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Oscillations of First-Order Neutral Delay Differential Equations

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Consider the neutral delay differential equation $(*)$ $(d/dt)[y(t)+py(t-\tau)]+qy(t-\sigma)=0$, $t\geqslant t_0$ where τ, q, and σ are positive constants, while $p\in(-\infty,-1)\cup(0,+\infty)$. (For the case $p\in[-1,0]$ see Ladas and Sficas, Oscillations of neutral delay differential equations (to appear)). The following results are then proved. Theorem 1. Assume $p<-1$. Then every nonoscillatory solution $y(t)$ of Eq. $(*)$ tends to $\pm\infty$ as $t\to\infty$. Theorem 2. Assume $p<-1$, $\tau>\sigma$, and $q(\sigma-\tau)/(1+p)>(1/e)$. Then every solution of Eq. $(*)$ oscillates. Theorem 3. Assume $p>0$. Then every nonoscillatory solution $y(t)$ of Eq. $(*)$ tends to zero as $t\to\infty$. Theorem 4. Assume $p>0$. Then a necessary condition for all solutions of Eq. $(*)$ to oscillate is that $\sigma>\tau$. Theorem 5. Assume $p>0$, $\sigma>\tau$, and $q(\sigma-\tau)/(1+p)>(1/e)$. Then every solution of Eq. $(*)$ oscillates. Extensions of these results to equations with variable coefficients are also obtained. © 1986 Academic Press, Inc.

1. INTRODUCTION

In a recent paper, Ladas and Sficas [9] studied the oscillatory behavior of solutions of neutral delay differential equations (NDDE) of the form

$$\frac{d}{dt}[y(t)+py(t-\tau)]+Q(t)\,y(t-\sigma)=0, \qquad t\geqslant t_0 \tag{1}$$

and the asymptotic behavior of nonoscillatory solutions in the case where τ and σ are positive constants, $Q\in C([t_0,\infty),\mathbb{R}^+)$ and the real parameter p lies in $[-1,0]$.

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In this paper, we investigate the behavior of the solutions of Eq. (1) for all other possible values of p, namely, for $p < -1$ and for $p > 0$. For the sake of simplicity, in Section 2 we state the asymptotic and oscillatory behavior of solutions of the NDDE (1) with $Q(t) \equiv q > 0$ constant, that is of the NDDE

$$\frac{d}{dt}[y(t) + py(t-\tau)] + qy(t-\sigma) = 0, \quad t \geqslant t_0. \tag{2}$$

In Section 3 we prove extensions of these results in the case where $Q \in C([t_0, \infty), \mathbb{R}^+)$.

Although the oscillatory theory of delay differential equations has been extensively developed during the last few years (see, for example, [1, 7, 8, 10–13]), there is hardly any work at this time (except for [9]) dealing with the oscillatory behavior of solutions of neutral delay differential equations. (For some results on second-order NDDE, see [14].) The problem of asymptotic and oscillatory behavior or solutions of neutral delay differential equations is of both theoretical and practical interest. It suffices to note that equations of this type appear in the study of networks containing lossless transmission lines. Such networks arise, for example, in high-speed computers where lossless transmission lines are used to interconnect switching circuits (see [3]).

Let $\phi \in C([t_0 - \rho, t_0], \mathbb{R})$, where $\rho = \max\{\tau, \sigma\}$. By a solution of Eq. (1), we mean a function $y \in C[t_0 - \rho, \infty), \mathbb{R})$ such that $y(t) = \phi(t)$ for $t_0 - \rho \leqslant t \leqslant t_0$, $y(t) + py(t-\tau)$ is continuously differentiable, and $y(t)$ satisfies Eq. (1) for all $t \geqslant t_0$.

Using the method of steps, it follows that for every continuous funtion ϕ, there exists a unique solution of Eq. (1) valid for $t \geqslant t_0$. For further questions on existence, uniqueness, and continuous dependence, see Driver [4, 5], Bellman and Cooke [2], and Hale [6].

As is customary, a solution of Eq. (1) is called *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or eventually negative.

2. Constant Coefficients

In this section, we shall state without proof our results in the case $Q(t) \equiv q$ is a positive constant. We shall also give several examples. Proofs of extensions of these results are given in Section 3. A schematic summary of the oscillatory and asymptotic results for Eq. (2), including the results of Ladas and Sficas [9], is given at the end of this section.

A. *The Case* $p < -1$

Consider the characteristic equation of Eq. (2), that is,

$$F(\lambda) = \lambda + \lambda p e^{-\lambda\tau} + q e^{-\lambda\sigma} = 0$$

where q, τ, and σ are positive constants and $p < -1$. For $\lambda \leqslant 0$ we have

$$F(\lambda) = -\lambda(-pe^{-\lambda\tau} - 1) + qe^{-\lambda\sigma} > 0.$$

Thus, the only real roots that the characteristic equation of Eq. (2) can have are positive. This observation provides the motivation for the following result, concerning the asymptotic behavior of nonoscillatory solutions.

THEOREM 1. *Consider the* NDDE (2). *Assume that* q, τ, *and* σ *are positive constants and* $p < -1$. *Then every nonoscillatory solution of Eq.* (2) *tends to* $+\infty$ *or* $-\infty$ *as* $t \to \infty$.

The next result provides sufficient conditions for oscillation of all solutions of Eq. (2).

THEOREM 2. *Consider the NDDE* (2). *Let* q, τ, *and* σ *be positive constants. Assume that* $p < -1$, $\tau > \sigma$, *and that*

(C_1) $q(\sigma - \tau)/(p+1) > 1/e.$

Then every solution of Eq. (2) oscillates.

B. *The Case* $p > 0$

As in Theorem 1, Theorem 3 is motivated by the characteristic equation. In this case with p, q, σ, and τ positive constants, the characteristic equation

$$F(\lambda) = \lambda + \lambda p e^{-\lambda\tau} + q e^{\lambda\sigma} = 0$$

cannot have nonnegative real zeros.

THEOREM 3. *Consider the NDDE* (2). *Assume that* p, q, σ, *and* τ *are positive constants. Then every nonoscillatory solution of Eq.* (2) *tends to zero as* $t \to \infty$.

If $\tau \geqslant \sigma > 0$, then Eq. (2) always has nonoscillatory solutions. Indeed, the characteristic equation of Eq. (2) is

$$F(\lambda) = \lambda + \lambda p e^{-\lambda\tau} + q e^{-\lambda\sigma} = 0$$

and $F(0) = q > 0$, while $\lim_{\lambda \to -\infty} F(\lambda) = -\infty$.

Thus, we have the following theorem.

THEOREM 4. *Consider the NDDE* (2). *Assume that* p, q, τ, *and* σ *are positive constants. Then a necessary condition for all solutions of Eq.* (2) *to oscillate is that*

$$\sigma > \tau.$$

As in the case of Theorem 2, the following result provides sufficient conditions for the oscillation of all solutions of Eq. (2).

THEOREM 5. *Consider the NDDE* (2). *Assume that* p, q, τ, *and* σ *are positive constants,* $\sigma > \tau$, *and that*

(C_1) $q(\sigma - \tau)/(1+p) > 1/e$.

Then every solution of Eq. (2) oscillates.

C. *Examples*

The following two examples illustrate the asymptotic behavior of nonoscillatory solutions.

EXAMPLE 1. The NDDE

$$\frac{d}{dt}[y(t) - 2ey(t-1)] + e^{1/2} y\left(t - \frac{1}{2}\right) = 0, \qquad t \geqslant 0 \tag{3}$$

satisfies the hypotheses of Theorem 1, and therefore every nonoscillatory solution of Eq. (3) tends to $+\infty$ or $-\infty$ as $t \to \infty$. For example, $y(t) = e^t$ and $y(t) = -e^t$ are such solutions.

EXAMPLE 2. The NDDE

$$\frac{d}{dt}\left[y(t) + \left(e - \frac{1}{e}\right) y(t-1)\right] + y(t-2) = 0, \qquad t \geqslant 0 \tag{4}$$

satisfies the hypotheses of Theorem 3, and hence every nonoscillatory solution of Eq. (4) tends to zero as $t \to \infty$. For example, $y(t) = e^{-t}$ is such a solution.

In the paper by Ladas and Sficas [9], it was shown that if $p \in [-1, 0]$, then $q\sigma > 1/e$ implies that every solution of Eq. (2) oscillates. Examples 1 and 2 also illustrate the fact that in the case $p < -1$ and $p > 0$, the condition $q\sigma > 1/e$ is not sufficient to imply that every solution of Eq. (2) oscillates.

EXAMPLE 3. The NDDE

$$\frac{d}{dt}[y(t) + py(t-2\pi)] + (-p-1)\, y\left(t - \frac{3\pi}{2}\right) = 0, \qquad t \geqslant 0 \tag{5}$$

where $p < -1$, satisfies the hypotheses of Theorem 2, and so every solution of Eq. (5) oscillates. For example, $y(t) = \sin t$ is such a solution.

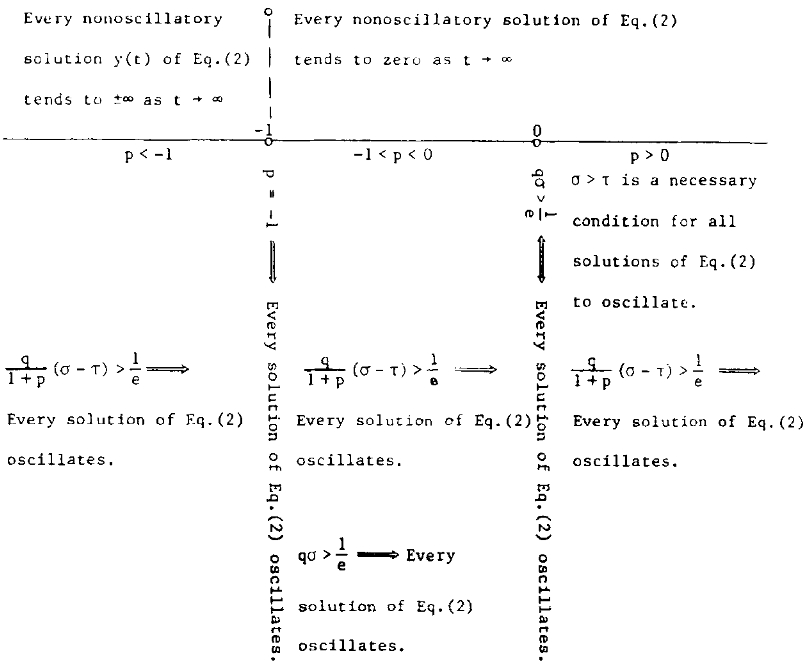
EXAMPLE 4. The NDDE

$$\frac{d}{dt}[y(t) + py(t - 2\pi)] + e^{3\pi/4}(e^{2\pi} + p)\sqrt{2}y\left(t - \frac{11}{4}\pi\right) = 0, t \geq 0 \quad (6)$$

where $p > 0$, satisfies the hypotheses of Theorem 5. Hence every solution of Eq. (6) is oscillatory. For example, $y(t) = e^t \sin t$ is such a solution.

D. *A Schematic Summary of the Oscillation and asymptotic Results of* NDDE(2)

See Scheme 1.



SCHEME 1.

3. VARIABLE COEFFICIENTS

In this section we assume that τ and σ are positive constants and $Q \in C([t_0, \infty), \mathbb{R})$. We assume further that there exists $q > 0$ so that

$Q(t) \geqslant q$ eventually. We shall study the neutral delay differential equation

$$\frac{d}{dt}[y(t)+py(t-\tau)]+Q(t)\,y(t-\sigma)=0, \qquad t \geqslant t_0 \tag{1}$$

where the real parameter p lies in $(-\infty, -1) \cup (0, +\infty)$.

The results of Section 2 will be immediate consequences of the results of this section.

The following lemma is an improved version of a result due to Ladas and Stavroulakis [11] concerning delay differential inequalities. The improvement is due to Koplatadze and Chanturia [8], who removed one of the hypotheses. For advanced inequalities see Onose [12].

LEMMA 1. *Assume that μ is a positive constant. Let $p \in C([t_0, \infty), \mathbb{R}^+)$, and suppose that*

$$\liminf_{t \to \infty} \int_{t-\mu}^{t} p(s)\,ds > \frac{1}{e}.$$

Then:

(i) the inequality

$$x'(t) - p(t)\,x(t+\mu) \leqslant 0, \qquad t \geqslant t_0$$

has no eventually negative solutions;

(ii) the inequality

$$x'(t) - p(t)\,x(t+\mu) \geqslant 0, \qquad t \geqslant t_0$$

has no eventually positive solutions;

(iii) the inequality

$$x'(t) + p(t)\,x(t-\mu) \leqslant 0, \qquad t \geqslant t_0$$

has no eventually positive solutions;

(iv) the inequality

$$x'(t) + p(t)\,x(t-\mu) \geqslant 0, \qquad t \geqslant t_0$$

has no eventually negative solutions.

A. *The Case $p < -1$*

The following theorem is the variable coefficient analogue for Theorem 1.

THEOREM 6. *Consider the NDDE* (1). *If* $p<-1$, *then every nonoscillatory solution of Eq.* (1) *tends to* $+\infty$ *or* $-\infty$ *as* $t\to\infty$.

Proof. Since the negative of a solution of Eq. (1) is again a solution of Eq. (1), it suffices to prove the theorem in the case of an eventually positive solution. So, suppose that $y(t)$ is an eventually positive solution of Eq. (1). Set

$$z(t)=y(t)+py(t-\tau). \tag{7}$$

Then

$$z'(t)=-Q(t)\,y(t-\sigma) \tag{8}$$

and eventually $z'(t)<0$.

We wish to show that $z(t)$ is eventually negative. Suppose, for the sake of contradiction, that eventually $z(t)\geqslant 0$. Then from (7) it follows that eventually

$$-py(t-\tau)\leqslant y(t)$$

which implies that

$$0<y(t)\leqslant\left(\frac{1}{-p}\right)^n y(t+n\tau),\qquad n=1,2,...,$$

eventually. So, as $-p>1$, we see that $y(t)\to+\infty$ as $t\to\infty$. But

$$z'(t)=-Q(t)\,y(t-\sigma)\leqslant -qy(t-\sigma)$$

and so $z'(t)\to-\infty$ as $t\to\infty$ which contradicts the assumption that eventually $z(t)\geqslant 0$.

Thus, we see that eventually $z(t)<0$. Since $z'(t)<0$ eventually, we have

$$0>\lim_{t\to\infty} z(t)=l\geqslant-\infty.$$

We wish to show that $l=-\infty$, and so shall assume, for the sake of contradiction, that $l>-\infty$. Then, by integrating (8) from t_0 to t, we find

$$z(t)-z(t_0)+\int_{t_0}^{t} Q(s)\,y(s-\sigma)\,ds=0$$

and so

$$\int_{t_0}^{+\infty} Q(s)\,y(s-\sigma)\,ds=z(t_0)-l.$$

So, since $Q(t) \geqslant q > 0$ eventually, we see that $y \in L_1(t_0, \infty)$. Thus, from (7) we conclude that $z \in L_1(t_0, \infty)$. This implies that $l = 0$ which is impossible. Hence we have that $\lim_{t\to\infty} z(t) = -\infty$, and as eventually $z(t) > py(t-\tau)$, we conclude that $\lim_{t\to\infty} y(t) = +\infty$. The proof is complete.

The next two theorems provide sufficient conditions for all solutions of Eq. (1) to oscillate.

THEOREM 7. *Consider the NDDE* (1). *Assume that* $p < -1$, $\tau > \sigma$, *and that*

$$(C_2) \qquad -\frac{1}{p} \liminf_{t\to\infty} \int_t^{t+(\tau-\sigma)} Q(s)\, ds > \frac{1}{e}.$$

Then every solution of Eq. (1) *oscillates.*

Proof. Othewise there is an eventually positive solution $y(t)$ of Eq. (1). Set

$$z(t) = y(t) + py(t-\tau).$$

Then, as in the proof of Theorem 6, it follows that eventually $z(t) < 0$, $z'(t) < 0$, and that

$$z(t) > py(t-\tau).$$

From this last inequality, we find that eventually

$$-\frac{1}{p} Q(t)\, z(t+(\tau-\sigma)) > -Q(t)\, y(t-\sigma) = z'(t)$$

and hence

$$z'(t) - \left(\frac{1}{-p}\right) Q(t)\, z(t+(\tau-\sigma)) < 0.$$

But, by (C_2) and Lemma 1 (i), it is impossible for this inequality to have an eventually negative solution. The proof is complete.

THEOREM 8. *Consider the NDDE* (1). *Assume that* $p < -1$, $\tau > \sigma$, *and* Q *is periodic with period* τ. *Finally, suppose that*

$$(C_3) \qquad -\frac{1}{1+p} \liminf_{t\to\infty} \int_t^{t+(\tau-\sigma)} Q(s)\, ds > \frac{1}{e}.$$

Then every solution of Eq. (1) *oscillates.*

Proof. Suppose, for the sake of contradiction, that there is an eventually positive solution $y(t)$. Set

$$z(t) = y(t) + py(t-\tau)$$

and

$$w(t) = z(t) + pz(t-\tau).$$

Then, since Q is periodic with period τ, it is easy to see that z and w are also solutions to Eq. (1).

As in the proof of Theorem 6, we eventually have $z(t) < 0$ and $z'(t) < 0$. The same argument, when applied to $-z(t)$, implies that eventually $w(t) > 0$ and $w'(t) > 0$. Hence

$$w(t) = z(t) + pz(t-\tau) < (1+p)\, z(t-\tau)$$

eventually, and so

$$-\frac{1}{1+p} Q(t)\, w(t + (\tau - \sigma)) \leqslant -Q(t)\, z(t-\sigma) = w'(t)$$

from which we find that eventually

$$w'(t) - \left(-\frac{1}{1+p}\right) Q(t)\, w(t + (\tau - \sigma)) \geqslant 0.$$

But, by (C_3) and Lemma 1 (ii), this is impossible.

The proof of the theorem is complete.

B. *The Case* $p > 0$

The following two theorems are the variable coefficient analogues for Theorems 3 and 5, respectively.

THEOREM 9. *Consider the NDDE* (1). *Assume that* $p > 0$. *Then every nonoscillatory solution of Eq.* (1) *tends to zero as* $t \to \infty$.

Proof. It sufficies to show that every eventually positive solution $y(t)$ of Eq. (1) tends to zero as $t \to \infty$. So, let $y(t)$ be an eventually positive solution of Eq. (1). Set

$$z(t) = y(t) + py(t-\tau). \tag{9}$$

Then

$$z'(t) = -Q(t)\, y(t-\sigma) \tag{10}$$

and so $z'(t)$ is eventually negative. Since $p > 0$, $z(t)$ is eventually positive. Thus, $\lim_{t \to \infty} z(t) = l$ exists, is finite, and $l \geqslant 0$.

By integrating (10) from t_0 to t, we obtain

$$z(t) - z(t_0) + \int_0^t Q(s)\, y(s-\sigma)\, ds = 0$$

and therefore

$$\int_{t_0}^t Q(s)\, y(s-\sigma)\, ds = z(t_0) - l.$$

So, as $Q(t) \geqslant q > 0$ eventually, we see that $y \in L_1(t_0, \infty)$, and so, by (9), $z \in L_1(t_0, \infty)$. Hence $l = 0$ which implies that $\lim_{t \to \infty} y(t) = 0$, and the proof is complete.

THEOREM 10. *Consider the NDDE* (1). *Assume that* $p > 0$, $\sigma > \tau$, *and* Q *is* τ *periodic. Finally, suppose that*

$$(\mathrm{C}_4) \qquad \frac{1}{1+p} \liminf_{t \to \infty} \int_{t-(\sigma-\tau)}^t Q(s)\, ds > \frac{1}{e}.$$

Then every solution of Eq. (1) *oscillates.*

Proof. Otherwise there is an eventually positive solution $y(t)$ of Eq. (1). Set

$$z(t) = y(t) + py(t-\tau)$$

and

$$w(t) = z(t) + pz(t-\tau).$$

Then eventually

$$z(t) > 0,\ z'(t) < 0, \qquad w(t) > 0, \quad \text{and} \quad w'(t) < 0.$$

Hence

$$w(t) = z(t) + pz(t-\tau) < (1+p)\, z(t-\tau),$$

and so

$$-\frac{1}{1+p} Q(t)\, w(t-(\sigma-\tau)) \geqslant -Q(t)\, z(t-\sigma).$$

But the fact that Q is periodic with period τ implies that z, and also w, are solutions of Eq. (1), and so

$$-Q(t)\, z(t-\sigma) = w'(t),$$

that is to say,

$$w'(t) + \frac{1}{1+p}\, Q(t)\, w(t-(\sigma-\tau)) \leqslant 0.$$

In view of (C_4) and Lemma 1 (iii), this is impossible.

The proof of the theorem is complete.

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