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C-Cosine Functions and the Abstract Cauchy Problem, I*

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If A is the generator of an exponentially bounded C-cosine function on a Banach space X, then the abstract Cauchy problem (ACP) for A has a unique solution for every pair (x, y) of initial values from $(\lambda - A)^{-1}C(X)$. The main result is a characterization of the generator of a C-cosine function, which may not be exponentially bounded and may have a nondensely defined generator, in terms of the associated ACP. © 1997 Academic Press

INTRODUCTION

Let A be a closed linear operator with domain $D(A)$ and range $R(A)$ in a Banach space X. The second order abstract Cauchy problem associated with A is the initial value problem,

$$\begin{cases} u''(t) = Au(t) + f(t), & t \in R, \\ u(0) = x, \quad u'(0) = y, \end{cases} \qquad \text{ACP}(f; x, y)$$

where $f \in C(R; X)$. A function u is a (strong) solution of ACP$(f; x, y)$ if $u \in C^2(R; X)$, $u(t) \in D(A)$ for all $t \in R$, and ACP$(f; x, y)$ is satisfied. It is well known that the ACP is closely related to the theory of strongly continuous cosine operator functions (see [4–6, 13]); ACP is well-posed if and only if A is the generator of a strongly continuous cosine operator function. Since the generator of a cosine operator function is necessarily densely defined, the well-established theory of cosine operator functions is

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not applicable to those ACPs which have a nondensely defined operator A. Many partial differential operators, e.g., the Laplacian Δ on $L^p(R^n)$ ($n > 1$ and $p \neq 2$) do not generate strongly continuous cosine operator functions.

Recently, to deal with the case which cannot be treated by cosine operator functions, a generalization, the so-called n-times integrated C-cosine function, has been investigated in [9, 10, 12]. See also [8, 11, 16] for the case $n = 0$. When A is the generator of an exponentially bounded (0-times integrated) C-cosine function, the ACP$(0; x, y)$ has the unique solution $u(t; x, y) = C^{-1}[C(t)x + \int_0^t C(s)y\,ds]$ for every pair of initial values $x, y \in C(D(A))$ [9, Corollary 4.3]. Many applications of C-cosine functions to abstract differential operators and concrete partially differential operators have been discussed in [16]. For instance, if a polynomial $P(\cdot)$ on R^n is r-coercive for some $r \in (0, m]$, and if $\alpha > (mn/2)|1/2 - 1/p|$ ($1 \leq p \leq \infty$), then the differential operator $P(D)$ generates a norm continuous $(1 - \Delta)^{-\alpha}$-cosine function on $L^p(R^n)$.

A cosine operator function (i.e., the case $C = I$) is necessarily exponentially bounded (see [13]). But, as is shown by an example in Section 1, a general C-cosine function need not be exponentially bounded. The above mentioned papers mainly deal with C-cosine functions which are exponentially bounded, and the relationship between a C-cosine function $C(\cdot)$ and the corresponding ACP has not been fully investigated, even in the case where $C(\cdot)$ is exponentially bounded. In this paper and the subsequent paper [7] we attempt to give thorough discussions on this subject. The main purpose in this paper is to investigate a certain relationship between $C(\cdot)$ and its associated ACP$(0; x, y)$. Further characterizations of the generator of $C(\cdot)$ in terms of the existence and uniqueness of strong and weak solutions of ACP$(f; x, y)$, as well as an application to perturbation of generators, will be discussed in [7].

Some basic properties of a C-cosine function and the existence and uniqueness of a solution of the ACP associated with the generator will be investigated in Section 1, and a characterization of the generator of a C-cosine function in terms of the ACP will be established in Section 2.

More precisely, after preparing some basic properties we shall show that if A is the generator of a (not necessarily exponentially bounded) C-cosine function, then $u(t) = C^{-1}C(t)x + C^{-1}\int_0^t C(s)y\,ds$ is the unique strong solution of the ACP$(0; x, y)$ for every pair (x, y) of initial values from $C(D(A))$ (Theorem 1.5(i)). In the case where $C(\cdot)$ is exponentially bounded, one can take initial values from the set $(\lambda - A)^{-1}C(X)$ (Corollary 1.6), which is larger than $C(D(A))$ in general, but is equal to the latter when $\lambda \in \rho(A)$, the resolvent set of A (see [15, Proposition 1.4]). Conversely, if A is a closed linear operator such that $R(C) \subset D((\lambda - A)^{-1})$ for some real number λ and A commutes with C, and if

ACP(0; $x, 0$) has a unique strong solution for every initial value $x \in (\lambda - A)^{-1}C(X)$, then $C^{-1}AC$ is the generator of a C-cosine function (Theorem 2.1). Since $C^{-1}AC = A$ in case $\rho(A) \neq \phi$ [15, Proposition 1.4], it follows that a closed linear operator A with $\rho(A) \neq \phi$ is the generator of a C-cosine function if and only if A commutes with C and ACP(0; x, y) has a unique strong solution for every pair (x, y) of initial values in $C(D(A))$, if and only if A commutes with C and ACP(0; $x, 0$) has a unique strong solution for every initial value $x \in C(D(A))$ (Corollary 2.2). In the case where C is the identity operator I, this result reduces to Fattorini's theorem [4, 5]. We also characterize the generator of an exponentially bounded C-cosine function in terms of the ACP (see Theorem 2.3). These results are analogous to some results of Tanaka and Miyadera [15] on the relationship between a C-semigroup and its associated Cauchy problem of the first order. The reader can refer to [1–3, 14, 15] and references therein for recent research on C-semigroups.

1. C-COSINE FUNCTIONS

Let X be a Banach space and let $B(X)$ be the set of all bounded linear operators from X into itself. Throughout this paper, $C \in B(X)$ will be injective. A family $\{C(t); t \in R\}$ in $B(X)$ is called a *C-cosine function* on X if

$$C(\cdot)x : R \to X \text{ is continuous for each } x \in X, \tag{1.1}$$

$$C[C(t+s) + C(t-s)] = 2C(t)C(s) \quad \text{for all } t, s \in R \text{ and } C(0) = C. \tag{1.2}$$

The associated *C-sine function* (or *integrated C-cosine function*) is the family $\{S(t); t \in R\}$ of operators defined by $S(t)x = \int_0^t C(s)x\,ds$, $x \in X$, $t \in R$. Equation (1.2) implies that C commutes with $C(t)$ and $S(t)$. $C(\cdot)$ is said to be *exponentially bounded* if

$$\text{there are } M \geq 0 \text{ and } w \geq 0 \text{ such that } \|C(t)\| \leq Me^{w|t|} \text{ for all } t \in R. \tag{1.3}$$

When $C = I$, a C-cosine function is a classical *cosine operator function*, and it is necessarily exponentially bounded (see [13]). In general, just like in the case of C-semigroups, this may not be true. For examples, the C-group $T(\cdot)$ on $X := L^2(R)$, defined by $(T(t)f)(s) = e^{st-s^2}f(s)$, $f \in X$, $s, t \in R$, is not exponentially bounded (see [1]). Consider the associated family $\{C(t); t \in R\}$ of operators defined by

$$(C(t)f)(s) = 2^{-1}(e^{st} + e^{-st})e^{-s^2}f(s), \qquad f \in X, s \in R, t \in R.$$

It is clear that $C(\cdot)$ is a C-cosine function on X, with C being the multiplication operator defined by $(Cf)(s) = e^{-s^2}f(s)$ and with the generator A defined by $(Af)(s) = sf(s)$. We have

$$\begin{aligned}\|C(t)\| &= \sup_{s\in R}\left\{2^{-1}(e^{-st}+e^{st})e^{-s^2}\right\}\\ &= \sup_{s\in R}\left\{e^{s^2/4}\frac{1}{2}\left(e^{-(s-t/2)^2}+e^{-(s+t/2)^2}\right)\right\}\\ &\geq \frac{1}{2}e^{t^2/4}.\end{aligned}$$

One can define the *infinitesimal generator* A of $C(\cdot)$ by

$$\begin{cases} D(A) = \left\{x \in X;\ \lim_{t\to 0} 2t^{-2}(C(t)x - Cx) \in R(C)\right\}, \\ Ax = C^{-1}\lim_{t\to 0} 2t^{-2}(C(t)x - Cx), \qquad x \in D(A). \end{cases} \tag{1.4}$$

PROPOSITION 1.1. *Let $\{C(t); t \in R\}$ be a C-cosine function on X. The following assertions hold*

$$C(t) = C(-t) \qquad \text{for all } t \in R; \tag{1.5}$$

$$S(-t) = -S(t) \qquad \text{for all } t \in R. \tag{1.6}$$

$$C(s), S(s), C(t), \text{ and } S(t) \text{ commute for all } t, s \in R; \tag{1.7}$$

$$S(\cdot)x \in C^1(R, X) \qquad \text{for each } x \in X; \tag{1.8}$$

$$[S(s+t) + S(s-t)]C = 2C(t)S(s) \qquad \text{for all } t, s \in R; \tag{1.9}$$

$$S(t+s)C = S(t)C(s) + C(t)S(s) \qquad \text{for all } t, s \in R. \tag{1.10}$$

Proof. Assertion (1.5) follows from (1.2) by setting $t = 0$; (1.6) follows from (1.5); (1.7) follows from (1.2) and (1.5); (1.8) follows from (1.1); (1.9) follows by integrating (1.2) with respect to s; (1.10) follows from (1.9), (1.7), and (1.6).

Some important properties of the generator of a C-cosine function are provided by the following proposition. Note that it was proved in [9] under the assumption that $C(\cdot)$ is exponentially bounded. We give a proof of it without that assumption.

PROPOSITION 1.2. *Let C be an injection and $\{C(t); t \in R\}$ be a C-cosine function with generator A. The following assertions hold*

$$C(t)x \in D(A) \text{ and } AC(t)x = C(t)Ax \text{ for } x \in D(A) \text{ and } t \in R; \tag{1.11}$$

$$S(t)x \in D(A) \text{ and } AS(t)x = S(t)Ax \text{ for } x \in D(A) \text{ and } t \in R; \tag{1.12}$$

$$\int_0^t S(s)x\,ds \in D(A) \text{ and } A\int_0^t S(s)x\,ds = C(t)x - Cx \quad \text{for } x \in X \text{ and } t \in R; \tag{1.13}$$

$$[C(t+s) - C(t-s)]C = 2AS(t)S(s) \qquad \text{for all } s, t \in R; \tag{1.14}$$

$$C(t)x - Cx = \int_0^t S(s)Ax\,ds \qquad \text{for } x \in D(A) \text{ and } t \in R; \tag{1.15}$$

$$C(\cdot)x \in C^2(R, X) \qquad \text{for each } x \in D(A); \tag{1.16}$$

$$A \text{ is a closed linear operator in } X; \tag{1.17}$$

$$C^{-1}AC = A; \tag{1.18}$$

$$R(C) \subset \overline{D(A)}. \tag{1.19}$$

Proof. To show that (1.11) holds, let $x \in D(A)$ and $t \in R$. Then for all $s \in R \setminus \{0\}$ we have

$$\begin{aligned} 2s^{-1}[C(s)C(t)x - CC(t)x] &= C(t)\left[2s^{-2}(C(s)x - Cx)\right] \\ &\to C(t)CAx = CC(t)Ax \in R(C) \end{aligned}$$

as $s \to 0$. This means that $C(t)x \in D(A)$ and $AC(t)x = C(t)Ax$. Assertion (1.12) follows from the definition of $S(\cdot)$ and the closedness of A. We next prove that (1.13) holds. Using (1.9) we have for all $x \in X$

$$\begin{aligned} &2s^{-2}\left[C(s)\int_0^t S(\tau)x\,d\tau - C\int_0^t S(\tau)x\,d\tau\right] \\ &\quad = 2s^{-2}\left[\frac{1}{2}\left(\int_0^t S(s+\tau)Cx\,d\tau - \int_0^t S(s-\tau)Cx\,d\tau\right) - \int_0^t S(\tau)Cx\,d\tau\right] \\ &\quad = 2s^{-2}\left[\frac{1}{2}\left(\int_s^{t+s} S(\tau)Cx\,d\tau + \int_{-s}^{t-s} S(\tau)Cx\,d\tau\right) - \int_0^t S(\tau)Cx\,d\tau\right] \\ &\quad \to C(t)Cx - C^2x = C(C(t)x - Cx) \qquad \text{as } s \to 0. \end{aligned}$$

Thus $\int_0^t S(\tau)x\,d\tau \in D(A)$ and $A\int_0^t S(\tau)x\,d\tau = C(t)x - Cx$ for $x \in X$ and $t \in R$.

By (1.2), we have for $x \in X$

$$
\begin{aligned}
C(r)S(t)S(s)x &= C(r)\int_0^t\int_0^s C(u)C(v)x\,du\,dv \\
&= \int_0^t\int_0^s \frac{1}{2}[C(r+u)+C(r-u)]CC(v)x\,du\,dv \\
&= \frac{1}{2}\int_0^t\left[\int_r^{r+s} C(u)C(v)Cx\,du + \int_{r-s}^{r} C(u)C(v)Cx\,du\right]dv \\
&= \frac{1}{2}\int_0^t\int_{r-s}^{r+s} C(u)C(v)Cx\,du\,dv.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{d}{dr}C(r)S(t)S(s)x &= \frac{1}{2}\int_0^t [C(r+s)-C(r-s)]C(v)Cx\,dv \\
&= \frac{1}{2}\int_0^t \frac{1}{2}[C(r+s+v)+C(r+s-v) \\
&\qquad -C(r-s+v)-C(r-s-v)]C^2x\,dv \\
&= \frac{1}{4}\left(\int_{r+s}^{r+s+t} + \int_{r+s-t}^{r+s} - \int_{r-s}^{r-s+t} - \int_{r-s-t}^{r-s}\right)C(v)C^2x\,dv \\
&= \frac{1}{4}\left(\int_{r+s-t}^{r+s+t} - \int_{r-s-t}^{r-s+t}\right)C(v)C^2x\,dv
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d^2}{dr^2}C(r)S(t)S(s) &= \frac{1}{4}[C(r+s+t)-C(r+s-t)-C(r-s+t) \\
&\qquad +C(r-s-t)]C^2x.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
&\lim_{r\to 0} 2r^{-2}[C(r)S(t)S(s)x - CS(t)S(s)x] \\
&\quad = \frac{d^2}{dr^2}C(r)S(t)S(s)x\,|_{r=0} \\
&\quad = 2^{-1}[C(s+t)-C(s-t)]C^2x \\
&\quad = C2^{-1}[C(s+t)-C(s-t)]Cx
\end{aligned}
$$

for all $x \in X$ and $t, s \in R$.

It follows that $S(t)S(s)x \in D(A)$ and $AS(t)S(s)x = 2^{-1}[C(t+s) - C(t-s)]Cx$ for all $x \in X$ and $t, s \in R$; to show that (1.15) holds, we first claim that $C(\cdot)Cx \in C^1(R, X)$ and $(d/dt)C(t)Cx = CS(t)Ax$ for $x \in D(A)$. In fact, using (1.2), (1.14), and (1.12) we have

$$\begin{aligned} s^{-1}&[C(t+s)Cx - C(t)Cx] \\ &= s^{-1}[C(t)C(s)x + AS(t)S(s)x - C(t)Cx] \\ &= C(t)\left[s^{-1}(C(s)x - Cx)\right] + s^{-1}S(s)S(t)Ax \\ &= C(t)\left[2s^{-2}(C(s)x - Cx)\cdot s/2\right] + s^{-1}S(s)S(t)Ax, \end{aligned}$$

which converges to $CS(t)Ax$ as $s \to 0$. It follows that

$$\begin{aligned} C\int_0^t S(s)Ax\,ds = \int_0^t CS(s)Ax\,ds = \int_0^t \frac{d}{ds}C(s)Cx\,ds \\ = C(C(t)x - Cx), \end{aligned}$$

which proves that $C(t)x - Cx = \int_0^t S(s)Ax$ for each $x \in D(A)$ and $t \in R$ because C is injective. Assertion (1.16) follows from (1.15). To show that (1.17) holds, let $x_n \in D(A)$, $x_n \to x$, and $Ax_n \to y$. Then from the equality

$$C(t)x_n - Cx_n = \int_0^t S(s)Ax_n\,ds$$

it follows that as $n \to \infty$, $C(t)x - Cx = \int_0^t S(s)y\,ds$ for all $t \in R$ and

$$2t^{-2}(C(t)x - Cx) = 2t^{-2}\int_0^t S(s)y\,ds \to Cy \qquad as\ t \to 0.$$

This shows that $x \in D(A)$ and $Ax = y$, and so A is a closed linear operator in X. Finally we show that $C^{-1}AC = A$. The relation $A \subset C^{-1}AC$ immediately follows from (1.11) with $t = 0$. To show the converse, let $x \in D(C^{-1}AC)$, that is, $Cx \in D(A)$ and $ACx \in R(C)$. Then, by (1.15) we have

$$\begin{aligned} C(C(t)x - Cx) = C(t)Cx - C^2x = \int_0^t S(\tau)ACx\,d\tau \\ = C\int_0^t S(\tau)C^{-1}ACx\,d\tau \end{aligned}$$

from which it follows that

$$2t^{-2}(C(t)x - Cx) = 2t^{-2}\int_0^t S(\tau)C^{-1}ACx\,d\tau \to ACx \in R(C)$$

as $t \to 0$. This means that $x \in D(A)$ and $Ax = C^{-1}ACx$. Finally, the facts that $\int_0^t S(s)x\,ds \in D(A)$ (see (1.13)) and $2t^{-2}\int_0^t S(s)x\,ds \to Cx$ as $t \to 0$ imply (1.19).

LEMMA 1.3. *Let D_λ denote the set*

$$D_\lambda := \left\{ x \in X \mid \text{both } L_\lambda x = \int_0^\infty e^{-\lambda t} C(t) x\,dt \text{ and } \int_0^\infty e^{-\lambda t} S(t) x\,dt \text{ exist} \right\}.$$

The following assertions hold.

(i) $C(t)L_\lambda D_\lambda \subset R(C)$ *and* $(d^2/dt^2)C^{-1}C(t)L_\lambda x = \lambda^2 C^{-1}C(t)L_\lambda x - \lambda C(t)x$ *for* $x \in D_\lambda$ *and* $t \in R$.

(ii) $L_\lambda D_\lambda \subset D(A)$ *and* $(\lambda^2 - A)L_\lambda x = \lambda Cx$ *for* $x \in D_\lambda$.

Proof. Using (1.2) and (1.9), we easily see that for $x \in D_\lambda$

$$C(t)L_\lambda x = C\frac{1}{2}\left[e^{\lambda t}\int_t^\infty e^{-\lambda s}C(s)x\,ds + e^{-\lambda t}\int_{-t}^\infty e^{-\lambda s}C(s)x\,ds \right] \in R(C),$$

so that

$$\frac{d}{dt}C^{-1}C(t)L_\lambda x = \frac{\lambda}{2}\left[e^{\lambda t}\int_t^\infty e^{-\lambda s}C(s)x\,ds - e^{-\lambda t}\int_{-t}^\infty e^{-\lambda s}C(s)x\,ds \right]$$

and

$$\begin{aligned}\frac{d^2}{dt^2}&C^{-1}C(t)L_\lambda x \\ &= \frac{\lambda^2}{2}\left[e^{\lambda t}\int_t^\infty e^{-\lambda s}C(s)x\,ds + e^{-\lambda t}\int_{-t}^\infty e^{-\lambda s}C(s)x\,ds \right] - \lambda C(t)x \\ &= \lambda^2 C^{-1}C(t)L_\lambda x - \lambda C(t)x.\end{aligned}$$

It follows that

$$\frac{d^2}{dt^2}C(t)L_\lambda x = C\frac{d^2}{dt^2}C^{-1}C(t)L_\lambda x = \lambda^2 C(t)L_\lambda x - \lambda CC(t)x$$

for all $t \in R$. In particular, $(d^2/dt^2)C(t)L_\lambda x\,|_{t=0} = C(\lambda^2 L_\lambda x - \lambda Cx)$. Thus $(\lambda^2 - A)L_\lambda x = \lambda Cx$.

Since the generator A is closed, its domain $D(A)$, equipped with the graph norm $|x|_A = \|x\| + \|Ax\|$, is a Banach space. We shall denote it by $[D(A)]$.

When $C(\cdot)$ is exponentially bounded, for large λ the set D_λ as defined in Lemma 1.3 is clearly equal to X. Then (ii) of Lemma 1.3 together with (1.11) yields the next lemma, which was also proved in [9, Proposition 3.2] by different methods.

LEMMA 1.4. *Let $C(\cdot)$ be an exponentially bounded C-cosine function satisfying condition* (1.3). *Then for each $\lambda > w$, $\lambda^2 - A$ is injective, $R(C) \subset R(\lambda^2 - A)$, $L_\lambda \in B(X)$, $R(L_\lambda) \subset D(A)$, and $L_\lambda(\lambda^2 - A) \subset (\lambda^2 - A)L_\lambda = \lambda C$.*

THEOREM 1.5. *Let A be the generator of a C-cosine function $\{C(t); t \in R\}$. The following assertions hold*

(i) *The function $u(t) := C(t)x + S(t)y$ is the unique solution of* ACP$(0; Cx, Cy)$ *for each pair x, y in $D(A)$.*

(ii) *The function $u(t; L_\lambda x, L_\lambda y) := C^{-1}C(t)L_\lambda x + C^{-1}S(t)L_\lambda y$ is the unique solution of* ACP$(0; L_\lambda x, L_\lambda y)$ *for each pair x, y in D_λ.*

Proof. (i) Set $u(t) = C(t)x + S(t)y$ for $x, y \in D(A)$ and $t \in R$. Then by (1.11), (1.12), and (1.15) we have $u(t) \in C^2(R, X) \cap C(R, [D(A)])$,

$$\frac{d^2}{dt^2}u(t) = \frac{d}{dt}(S(t)Ax + C(t)y) = C(t)Ax + S(t)Ay$$
$$= A(C(t)x + S(t)y) = Au(t),$$

$u(0) = Cx$ and $u'(0) = Cy$. Hence u is a solution of ACP$(0; Cx, Cy)$. To show that it is unique, let v be a solution of ACP$(0; Cx, Cy)$. Then $(d/dt)(C(s-t)v(t) + S(s-t)v'(t)) = 0$. Integrating this equality from 0 to s yields that $v(s) = C(s)x + S(s)y$ for each $s \in R$.

(ii) Using Lemma 1.3, (1.11), and (1.15) we have for $x \in D_\lambda$

$$C\big(C^{-1}C(t)L_\lambda x\big) = C(t)L_\lambda x \in D(A)$$

and

$$AC\big(C^{-1}C(t)L_\lambda x\big) = AC(t)L_\lambda x = \frac{d^2}{dt^2}C(t)L_\lambda x$$
$$= C\frac{d^2}{dt^2}C^{-1}C(t)L_\lambda x \in R(C).$$

This means that

$$C^{-1}C(t)L_\lambda x \in D(C^{-1}AC) = D(A)$$

and

$$\frac{d^2}{dt^2}C^{-1}C(t)L_\lambda x = C^{-1}AC\big(C^{-1}C(t)L_\lambda x\big) = AC^{-1}C(t)L_\lambda x.$$

Moreover,

$$\begin{aligned}\frac{d^2}{dt^2}C^{-1}S(t)L_\lambda y &= C^{-1}\frac{d^2}{dt^2}S(t)L_\lambda y = C^{-1}\frac{d}{dt}C(t)L_\lambda y\\ &= C^{-1}AS(t)L_\lambda y = (C^{-1}AC)C^{-1}S(t)L_\lambda y\\ &= A\big(C^{-1}S(t)L_\lambda y\big).\end{aligned}$$

Combining these facts we obtain that $u(0; L_\lambda x, L_\lambda y) = L_\lambda x$, $u'(0; L_\lambda x, L_\lambda y) = L_\lambda y$ and

$$\frac{d^2}{dt^2}u(t; L_\lambda x, L_\lambda y) = Au(t; L_\lambda x, L_\lambda y), \qquad t \in R,$$

if we put $u(t; L_\lambda x, L_\lambda y) = C^{-1}C(t)L_\lambda x + C^{-1}S(t)L_\lambda y$. Similarly we can prove that $u(t; L_\lambda x, L_\lambda y)$ is the unique solution of ACP$(0; L_\lambda x, L_\lambda y)$.

COROLLARY 1.6. *Let A be the generator of an exponentially bounded C-cosine function $\{C(t); t \in R\}$ with $\|C(t)\| \le Me^{w|t|}$ and let $\lambda > w$. Then for each pair (x, y) of elements of $(\lambda^2 - A)^{-1}C(X) = L_\lambda(X)$, $u(t; x, y) = C^{-1}(C(t)x + S(t)y)$ is a unique solution of* ACP$(0; x, y)$ *satisfying*

$$\begin{aligned}\|u(t; x, y)\| &\le M(\lambda - w)^{-1}e^{w|t|}\big(\|L_\lambda^{-1}x\| + |t|\|L_\lambda^{-1}y\|\big),\\ \|u''(t; x, y)\| &\le M(2\lambda^2 - \lambda w)(\lambda - w)^{-1}e^{w|t|}\big(\|L_\lambda^{-1}x\| + |t|\,\|L_\lambda^{-1}y\|\big),\\ & t \in R.\end{aligned} \tag{1.20}$$

Proof. The first part of the corollary is a direct consequence of Lemma 1.4 and Theorem 1.5. The estimate (1.20) follows from

$$\begin{aligned}u(t; x, y) &= e^{\lambda t}\frac{1}{2}\int_t^\infty e^{-\lambda s}C(s)\tilde{x}\,ds + e^{-\lambda t}\frac{1}{2}\int_{-t}^\infty e^{-\lambda s}C(s)\tilde{x}\,ds\\ &\quad + e^{\lambda t}\frac{1}{2}\int_t^\infty e^{-\lambda s}S(s)\tilde{x}\,ds - e^{-\lambda t}\frac{1}{2}\int_{-t}^\infty e^{-\lambda s}S(s)\tilde{x}\,ds,\end{aligned}$$

and

$$\frac{d^2}{dt^2}u(t; x, y) = \lambda^2 u(t; x, y) - \lambda\big(C(t)\tilde{x} + S(t)\tilde{y}\big),$$

where

$$x = \lambda(\lambda^2 - A)^{-1}C\tilde{x} = L_\lambda\tilde{x}, \qquad y = \lambda(\lambda^2 - A)^{-1}C\tilde{y} = L_\lambda\tilde{y}.$$

In Section 2 we shall need the following lemma which is taken from [15, Proposition 1.4].

LEMMA 1.7. *Let* $\lambda \in R$ *and let* A *be a closed linear operator satisfying*

(a) *for* $x \in D(A)$, $Cx \in D(A)$ *and* $ACx = CAx$,

(b) $\lambda - A$ *is injective and* $D((\lambda - A)^{-1}) \supset R(C)$.

Then we have

(i) $C(D(A)) \subset C(D(C^{-1}AC)) \subset (\lambda - A)^{-1}C(X)$,

(ii) $C(D(A)) = (\lambda - A)^{-1}C(X)$ *if and only if* $\lambda \in \rho(A)$,

(iii) $C^{-1}AC = A$ *if* $\rho(A) \neq \varnothing$.

2. THE ABSTRACT CAUCHY PROBLEM

In this section we give a characterization of the generator of a C-cosine function in terms of the unique existence of strong solution of the ACP. The main theorem is stated as follows:

THEOREM 2.1. *Let* $\lambda \in R$ *and let* A *be a closed linear operator satisfying*

(a) *for* $x \in D(A)$, $Cx \in D(A)$ *and* $ACx = CAx$,

(b) $\lambda - A$ *is injective and* $D((\lambda - A)^{-1}) \supset R(C)$,

(c) *for each* $x \in (\lambda - A)^{-1}C(X)$, *the* ACP$(0; x, 0)$ *has a unique solution*.

Then there exists a C*-cosine function* $C(\cdot)$ *on* X *with* $C^{-1}AC$ *as its generator*.

Proof. We denote the unique solution of ACP$(0; x, 0)$ by $u(t; x, 0)$ and define the operator $C(t)\colon X \to X$ for $t \in R$ by $C(t)x = (\lambda - A)u(t; (\lambda - A)^{-1}Cx, 0)$ for $x \in X$. Then $C(\cdot)x : R \to X$ is continuous for $x \in X$. Now, the uniqueness of the solution implies that $C(t)$ is linear, $Cu(t; (\lambda - A)^{-1}Cx, 0) = u(t; (\lambda - A)^{-1}C^2x, 0)$, $u(-s; (\lambda - A)^{-1}C^2x, 0) = u(s; (\lambda - A)^{-1}C^2x, 0)$, and $C(0) = C$. For $x \in X$ we define $v(t) = 2^{-1}[u(t + s; (\lambda - A)^{-1}C^2x, 0) + u(t - s; (\lambda - A)^{-1}C^2x, 0)]$. Then $v \in C^2(R, X) \cap C(R, [D(A)])$, $v''(t) = Av(t)$, and

$$v(0) = \frac{1}{2}\Big[u\big(s; (\lambda - A)^{-1}C^2x, 0\big) + u\big(-s; (\lambda - A)^{-1}C^2x, 0\big)\Big]$$

$$= u\big(s; (\lambda - A)^{-1}C^2x, 0\big).$$

We also have

$$v'(0) = \frac{1}{2}\Big[u'\big(s; (\lambda - A)^{-1}C^2x, 0\big) + u'\big(-s; (\lambda - A)^{-1}C^2x, 0\big)\Big] = 0.$$

The uniqueness of solution implies that

$$\begin{aligned} v(t) &= u\big(t; u\big(s; (\lambda - A)^{-1}C^2x, 0\big)\big) = u\big(t; Cu\big(s; (\lambda - A)^{-1}Cx, 0\big)\big) \\ &= u\big(t; (\lambda - A)^{-1}C(\lambda - A)u\big(s; (\lambda - A)^{-1}Cx\big), 0\big) \\ &= u\big(t; (\lambda - A)^{-1}CC(s)x, 0\big). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{2}C[C(t+s) + C(t-s)]x &= (\lambda - A)v(t) \\ &= (\lambda - A)u\big(t; (\lambda - A)^{-1}CC(s)x, 0\big) \\ &= C(t)C(s)x. \end{aligned}$$

Let $t_0 > 0$ be arbitrarily given and consider the linear map $\eta_{t_0}: X \to C([0, t_0], [D(A)])$ given by

$$\eta_{t_0}(x) = u\big(\cdot; (\lambda - A)^{-1}Cx, 0\big) = (\lambda - A)^{-1}C(\cdot)x.$$

We show that η_{t_0} is a closed linear operator. In fact, let $x_n \to x$ in X and $\eta_{t_0}(x_n) = u(\cdot; (\lambda - A)^{-1}Cx_n, 0) \to v$ in $C([0, t_0], [D(A)])$. Then $u(t; (\lambda - A)^{-1}Cx_n, 0) = (\lambda - A)^{-1}Cx_n + \int_0^t \int_0^s Au(r; (\lambda - A)^{-1}Cx_n, 0)\,dr\,ds$. Letting $n \to \infty$ we obtain $v(t) = (\lambda - A)^{-1}Cx + \int_0^t \int_0^s Av(r)\,dr\,ds$ for $0 \le t \le t_0$. Let $\tilde{v}(t) = Cv(|t|)$ for $|t| \le t_0$ and $\tilde{v}(t) = u(|t| - t_0; Cv(t_0), 0)$ for $|t| > t_0$. Then $\tilde{v}(t)$ is a solution of ACP$(0; (\lambda - A)^{-1}C^2x, 0)$. Therefore, from the uniqueness of solution it follows that for $0 \le t \le t_0$, $Cv(t) = \tilde{v}(t) = u(t; (\lambda - A)^{-1}C^2x, 0) = Cu(t; (\lambda - A)^{-1}Cx, 0)$, and so $v = u(\cdot; (\lambda - A)^{-1}Cx, 0) = \eta_{t_0}(x)$ on $[0, t_0]$. We have shown that η_{t_0} is closed. By the closed graph theorem, η_{t_0} is a bounded linear operator. So there exists an $M_{t_0} > 0$ such that $\sup_{0 \le t \le t_0} |\eta_{t_0}(x)(t)|_A \le M_{t_0}\|x\|$ for $x \in X$, and we see that for $0 \le t \le t_0$ and $x \in X$,

$$\begin{aligned} \|C(t)x\| &= \|(\lambda - A)u\big(t; (\lambda - A)^{-1}Cx, 0\big)\| = \|(\lambda - A)\eta_{t_0}(x)(t)\| \\ &\le (|\lambda| + 1)|\eta_{t_0}(x)(t)|_A \le (|\lambda| + 1)M_{t_0}\|x\| \end{aligned}$$

for all $t \in R$.

Having shown that $C(\cdot)$ is a C-cosine function on X, we next show that $C^{-1}AC$ is the generator of $C(\cdot)$. Let B be the generator of $C(\cdot)$ and let $x \in D(B)$. We have

$$\begin{aligned} &2t^{-2}(C(t)x - Cx) \\ &\quad = (\lambda - A)2t^{-2}\big[u\big(t; (\lambda - A)^{-1}Cx, 0\big) - u\big(0; (\lambda - A)^{-1}Cx, 0\big)\big] \to CBx \end{aligned}$$

and

$$2t^{-2}\Big[u\big(t;(\lambda-A)^{-1}Cx,0\big)-u\big(0;(\lambda-A)^{-1}Cx,0\big)\Big]$$
$$\to Au\big(0;(\lambda-A)^{-1}Cx,0\big)$$
$$=A(\lambda-A)^{-1}Cx=\lambda(\lambda-A)^{-1}Cx-Cx$$

as $t\to 0$. It follows from the closedness of A that $Cx\in D(A)$ and $ACx=(\lambda-A)[\lambda(\lambda-A)^{-1}Cx-Cx]=CBx\in R(C)$. That shows that $B\subset C^{-1}AC$.

To prove the converse, let $x\in D(C^{-1}AC)$. Then $Cx\in(\lambda-A)^{-1}C(X)$ (Lemma 1.7), and so $u(\cdot;Cx,0)$ as well as $u(\cdot;(\lambda-A)^{-1}Cx,0)$ is well defined and

$$\frac{d^2}{dt^2}\Big(\lambda u\big(t;(\lambda-A)^{-1}Cx,0\big)-u(t;Cx,0)\Big)$$
$$=A\Big(\lambda u\big(t;(\lambda-A)^{-1}Cx,0\big)-u(t;Cx,0)\Big).$$

Let $v(t)=(\lambda-A)^{-1}Cx+\int_0^t\int_0^s(\lambda u(r;(\lambda-A)^{-1}Cx,0)-u(r;Cx,0))\,dr\,ds$, $t\in R$. Then we have $v(0)=(\lambda-A)^{-1}Cx$, $v'(0)=0$, and

$$\begin{aligned}Av(t)&=A(\lambda-A)^{-1}Cx\\&\quad+\int_0^t\int_0^s A\Big(\lambda u\big(r;(\lambda-A)^{-1}Cx,0\big)-u(r;Cx,0)\Big)\,dr\,ds\\&=A(\lambda-A)^{-1}Cx+\Big[\lambda u\big(t;(\lambda-A)^{-1}Cx,0\big)\\&\qquad-u(t;Cx,0)\Big]-\Big[\lambda(\lambda-A)^{-1}Cx-Cx\Big]\\&=\lambda u\big(t;(\lambda-A)^{-1}Cx,0\big)-u(t;Cx,0)\\&=v''(t)\end{aligned}$$

for all $t\in R$. Hence v is identical to the unique solution $u(\cdot;(\lambda-A)^{-1}Cx,0)$ of ACP$(0;(\lambda-A)^{-1}Cx,0)$, and we have

$$C(t)x=(\lambda-A)u\big(t;(\lambda-A)^{-1}Cx,0\big)=\lambda v(t)-Av(t)=u(t;Cx,0).$$

It follows that as $t\to 0$

$$2t^{-2}(C(t)x-Cx)=2t^{-2}(u(t;Cx,0)-u(0;Cx,0))$$
$$\to Au(0;Cx,0)=ACx\in R(C).$$

This shows that $x\in D(B)$ and $Bx=C^{-1}ACx$, and so $C^{-1}AC$ is the generator of $C(\cdot)$.

COROLLARY 2.2. *Let A be a closed linear operator with nonempty resolvent set. Then the following are equivalent*:

(i) *A is the generator of a C-cosine function.*

(ii) *A satisfies condition* (a) *and* ACP$(0; Cx, Cy)$ *has a unique solution for every $x, y \in D(A)$.*

(iii) *A satisfies condition* (a) *and* ACP$(0; Cx, 0)$ *has a unique solution for each $x \in D(A)$.*

Proof. The implication (i) ⇒ (ii) is a direct consequence of Proposition 1.2 and Theorem 1.5(i); (ii) ⇒ (iii) is obvious. We show the implication (iii) ⇒ (i). Since $C(D(A)) = (\lambda - A)^{-1}C(X)$ for $\lambda \in \rho(A)$ and $C^{-1}AC = A$ (Lemma 1.7(ii), (iii)), it follows from Theorem 2.1 that A is the generator of a C-cosine function.

A characterization of exponentially bounded C-cosine functions in terms of the ACP is given by

THEOREM 2.3. *Let A be a closed linear operator in X. Then the following are equivalent*:

(i) *A is the generator of an exponentially bounded C-cosine function $C(\cdot)$ with $\|C(t)\| \le Me^{wt}$ for $t \ge 0$.*

(ii) *A satisfies conditions*

(a*) $C^{-1}AC = A$,

(b) *for some $\lambda \ge 0$, $\lambda^2 - A$ is injective and $D((\lambda^2 - A)^{-1}) \supset R(C)$,*

(c*) *for every $x, y \in (\lambda^2 - A)^{-1}C(X)$, the* ACP$(0; x, y)$ *has a unique solution $u(t; x, y)$ such that $u(t; x, 0)$ and $(d^2/dt^2)u(t; x, 0)$ are of order $O(e^{w|t|})$ as $|t| \to \infty$.*

(iii) *A satisfies* (a*), (b) *and*

(c′) *for every $x \in (\lambda^2 - A)^{-1}C(X)$, the* ACP$(0; x, 0)$ *has a unique solution $u(t; x, 0)$ such that $u(t; x, 0)$ and $(d^2/dt^2)u(t; x, 0)$ are of order $O(e^{w|t|})$ as $|t| \to \infty$.*

Proof. The implication (i) ⇒ (ii) follows from Corollary 1.6 and (1.18); (ii) ⇒ (iii) is obvious. To show the implication (iii) ⇒ (i), by Theorem 2.1, it suffices to show that the family $\{C(t); t \in R\}$ defined by $C(t)x = (\lambda - A)u(t, (\lambda - A)^{-1}Cx)$ is exponentially bounded. From the condition (c′) we deduce that $\|e^{-w|t|}C(t)x\| < \infty$. Thus, the uniform boundedness principle implies that $\sup_{t \in R}\|e^{-w|t|}C(t)\| < \infty$.

In the case where A has nonempty resolvent set and (a) holds, the conditions (a*) and (b) are automatically satisfied. Thus we deduce the following corollary.

COROLLARY 2.4. *Let A be a closed linear operator with nonempty resolvent set. Then the following are equivalent*:

(i) *A is the generator of an exponentially bounded C-cosine function $C(\cdot)$ with $\|C(t)\| \le Me^{w|t|}$ for $t \in R$.*

(ii) *A satisfies condition* (a) *of Theorem* 2.1, *and for every $x, y \in D(A)$, the* ACP$(0; Cx, Cy)$ *has a unique solution $u(t; Cx, Cy)$ such that $\|u(t; Cx, 0)\| = O(e^{w|t|})$ and $\|(d^2/dt^2)u(t; Cx, 0)\| = O(e^{w|t|})$ as $|t| \to \infty$.*

(iii) *A satisfies condition* (a) *of Theorem* 2.1, *and for every $x \in D(A)$ the* ACP$(0; Cx, 0)$ *has a unique solution $u(t; Cx, 0)$ such that $\|u(t; Cx, 0)\| = O(e^{w|t|})$ and $\|(d^2/dt^2)u(t; Cx, 0)\| = O(e^{w|t|})$ as $|t| \to \infty$.*

Remark. In the special case that the operator C is the identity operator I, Theorem 2.1, Corollary 2.2, Theorem 2.3, and Corollary 2.4 all coincide, and we obtain Fattorini's theorem [4, 5].

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