JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 103, 134–138 (1984)

Successive Projections on Hyperplanes

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Any sequence of points in R^n obtained by successive projections of a point on elements of a finite set of hyperplanes is bounded.

INTRODUCTION AND PRELIMINARIES

There exists a family of methods for solving systems of equations by iterative projections of a point on sets (usually convex) in R^n (see, e.g., [1–3]). Often the first step in proving the convergence of the iterations is showing that they are bounded. This motivates the main theorem of this paper, which states that if the given sets are hyperplanes and their number is finite then the sequence of iterations is bounded regardless of the order in which the projections are chosen.

In this paper "hyperplane" means any translate of a proper subspace of R^n, i.e., it is not necessarily of co-dimension 1. If H is a family of hyperplanes we denote by $\Pi(\mathrm{H})$ the set of projections on members of H. A sequence $(x_k\colon 0 \leqslant k < \alpha)$ of points in R^n, where $0 \leqslant \alpha \leqslant \omega$, is called a *sequence of projections on* H if $x_{k+1} = p(x_k)$ for some $p \in \Pi(\mathrm{H})$ whenever $k+1 < \alpha$. The main result of this paper is:

THEOREM. *If* H *is a finite family of hyperplanes in* R^n *then any sequence of projections on* H *is bounded.*

* The two first named authors were partly supported by a Canadian SERC Grant at the University of Waterloo.

0022-247X/84 $3.00

It so turns out that it is more convenient to prove the conclusion of Theorem 1 for a more general class of families of hyperplanes, families which we name "quasi-finite." A family H of hyperplanes in R^n is called *quasi-finite* if there exist a finite set I and subspaces V_i and bounded subsets B_i of R^n for each $i \in I$ such that, denoting $\mathrm{H}_i = \{V_i + b\colon b \in B_i\}$, there holds: $\mathrm{H} = \bigcup\{\mathrm{H}_i\colon i \in I\}$. A family H of hyperplanes is *complete* if the intersection of any two members of H belongs to H.

We denote by $\bar{0}$ the point $(0, 0,..., 0)$ in R^n. If H is a hyperplane we write $H^{\perp}$ for the orthogonal subspace to H. If C is a bounded subset of R^n we write

$$r(C) = \sup\{|x|\colon x \in C\}.$$

We shall prove the following stronger version of Theorem 1.

THEOREM 2. *Let* H *be a quasi-finite family of hyperplanes in* R^n. *Then there exists* $r \geqslant 0$ *such that for any sequence* $(x_k\colon k < \omega)$ *of projections on* H *there holds*

$$|x_k| \leqslant |x_0| + r \tag{1}$$

for every $k < \omega$.

2. PROOF OF THEOREM 2

The family $\tilde{\mathrm{H}} = \{\bigcap \mathrm{F}\colon \varnothing \neq \mathrm{F} \subset \mathrm{H}\}$ is quasi-finite and complete. Since $\tilde{\mathrm{H}} \supset \mathrm{H}$ it suffices to prove Theorem 2 replacing H by $\tilde{\mathrm{H}}$, and hence we may assume that H itself is complete.

The proof of the theorem is by induction on n, and thus we are assuming that the theorem holds in R^k for every $k < n$.

LEMMA 1. *Let* H *be a family of hyperplanes in* R^k *and suppose that the conclusion of Theorem* 2 *holds for* H *for some* $r \geqslant 0$. *Let* C *be a bounded set in* R^k. *There exists then a bounded set* D *in* R^k *containing* C *such that* $p(D) \subset D$ *for every* $p \in \Pi(\mathrm{H})$.

Proof. Take $D = \bigcup\{\{x_j : j < \omega\} : (x_j : j < \omega)$ is a sequence of projections on H and $x_0 \in C\}$.

By the induction hypothesis we may assume that the conclusion of Lemma 1 holds for any quasi-finite family of hyperplanes in R^k for every $k < n$. We write $\mathrm{H} = \bigcup\{\mathrm{H}_i : i \in I\}$, where I is finite, $\mathrm{H}_i = \{V_i + b\colon b \in B_i\}$, V_i is a proper subspace of R^n and B_i is a bounded set in R^n for each $i \in I$. For each $i \in I$ let $S_i = \bigcup \mathrm{H}_i$. We define by induction on the co-dimension of V_i subsets C_i, D_i and E_i of R^n. If dim $V_i = n - 1$ let $C_i = D_i = S_i$ and

$E_i = V_i + \bar{E}_i$, where $\bar{E}_i$ is a ball centered at $\bar{0}$ in R^n large enough to satisfy $E_i \supset D_i$. Suppose now that $k < n-1$ and that C_i, D_i and E_i have been defined whenever dim $V_i > k$. Let dim $V_j = k$ for some $j \in I$. Let $U_j = V_j^{\perp}$. For each $i \in I$ such that $V_i \supset V_j$ we define $\mathrm{H}_{ji} = \{H \cap U_j : H \in \mathrm{H}_i\}$ and $E_{ji} = E_i \cap U_j$. Let C_j be a bounded subset of U_j containing $E_{ji} \cap E_{jm}$ whenever $E_{ji} \cap E_{jm}$ is bounded. The set $\mathrm{F}_j = \bigcup\{\mathrm{H}_{ji} : V_i \supset V_j\}$ is a quasi finite family of hyperplanes in U. Hence, by the induction hypothesis there exists a subset $\bar{D}_j$ of U_j containing C_j and satisfying the conclusion of Lemma 1 with respect to F_j. Let $\bar{E}_j$ be a ball in U_j centered at $\bar{0}$ and containing $\bar{D}_j$. Finally, let

$$D_j = V_j + \bar{D}_j \qquad \text{and} \qquad E_j = V_j + \bar{E}_j.$$

From the construction there readily follows:

$$p(D_j) \subset D_j \qquad \text{whenever} \qquad V_i \supset V_j \text{ and } p \in \Pi(\mathrm{H}_i) \tag{2}$$

Let B be a ball in R^n centered at $\bar{0}$ which contains $E_i \cap E_j$ whenever $V_i \cap V_j = \{\bar{0}\}$. By the induction hypothesis there exist numbers $r_j \geqslant 0$ for each $j \in I$ such that for any sequence $(y_k : k < \omega)$ of projections in U_j on F_j there holds

$$|y_k| \leqslant |y_0| + r_j \qquad \text{for every } k < \omega. \tag{3}$$

Define

$$s = \max\{r_i : i \in I\}, \qquad r = 2s + r(B). \tag{4}$$

LEMMA 2. *If* $x \notin B$ *and* $x \in E_i \cap E_j$ *then either* $V_i \subset V_j$ *or* $x \in D_k$ *for some* k *such that* dim $V_k <$ dim V_i.

Proof. Since B contains $E_i \cap E_j$ whenever $V_i \cap V_j = \{\bar{0}\}$ it follows that $\dim(V_i \cap V_j) > 0$. Let $V_k = V_i \cap V_j$. By its definition, D_k contains $E_i \cap E_j$ and hence $x \in D_k$.

For any $x \in \bigcup\{S_i : i \in I\}$ define: $n(x) = \min\{\dim V_j : x \in D_j\}$. By Lemma 2 there exists a unique j such that $x \in D_j$ and dim $V_j = n(x)$. We write then $V_j = V(x)$, $D_j = D(x)$ and $E_j = E(x)$.

LEMMA 3. *If* $x \notin B$ *and* $x \in E_j$ *then* $V(x) \subset V_j$.

Proof. Since $x \in D(x) \subset E(x)$ it follows that $x \in E(x) \cap E_j$. By Lemma 2 either $V(x) \subset V_j$ or $x \in D_k$ for some k such that dim $V_k <$ dim V_j. But the second of these possibilities contradicts the definition of $V(x)$, and hence $V(x) \subset V_j$.

DEFINITION. Let $(x_k : 0 \leqslant k < \alpha)$, where $\alpha \leqslant \omega$, be a sequence of projections on H. We say that x_k is a *turning point* of the sequence if $x_{k+1} \notin D(x_k)$.

LEMMA 4. *If* $(x_k : 0 \leqslant k < \alpha)$ *is a sequence of projections on* H *and* $x_m \notin B$ *and* x_m *is not a turning point of the sequence then* $V(x_m) \supset V(x_{m+1})$.

Proof. This follows directly from Lemma 3 and the definition of $V(x_m)$.

LEMMA 5. *Let* $(x_k : k < \alpha)$, $\alpha \leqslant \omega$, *be a sequence of projections on* H *disjoint from* B. *Suppose that* $i < j < \alpha$ *and that* x_i *is a turning point of the sequence and that there are no turning points between* x_i *and* x_j. *Then* $|x_i| > |x_j|$.

Proof. For each k such that $k+1 < \alpha$ let p_k be the projection for which $x_{k+1} = p_k(x_k)$ and let $m(k) \in I$ be such that $p_k \in \Pi(\mathrm{H}_{m(k)})$. By Lemma 4 it follows from the facts that $x_k \notin B$ and x_k is not a turning point for every $i < k \leqslant j$, that $V(x_k) \supset V(x_{k+1})$ for each such k. Therefore:

$$V(x_{i+1}) \supset V(x_{i+2}) \supset \cdots \supset V(x_j). \tag{5}$$

For each $i+1 \leqslant k < j$, since $x_k \notin B$ we have by Lemma 3:

$$V_{m(k)} \supset V(x_k) \tag{6}$$

Suppose, if possible, that $x_i \in E(x_j)$. Then, by (6) and (2), $x_{i+1} = p_i(x_i) \in p_i(D(x_i)) \subset D(x_i)$, contradicting the assumption that x_i is a turning point. We conclude, therefore, that

$$x_i \notin E(x_j). \tag{7}$$

By (5) and (6) each projection p_k is perpendicular to $V(x_j)$ and hence $x_i, x_{i+1}, \ldots, x_j$ all lie in one hyperplane orthogonal to $V(x_j)$. By the definition of $E(x_j)$ as $V(x_j) + \bar{E}$, where $\bar{E}$ is a ball centered at $\bar{0}$ in $V(x_j)^{\perp}$, and since $x_j \in E(x_j)$, it follows from (7) that $|x_j| < |x_i|$.

Lemma 5 implies:

LEMMA 6. *If* $(x_k : k < \alpha)$, $\alpha < \omega$, *is a sequence of projections on* H *disjoint from* B *and* x_i *is a turning point of the sequence then* $|x_j| < |x_i|$ *for each* $j > i$.

LEMMA 7. *If* $(x_k : k < \alpha)$, $\alpha \leqslant \omega$ *is a sequence of projections on* H *disjoint from* B *then* $|x_k| \leqslant |x_0| + s$ *for any* $k < \alpha$.

Proof. If $0 \leqslant k < \alpha$ is such that x_j is not a turning point for any $0 \leqslant j < k$ then, by Lemma 4, $V(x_0) \supset V(x_1) \supset \cdots \supset V(x_k)$. Hence the points $x_0, x_1, \ldots, x_k$ lie all in the same hyperplane orthogonal to $V(x_k)$. Let

$m \in I$ be such that $V(x_k) = V_m$. Then, by the definition of r_m, $|x_j| \leqslant |x_0| + r_m \leqslant |x_0| + s$ for every $j \leqslant k$. If there exists $j \leqslant k$ such that x_j is a turning point then choose j to be the first such index. By the above $|x_j| \leqslant |x_0| + s$, and by Lemma 6 $|x_k| \leqslant |x_j|$.

LEMMA 8. *If $p \in \Pi(\mathrm{H})$ and $y = p(x)$ then $|y| < |x| + s$.*

Proof. Let $V_m = V(y)$. If $|y| > |x|$ then there must hold $p \in \Pi(\mathrm{H}_m)$ and $x \in E_m$. Hence, by the definition of r_m we have $|y| < |x| + r_m$.

We can now complete the proof of Theorem 2. Let $(x_k : k < \omega)$ be a sequence of projections on H. If k is such that $x_j \notin B$ for any $0 \leqslant j \leqslant k$ then (2) holds for k by Lemma 7. If there exists $j \leqslant k$ such that $x_j \in B$ then let j be the last such index. By Lemma 8 $|x_{j+1}| \leqslant |x_j| + s$. By Lemma 7 $|x_k| \leqslant |x_{j+1}| + s$ and hence $|x_k| \leqslant |x_j| + 2s \leqslant r(B) + 2s$, and thus (2) holds for x_k also in this case.

3. A POSSIBLE STRENGTHENING

Quasi-finiteness is by no means a necessary condition for a family of hyperplanes to satisfy the conclusion of Theorems 1 or 2. For example any family of hyperplanes which pass all through a common point satisfies this property. The authors think that the "true" condition is that of a certain boundedness of the intersections of the hyperplanes in the family. This is made precise in the following

Conjecture. Let H be a family of hyperplanes and let $K = \bigcap\{H^{\perp\perp} : H \in \mathrm{H}\}$. (i.e., the maximal subspace parallel to all members of H). Then every sequence of projections on H is bounded if and only if the set

$$\{x \in K^{\perp} : \{x\} = \bigcap \mathrm{F} \cap K^{\perp} \text{ for some } \mathrm{F} \subset \mathrm{H}\}$$

is bounded.

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