorname{AG}(\operatorname{Tot}(R))$. This shows that $\operatorname{AG}(\operatorname{Tot}(R))$ is complemented and so as is remarked earlier in this proof, it follows that $n = 2$. □
Lemma 6.7. Let $T_1, T_2$ be commutative rings with identity. Suppose that $N_i$ is the unique maximal $N$-prime of $(0)$ in $T_i$ for each $i \in \{1, 2\}$ with $\text{nil}(T_i) = N_i$. Let $T = T_1 \times T_2$. Suppose that $AG(T)$ is complemented. If $N_2 \neq (0)$, then $N_2$ is a maximal ideal of $T_2$.

Proof. Since $\text{nil}(T_2) = N_2 \neq (0)$, there exists $t_2 \in N_2$ such that $t_2 \neq 0$ but $t_2^2 = 0$. By contradiction, suppose that $N_2$ is not a maximal ideal of $T_2$. Let $M$ be a maximal ideal of $T_2$ such that $N_2 \subset M$. Consider the ideal $I = T_1 \times T_2 t_2$. Note that $I \in A(T)^*$. As $AG(T)$ is complemented, there exists $J \in A(T)^*$ such that $I \perp J$. Observe that $J = J_1 \times J_2$ for some ideal $J_1$ of $T_1$ and an ideal $J_2$ of $T_2$. From $IJ = (0) \times (0)$, it follows that $J_1 = (0)$ and $(T_2t_2)J_2 = (0)$. Let $y \in M \setminus N_2$. Since $Z(T_2) = N_2$, we obtain that $y \notin Z(T_2)$. As $t_2 \neq 0$, it follows that $yt_2 \neq 0$. Note that the nonzero ideal $K = (0) \times T_2(yt_2)$ is such that $IK = JK = (0) \times (0)$. Since $I \perp J$, we obtain that $K \in \{I, J\}$. It is clear that $K \neq I$. Hence $K = J$. Therefore, we obtain that $T_2t_2 = T_2(yt_2)$. So there exists $s_2 \in T_2$ such that $t_2 = s_2yt_2$. This implies that $t_2(1 - s_2y) = 0$. Thus $1 - s_2y \in Z(T_2) = N_2 \subset M$. As $y \in M$, it follows that $1 = 1 - s_2y + s_2y = M$. This is impossible. Therefore, $N_2$ must be a maximal ideal of $T_2$. □

We also make use of the following lemma in the proof of Theorem 6.9.

Lemma 6.8. Let $(S, M)$ be a SPIR with $M \neq (0)$ but $M^2 = (0)$ and $D$ be an integral domain. Let $R = S \times D$. Then $AG(R)$ is complemented.

Proof. If $D$ is a field, then it is already verified in the proof of (ii) ⇒ (i) of Theorem 5.6 that $AG(R)$ is complemented. Suppose that $D$ is not a field. Observe that $A(R)^* = \{(0) \times I | I \text{ varies over all nonzero ideals of } D\} \cup \{M \times J | J \text{ varies over all ideals of } D\} \cup \{S \times (0)\}$. It is easy to verify that for any nonzero ideal $I$ of $D$, $(0) \times I \subset M \times (0)$, for any nonzero ideal $J$ of $D$, $M \times J \subset M \subset (0)$, and $S \times (0) \subset (0) \times D$. This proves that each element of $A(R)^*$ admits a complement in $AG(R)$ and hence we obtain that $AG(R)$ is complemented. □

With the help of Lemmas 6.4–6.8, we prove the following theorem.

Theorem 6.9. Let $R$ be a ring which is not reduced. Let $n \geq 2$ and let $\{P_1, P_2, \ldots, P_n\}$ be the set of all maximal $N$-primes of $(0)$ in $R$. Suppose that $\text{nil}(R) = \bigcap_{i=1}^{n} P_i$. Then $AG(R)$ is complemented if and only if either $R$ is isomorphic to $F \times S$ as rings, where $F$ is a field and $(S, M)$ is a SPIR with $M \neq (0)$ but $M^2 = (0)$, or is isomorphic to $S \times D$ as rings, where $(S, M)$ is a SPIR with $M \neq (0)$ but $M^2 = (0)$ and $D$ is an integral domain which is not a field.

Proof. Suppose that $AG(R)$ is complemented. We know from Lemma 6.6 that $n = 2$. Thus $\{P_1, P_2\}$ is the set of all maximal $N$-primes of $(0)$ in $R$. We know from Lemma 6.4(i) that there exist $x_1, x_2 \in R$ such that $P_1 = ((0):_R x_1)$, $P_2 = ((0):_R x_2)$ and moreover, $x_1 \in P_2$ and $x_2 \in P_1$. Furthermore, we know from Lemma 6.4(ii) that either $x_1 \in \text{nil}(R)$ or $x_2 \in \text{nil}(R)$. We may assume without loss of generality that $x_1 \in \text{nil}(R)$. In such a case, it follows from Lemma 6.4(ii) that $P_1$ is a maximal ideal of $R$. As $P_1, P_2$ are distinct maximal $N$-primes of $(0)$ in $R$, we obtain that $P_1 + P_2 = R$. We know from Lemma 6.5 that $(\text{nil}(R))^2 = (0)$ and so $(P_1 \cap P_2)^2 = (0)$. Hence $P_1^2P_2^2 = (0)$. As $P_1^2 + P_2^2 = R$, we obtain from the Chinese reminder theorem [3, Proposition 1.10(ii) and (iii)] that the mapping $f : R \rightarrow R/P_1^2 \times R/P_2^2$ given by $f(r) = (r + P_1^2, r + P_2^2)$ is an isomorphism
of rings. Let us denote $R/P_1^2$ by $T_1$ and $R/P_2^2$ by $T_2$. Moreover, let us denote $P_1/P_1^2 = N_1$ and $P_2/P_2^2$ by $N_2$. Note that $f(Z(R)) = f(P_1 \cup P_2) = (N_1 \times T_2) \cup (T_1 \times N_2)$. As $f$ is an isomorphism of rings, it follows that $f(Z(R)) = Z(T_1 \times T_2) = (Z(T_1) \times T_2) \cup (T_1 \times Z(T_2))$. Hence we obtain that $Z(T_1) = N_1$ and $Z(T_2) = N_2$. Therefore, $N_i$ is the unique maximal N-prime of the zero ideal of $T_i$ for each $i \in \{1, 2\}$. Moreover, $f(nil(R)) = f(P_1 \cap P_2) = P_1/P_1^2 \times P_2/P_2^2 = nil(T_1) \times nil(T_2)$. Hence it follows that $nil(T_1) = P_1/P_1^2 = N_1$ and $nil(T_2) = P_2/P_2^2 = N_2$.

We consider two cases.

**Case (i)** $P_2$ is a maximal ideal of $R$.

As $P_1$ is already a maximal ideal of $R$ and $nil(R) = P_1 \cap P_2$, it follows that $R$ is a zero-dimensional quasileocal ring with $\{P_1, P_2\}$ as its set of all prime ideals of $R$. Now $AG(R)$ is complemented and $R$ is not reduced. Hence it follows from (i) $\Rightarrow$ (ii) of Theorem 5.6 that $R$ must be isomorphic to $S \times F$ as rings, where $(S, M)$ is a SPIR with $M \neq (0)$ but $M^2 = (0)$ and $F$ is a field.

**Case (ii)** $P_2$ is not a maximal ideal of $R$.

Note that $N_2 = P_2/P_2^2$ is not a maximal ideal of $T_2$. Since $AG(R)$ is complemented and $R$ is isomorphic to $T_1 \times T_2$ as rings, we obtain that $AG(T_1 \times T_2)$ is complemented. Now it follows from Lemma 6.7 that $nil(T_2)$ is the zero ideal of $T_2$. Hence we obtain that $P_2 = P_2^2$ and so $T_2 = R/P_2^2 = R/P_2$ is an integral domain. By assumption, $P_2$ is not a maximal ideal of $R$ and so $T_2$ is not a field. Let us denote $T_1 \times T_2$ by $T$. Since $T$ is not reduced, it follows that $nil(T_1)$ is a nonzero ideal of $T_1$. Hence $P_1 \neq P_1^2$. We assert that any $x \in P_1 \setminus P_2^2$, $P_1/P_1^2 = T_1(x + P_1^2)$. Observe that $I = T_1(x + P_1^2) \times T_2 \in A(T)^*$. As $AG(T)$ is complemented, there exists an ideal $J_1$ of $T_1$ and an ideal $J_2$ of $T_2$ such that $I = T_1(x + P_1^2) \times T_2 \perp J = J_1 \times J_2$. Hence $J_2 = (0 + P_2^2)$ and from $T_1(x + P_1^2)J_1 = (0 + P_1^2)$, it follows that $J_1 \subseteq P_1/P_1^2$. Note that the ideal $K = P_1/P_1^2 \times (0 + P_2^2)$ is such that $IK = JK = (0 + P_1^2) \times (0 + P_2^2)$. Since $I \perp J$ and as $K \notin \{(0 + P_1^2) \times (0 + P_2^2), I\}$, it follows that $K = J$. Hence we obtain that $I \perp K$. Now the ideal $B = T_1(x + P_1^2) \times (0 + P_2^2)$ is such that $BI = BK = (0 + P_1^2) \times (0 + P_2^2)$. Since $I \perp K$ and $B \notin \{(0 + P_1^2) \times (0 + P_2^2), I\}$, we obtain that $B = K$. Hence $P_1/P_1^2 = T_1(x + P_1^2)$. As $P_1/P_1^2$ is a maximal ideal of $T_1$, it is clear that $(T_1, N_1)$ is a SPIR with $N_1$ is a nonzero ideal of $T_1$ but $N_1^2$ is the zero ideal of $T_1$. Let $S = T_1$, $M = N_1$, and $D = T_2$. Note that $(S, M)$ is a SPIR with $M \neq (0)$ but $M^2 = (0)$, $D$ is an integral domain which is not a field and moreover, $R \cong S \times D$ as rings.

The converse follows immediately from Lemma 6.8. □

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**REFERENCES**


When is the annihilating ideal graph of a zero-dimensional quasisemilocal commutative ring complemented?