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The ring of differential Fourier expansions

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ABSTRACT

For a fixed prime we prove structure theorems for the kernel and the image of the map that attaches to any differential modular function its differential Fourier expansion. The image of this map, which is the ring of differential Fourier expansions, plays the role of ring of functions on a "differential Igusa curve". Our constructions are then used to perform an analytic continuation between isogeny covariant differential modular forms on the differential Igusa curves belonging to different primes.

1. Introduction

1.1. Background

The concept of *differential modular form* (δ-*modular form* for short) was introduced in [4] and further developed and applied in subsequent work, in particular in [1,5–8]. The present work is a continuation of this study; however, for the convenience of the reader, we will not assume here familiarity with the above cited papers. Indeed, for the purpose of this Introduction we shall begin with an informal discussion of the main concepts of this theory while later, in the body of the paper, we shall provide a quick, yet formal, self-contained review of the necessary background.

We start by fixing a prime $p \geqslant 5$ and considering the ring $R := \hat{\mathbb{Z}}_p^{ur}$ obtained by completing the maximum unramified extension of the ring of p-adic integers. Let $\phi : R \to R$ be the unique lift of the p-power Frobenius on $k := R/pR$, and let $\delta_p = \delta : R \to R$ be the *Fermat quotient operator* defined by

$$\delta x := \frac{\phi(x) - x^p}{p} \tag{1.1}$$

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which, following [3,6], we view as a substitute for a “derivative operator with respect to p”. Let V be an affine smooth scheme over R and fix a closed embedding $V \subset \mathbb{A}^m$ into an affine space over R. Then a map $f : V(R) \to R$ is called a *δ-function of order* r [3] if there exists a restricted power series Φ in $m(r+1)$ variables, with R-coefficients such that

$$f(a) = \Phi\big(a, \delta a, \ldots, \delta^r a\big),$$

for all $a \in V(R) \subset R^m$. (Recall that *restricted* means *with coefficients converging p-adically to* 0; also the definition above does not depend on the embedding $V \subset \mathbb{A}^m$.)

Let $X_1(N)$ be the modular curve of level $\Gamma_1(N)$ over R with N not divisible by p; cf. [13]. In [4] we considered the level one situation $N = 1$ but here (as in [5–7]) we will assume $N > 3$. Let X be an affine open set of $X_1(N)$ disjoint from the cusps, let L be the line bundle on X, direct image of the sheaf of relative differentials on the universal elliptic curve over X, and let

$$V = Spec\left(\bigoplus_{n\in\mathbb{Z}} L^{\otimes n}\right) \to X$$

be the $\mathbb{G}_m$-torsor associated to L. Then a *δ-modular function* of order r and level $\Gamma_1(N)$ (holomorphic on X) is, by definition, a δ-function $f : V(R) \to R$ of order r.

Let $W := \mathbb{Z}[\phi]$ be the ring generated by ϕ. For $w = \sum a_i\phi^i \in W$ ($a_i \in \mathbb{Z}$) set $deg(w) = \sum a_i \in \mathbb{Z}$; for $\lambda \in R^\times$ we set $\lambda^w := \prod \phi^i(\lambda)^{a_i}$. A *$\delta$-modular form of weight* w is a δ-modular function $f : V(R) \to R$ such that

$$f(\lambda \cdot a) = \lambda^w f(a),$$

for all $\lambda \in R^\times$ and $a \in V(R)$, where $(\lambda, a) \mapsto \lambda \cdot a$ is the natural action $R^\times \times V(R) \to V(R)$.

We will assume in this Introduction that the reduction mod p of X is contained in the ordinary locus of the modular curve. We denote by M^∞ the ring of all δ-modular functions and by S^∞ the ring of modular forms of weight 0. There exists a natural *δ-Fourier expansion map*

$$M^\infty \to S^\infty_{for},$$

where S^∞_{for} is the ring of *δ-power series*, direct limit of the rings $R((q))[q', \ldots, q^{(n)}]\hat{}$, where $q, q', \ldots$ are variables and $\hat{}$ means p-adic completion. We may also consider the composition

$$M^\infty \to S^\infty_{for} \xrightarrow{\pi} \widehat{S_{for}},$$

where $S_{for} := R((q))$ and the map π sends $q', q'', \ldots$ into 0; we refer to this composition as the *Fourier expansion map*.

One of the main features of this theory [4–6] is that there exist δ-modular forms which possess a remarkable covariance property with respect to isogenies. These forms were called in [4–6] *isogeny covariant δ-modular forms* and have no analogue in the classical [13] or p-adic [16] theory of modular forms. The ring spanned by the isogeny covariant forms is generated by two fundamental forms f^∂ and f^1 of weight $\phi - 1$ and $-\phi - 1$ respectively [1,6]; this ring can be viewed, in a sense explained in [1,6], as the “projective coordinate ring” for the “quotient of the modular curve by the Hecke correspondences”. This quotient does not exist, of course, in usual algebraic geometry but, rather, in *δ-geometry*; cf. [6].

As shown in [4–6], a fundamental role is played by the isogeny covariant δ-modular forms of weights w with $deg(w) = -2$. If δ-modular forms are morally viewed as *not necessarily linear* “arithmetic differential operators” (on certain line bundles over modular curves) then isogeny covariant

δ-modular forms of weight w of degree $deg(w) = -2$ should correspond to the *linear* "arithmetic differential operators". So, morally, for δ-modular forms,

$$(\text{isogeny covariance}) + (\text{weight of degree } -2) \quad \Longleftrightarrow \quad (\text{linearity}).$$

Hence f^1 is "linear" whereas f^∂ is not.

1.2. Aim of the paper

The aim of the paper is two-fold namely:

(1) For a fixed prime, we prove a series of results about the kernel and the image of the δ-Fourier (respectively Fourier) expansion map; these settle some central issues left open in [4] and will permit us, in particular, to introduce and study the "ring of functions on the δ-Igusa curve" and the ring of "Igusa δ-modular functions".

(2) We develop a "partial differential" analogue of the above theory relative to a set of primes $\mathcal{P} = \{p_1, \dots, p_d\}$; this will allow us to introduce and determine all the "linear partial differential operators" on (the appropriate bundles over) "$\delta_{\mathcal{P}}$-Igusa curves". (No such operators exist on the appropriate bundles over the modular curves themselves; this was indeed our main motivation for introducing δ-Igusa curves.)

1.3. The theory for one prime

Before explaining our main results let us recall from [14, p. 462], the classical picture of modular forms mod p (of which the "δ-picture" is an analogue). Let $\overline{X}$ be the ordinary locus of the modular curve $X_1(N) \otimes k$ over the field k ($=$ algebraic closure of $\mathbb{F}_p$), let $\overline{S}$ be the affine ring of $\overline{X}$, and let $\overline{M}$ be the ring of *modular functions* on $\overline{X}$ (by which we mean here the k-algebra generated by all modular forms over k together with the inverse of the Hasse invariant $\overline{H}$). In particular, $\overline{S} \subset \overline{M}$. Furthermore let $\overline{S_{for}} = k((q))$ be the ring of Laurent power series over k. Consider the Fourier expansion map $\overline{M} \to \overline{S_{for}}$. This map is not injective (although its restriction to $\overline{S}$ is injective); by a theorem of Swinnerton–Dyer and Serre the kernel of this map is generated by $\overline{H} - 1$. Let $\overline{S_\heartsuit}$ be the image of $\overline{M} \to \overline{S_{for}}$. This is the *ring of Fourier expansions* and it turns out to be a $(\mathbb{Z}/p\mathbb{Z})^\times$-extension of $\overline{S}$; the spectrum of $\overline{S_\heartsuit}$ is birationally equivalent to the Igusa curve. Moreover the ring $\overline{M_\heartsuit} := \overline{S_\heartsuit} \otimes_{\overline{S}} \overline{M}$ corresponds, birationally, to the appropriate ring of modular functions on the Igusa curve.

Our main idea in the first part of the paper is to imitate the above construction with

$$\overline{S}, \quad \overline{M}, \quad \overline{S_{for}}$$

replaced by the rings

$$S^\infty, \quad M^\infty, \quad S^\infty_{for},$$

where M^∞ is the ring of δ-modular functions, S^∞ is the ring of δ-modular forms of weight 0, and S^∞_{for} is the ring of δ-*power series*; cf. Section 1.1. Let $S^\infty_\heartsuit$ be the image of the δ-Fourier expansion map $M^\infty \to S^\infty_{for}$. This is the *ring of δ-Fourier expansions* and will be viewed as playing the role of ring of functions on a "δ-Igusa curve". Moreover a certain "partially completed version", $M^\infty_\heartsuit$, of the ring $S^\infty_\heartsuit \otimes_{S^\infty} M^\infty$ will play the role of ring of "Igusa δ-modular functions". (We will not introduce, in this paper, an object called the *δ-Igusa curve*; such an object can be formally introduced in the sense of δ-algebraic geometry [6] but we shall not pursue this here. All we shall be working with are certain rings that play the roles of rings of functions, or rings of sections of bundles, on such a δ-geometric object.)

Here are (somewhat rough formulations of) our main results about the kernel and the image of the δ-Fourier (respectively Fourier) expansion map. For some of the terminology involved, and for more precise formulations of the results we refer to the body of the paper, as we shall explain presently.

The first two theorems below should be viewed as δ-analogues of the Swinnerton–Dyer and Serre Theorem about the kernel of the Fourier expansion map in positive characteristic.

Theorem 1.1. *The kernel of the δ-Fourier expansion map, $M^\infty \to S^\infty_{for}$, is the p-adic closure of the ideal generated by the elements $\delta^i(f^\partial - 1)$, where $i \geqslant 0$.*

Theorem 1.2. *The kernel of the Fourier expansion map, $M^\infty \to \widehat{S_{for}}$, is the p-adic closure of the ideal generated by the elements $\delta^i(f^\partial - 1)$ and $\delta^i f^1$, where $i \geqslant 0$.*

In the next statement, for any ring A, we denote by $\overline{A}$ the ring A/pA.

Theorem 1.3. *The cokernel of the δ-Fourier expansion map $M^\infty \to S^\infty_{for}$ is torsion free. In particular the ring $\overline{S^\infty_\heartsuit}$ is an integral domain. Moreover $\overline{S^\infty_\heartsuit}$ is an ind-étale $\mathbb{Z}_p^\times$-extension of $\overline{S^\infty}$.*

The statement about the cokernel should be viewed as a *δ-expansion principle* for Igusa δ-modular forms. The rest of the theorem shows that, morally, the "δ-Igusa curve" is irreducible and is a pro-étale "formal $\mathbb{Z}_p^\times$-cover" of the "δ-modular curve" (i.e. of the modular curve viewed as an object of "δ-geometry"). By the way, we will also show, in the body of the paper, that the ring $S^\infty_\heartsuit$ comes equipped with a sequence of derivations naturally associated to the Serre derivation operator; these derivations "topologically generate" the space of all derivations on $S^\infty_\heartsuit$. Theorems 1.3, 1.1, 1.2 are consequences of the (more precisely formulated) Theorem 2.30 in the body of the paper.

In the following Theorem 1.4 we assume that the reduction mod p of X is the whole of the ordinary locus of the modular curve. To state this theorem recall Katz' rings

$$\mathbb{D} \subset \mathbb{W} \subset \widehat{S_{for}} = R((q))\hat{}$$

where $\mathbb{D}$ is the ring of *divided congruences* and $\mathbb{W}$ is the ring of *generalized p-adic modular forms*, both with coefficients in R, and both viewed as embedded into $R((q))\hat{}$ via the Fourier expansion. Cf. [15,17], and also the review in the present paper. Recall that $\mathbb{D} \subset R[[q]]$; also, if $\Delta \in R((q))$ is the discriminant then $\mathbb{D} + R[\Delta^{-1}]$ is p-adically dense in $\mathbb{W}$.

Theorem 1.4. *The image of the Fourier expansion map $M^\infty \to \widehat{S_{for}}$ contains $\mathbb{D}$ and hence is p-adically dense in $\mathbb{W}$.*

Morally Theorem 1.4 (which is Corollary 2.35 in the body of the paper) exhibits the "world of δ-modular functions" as a lift (with "huge" kernel described in Theorem 1.2) of the "world of generalized p-adic modular functions" of Katz. One can then ask if some of the basic constructions with values in Katz' ring $\mathbb{W}$ (such as the U-operator, various measures, Galois representations, etc.) can be lifted naturally to the world of δ-modular functions.

Note that Theorem 1.4 has the following consequence that is independent of our theory. Let $\delta_0 : R((q))\hat{} \to R((q))\hat{}$ be the operator

$$\delta_0\Big(\sum a_n q^n\Big) := \frac{\sum \phi(a_n)q^{np} - (\sum a_n q^n)^p}{p},$$

let $K = R[1/p]$, let $M(R, \kappa, N)$ denote the space of modular forms over R of weight κ and level $\Gamma_1(N)$, let $\Delta \in M(R, 12, N)$ be the discriminant form, and let $E_{p-1} \in M(R, p-1, N)$ be the normalized Eisenstein form of weight $p-1$. Then Theorem 1.4 implies that any series $f(q) \in \mathbb{D}$ can be represented in $R((q))\hat{}$ as

$$f(q) = \Phi\big(f_1(q), \ldots, f_n(q), \ldots, \delta_0^r\big(f_1(q)\big), \ldots, \delta_0^r\big(f_n(q)\big)\big), \tag{1.2}$$

where $f_j = F_j \Delta^{\mu_j} E_{p-1}^{\nu_j}$, $F_j \in M(R, \kappa_j, N)$, $\kappa_j, \mu_j, \nu_j \in \mathbb{Z}$, $f_j(q) \in R((q))$ are the Fourier expansions of f_j, and Φ is a restricted power series in $n(r+1)$ variables with R-coefficients. So, morally, the elements of Katz's ring $\mathbb{D}$ of divided congruences can be realized as limits of very special divided congruences that arise by iterating the Fermat quotient operation. It is not clear to us if this statement can be proved directly, independently of our theory.

1.4. The theory for several primes

As an application of the above one prime constructions we will introduce and study *linear partial differential operators* in the setting of modular curves, with respect to d arithmetic directions; these directions are represented by a set of primes $\mathcal{P} = \{p_1, \dots, p_d\}$ (along which the "derivatives" will be the corresponding Fermat quotient operators). Such a theory for algebraic groups (rather than modular curves) was developed in [9] where arithmetic analogues of Laplacians were constructed on the additive group, the multiplicative group, and on elliptic curves over $\mathbb{Q}$. The basic idea in [9] was, very roughly speaking, to construct *linear arithmetic partial differential operators* along the "vertical divisors" corresponding to each of the primes $p_1, \dots, p_d$ and then to perform a sort of *analytic continuation* between the various primes along a "horizontal divisor". We will keep this point of view of analytic continuation in the present paper. (For the convenience of the reader, we will make the present paper essentially independent of [9]; but for an informal explanation of analytic continuation, the reader may want to consult the Introduction to [9].) With this point of view we will be able to achieve our program of constructing linear partial differential operators in the modular setting; but for this we will have to pay the price of passing from modular curves to *$\delta_{\mathcal{P}}$-Igusa curves* (a several primes generalization of the δ-Igusa curves). The main reason why passing from modular curves to $\delta_{\mathcal{P}}$-Igusa curves is crucial is that, unlike the former, the latter carry a certain "tautological" weight one, order zero form, which we shall call f^0, and which will be the key to performing "analytic continuation" between various primes; we will show that such an analytic continuation cannot be performed in the context of the modular curves themselves.

Our main result here is, morally, a complete determination of all *linear arithmetic partial differential operators* on the appropriate bundles over the "$\delta_{\mathcal{P}}$-Igusa curves". The technical way to express this is the following (roughly formulated):

Theorem 1.5. *Let w be a weight of degree $deg(w) = -2$. Then the module of all weight w isogeny covariant Igusa $\delta_{\mathcal{P}}$-modular forms of weight w and order $r = (r_1, \dots, r_d)$ is free of rank $r_1 r_2 \dots r_d$.*

Cf. Theorem 3.20 in the body of the paper (and the discussion preceding it) for a precise formulation. We mention that, intuitively, "order $r = (r_1, \dots, r_d)$" means "order r_k with respect to p_k for each $k = 1, \dots, d$".

The forms in the above theorem are constructed using the tautological form f^0 and another basic form f^e of weight 0 and order $e := (1, \dots, 1)$. The form f^e itself is constructed using the forms f^1 corresponding to the various primes and can be viewed as a modular analogue of the arithmetic Laplacians in [9].

By the way we will also determine all isogeny covariant Igusa $\delta_{\mathcal{P}}$-modular forms of any order and any weight w with $deg(w) = 0$; they are all obtained from f^0. Finally we shall be able to analytically continue the differential eigenforms of [7]; the resulting form will be called f^{2e}, will have weight 0 and order $2e = (2, \dots, 2)$, will *not* be isogeny covariant, and will live on the modular curves themselves (rather than on the "$\delta_{\mathcal{P}}$-Igusa curves").

1.5. Variants

In all the discussion above, we considered the *modular curves* $X_1(N)$ parameterizing elliptic curves with $\Gamma_1(N)$ level structures; nevertheless a substantial part of the theory will be developed also in the case of Shimura curves $X^D(\mathcal{U})$ parameterizing false elliptic curves with *level $\mathcal{U}$ structures* [12]. The role of Fourier expansions in the Shimura curve case will be played by Serre–Tate expansions, in the sense of [19,5,6]. We will also consider the theory over modular curves with respect to Serre–Tate expansions.

Finally note that our theory in d arithmetic dimensions can be viewed as a $(0+d)$-dimensional theory (where 0 indicates that we have no "geometric direction" on the base). On the other hand we developed in [10,11] a $(1+1)$-dimensional theory (where the base has one geometric and one arithmetic direction.) One can then ask if the $(0+d)$-dimensional theory in the present paper and the $(1+1)$-dimensional theory in [10,11] can be "unified" as parts of a $(1+d)$-dimensional theory. At this point it is not clear whether this is possible; cf. the last section of [9] for comments on the difficulties arising from such an attempt at unification.

1.6. *Analytic analogues*

One can ask if the basic forms f^e, f^{2e} referred to in Section 1.4 have analogues in (real/complex) analysis. The forms f^{2e} are intimately related to the arithmetic Laplacians in [9] whose analytic analogues are discussed in the Introduction of that paper. The form f^e, on the other hand, can be loosely viewed as having an analytic analogue which we now describe.

Let D be a domain in the complex z-plane. Let $\mathbb{H} := \{\tau \in \mathbb{C};\ Im(\tau) > 0\}$ be the upper half plane. Let $F \subset \mathbb{H}$ be a domain containing none of the fixed points of the $SL_2(\mathbb{Z})$-action, such that any two $SL_2(\mathbb{Z})$-conjugate points in F are conjugate under a translation by an integer. (E.g. one can take $F = \{\tau \in \mathbb{C};\ Im(\tau) > 1\}$.) Let $j : \mathbb{H} \to \mathbb{C}$ be the classical j-function, let $G = j(F)$, and let $j^{-1} : G \to F$ be the (multivalued) inverse of j. For any C^∞-function $u \in C^\infty(D, G)$ we denote by $q_u \in C^\infty(D, \mathbb{C}^\times)$ the (well-defined !) function

$$q_u(z) = e^{2\pi\sqrt{-1}\cdot j^{-1}(u(z))}.$$

Then our form f^e can be viewed as an arithmetic analogue of the Laplace-type operator

$$C^\infty(D, G) \to C^\infty(D, \mathbb{C})$$

defined by

$$u \mapsto \partial_z \partial_{\bar{z}} \log q_u = \partial_z \left(\frac{\partial_{\bar{z}} q_u}{q_u} \right) = \partial_{\bar{z}} \left(\frac{\partial_z q_u}{q_u} \right).$$

The last equalities can be viewed as a *Dirac decomposition* for our operator and this decomposition will have an arithmetic analogue in the case of f^e. Cf. [9] for more on Dirac decompositions.

1.7. *Plan of the paper*

The paper has two parts. In the first part we develop the theory for one prime, first in an axiomatic setting, for an arbitrary curve equipped with an arbitrary line bundle, and then in the concrete setting of modular or Shimura curves equipped with their bundles of modular forms. The main result here is Theorem 2.30 (which implies Theorems 1.3, 1.1, 1.2) and Corollary 2.35 (whose content is that of Theorem 1.4).

Our one prime constructions will be used in the second part of the paper, where the theory for at least two primes is developed. In this second part analytic continuation in the modular/Shimura context is introduced and the main results on the space of isogeny covariant Igusa δ-modular forms, referred to in Section 1.4, are stated and proved. Cf. Theorems 3.20, 3.24.

2. The theory for one prime

2.1. Review of concepts and terminology from [3,6]

Unless otherwise stated all rings and algebras will be commutative with unit element. For any A-algebra $\varphi : A \to B$ and any element $a \in A$ we continue to denote by a the element $\varphi(a) = a1_B$. We fix, throughout this paper, a prime integer $p \geqslant 5$. For any $\mathbb{Z}$-module M we set $M\hat{} = \varprojlim M/p^n M$, the p-adic completion of M and $\overline{M} := M/pM = M \otimes \mathbb{Z}/p\mathbb{Z}$, the reduction of M mod p. We say M is p-adically complete if $M \to M\hat{}$ is an isomorphism. For $m \in M$ we let $\overline{m} \in \overline{M}$ be the image of m. For any scheme X we set $\overline{X} := X \otimes \mathbb{Z}/p\mathbb{Z}$. We denote by $\mathbb{Z}_{(p)}$ the local ring of $\mathbb{Z}$ at (p). We will repeatedly use the fact that if $M \to N$ is a homomorphism of $\mathbb{Z}_{(p)}$-modules such that $\overline{M} \to \overline{N}$ is injective, p is a non-zero divisor in N, and M is p-adically separated then $M \to N$ is injective and has torsion free cokernel; and that, conversely, if $M \to N$ is a morphism of $\mathbb{Z}_{(p)}$-modules which is injective and has torsion free cokernel then $\overline{M} \to \overline{N}$ is injective.

2.1.1. p-derivations

Let $C_p(X, Y) \in \mathbb{Z}[X, Y]$ be the polynomial with integer coefficients

$$C_p(X, Y) := \frac{X^p + Y^p - (X+Y)^p}{p}.$$

A *p-derivation* from a ring A into an A-algebra $\varphi : A \to B$ is a map $\delta : A \to B$ such that $\delta(1) = 0$ and

$$\delta(x+y) = \delta x + \delta y + C_p(x, y),$$
$$\delta(xy) = x^p \cdot \delta y + y^p \cdot \delta x + p \cdot \delta x \cdot \delta y,$$

for all $x, y \in A$. Given a p-derivation we always denote by $\phi : A \to B$ the map $\phi(x) = \varphi(x)^p + p\delta x$; then ϕ is a ring homomorphism. A *prolongation sequence* is a sequence $S^* = (S^n)_{n \geqslant 0}$ of rings S^n, $n \geqslant 0$, together with ring homomorphisms $\varphi_n : S^n \to S^{n+1}$ and p-derivations $\delta_n : S^n \to S^{n+1}$ such that $\delta_{n+1} \circ \varphi_n = \varphi_{n+1} \circ \delta_n$ for all n. We usually denote all φ_n by φ and all δ_n by δ and we view S^{n+1} as an S^n-algebra via φ. A morphism of prolongation sequences, $u^* : S^* \to \tilde{S}^*$ is a sequence $u^n : S^n \to \tilde{S}^n$ of ring homomorphisms such that $\delta \circ u^n = u^{n+1} \circ \delta$ and $\varphi \circ u^n = u^{n+1} \circ \varphi$. Let W be the ring of polynomials $\mathbb{Z}[\phi]$ in the indeterminate ϕ. Then, for $w = \sum_{i=0}^r a_i \phi^i \in W$, we set $deg(w) := \sum a_i$. If $a_r \neq 0$ we set $ord(w) = r$; we also set $ord(0) = 0$. For w as above (respectively for $w \in W_+ := \{\sum b_i \phi^i \mid b_i \geqslant 0\}$), S^* a prolongation sequence, and $x \in (S^0)^\times$ (respectively $x \in S^0$) we can consider the element $x^w := \prod_{i=0}^r \varphi^{r-i} \phi^i(x)^{a_i} \in (S^r)^\times$ (respectively $x^w \in S^r$). We let $W(r) := \{w \in W \mid ord(w) \leqslant r\}$.

Let $R := R_p := \hat{\mathbb{Z}}_p^{ur}$ be the completion of the maximum unramified extension of the ring of p-adic integers $\mathbb{Z}_p = (\mathbb{Z}_{(p)})\hat{}$ and we denote by k its residue field, $k = R/pR$. Then R has a unique p-derivation $\delta : R \to R$ given by

$$\delta x = \big(\phi(x) - x^p\big)/p,$$

where $\phi : R \to R$ is the unique lift of the p-power Frobenius map on k. One can consider the prolongation sequence R^* where $R^n = R$ for all n. By a *prolongation sequence over R* we understand a prolongation sequence S^* equipped with a morphism $R^* \to S^*$. From now on all our prolongation sequences are assumed to be over R.

By a δ-ring we mean a ring together with a p-derivation on it. A morphism of δ-rings is a ring homomorphism that commutes with the given p-derivations. In what follows all δ-rings will be assumed *over R* (i.e. equipped with δ-ring homomorphisms from R). If a is an element of a δ-ring we will sometimes denote by $a', a'', \dots, a^{(r)}$ the sequence $\delta a, \delta^2 a, \dots, \delta^r a$.

2.1.2. Conjugate derivations

Let A be a δ-ring in which p is a non-zero divisor and let $u : A^0 \to A$ be a ring homomorphism. Let $\partial : A^0 \to A^0$ be an R-derivation and let $j \geqslant 0$ be an integer. An R-derivation $\partial_j : A \to A$ will be called a *j-conjugate* of ∂ if for any integer $s \geqslant 0$ we have

$$\partial_j \circ \phi^s \circ u = \delta_{js} \cdot p^j \cdot \phi^s \circ u \circ \partial : A^0 \to A,$$

where δ_{js} is the Kronecker symbol. A sequence $(\partial_j)_{j\geqslant 0}$ where for each j, $\partial_j : A \to A$ is a j-conjugate of ∂, will be referred to as a *complete sequence of conjugates* of ∂. Let us say that A is *topologically δ-generated* by A^0 if the smallest δ-subring of A that contains $u(A^0)$ is p-adically dense in A. It is then trivial to see that if A is p-adically separated and topologically δ-generated by A^0 then (1) any derivation $\partial : A^0 \to A^0$ has at most one j-conjugate $\partial_j : A \to A$ for each j and (2) if $(\partial_j)_{j\geqslant 0}$ is a complete sequence of conjugates of ∂ then, for all $j \geqslant 0$,

$$\partial_j \circ \phi = p \cdot \phi \circ \partial_{j-1} : A \to A,$$
$$\partial_j \circ \delta^s \circ u = 0 : A^0 \to A, \quad \text{for } s < j,$$
$$\partial_j \circ \delta^j \circ u = \phi^j \circ u \circ \partial : A^0 \to A.$$

Here $\partial_{-1} = 0$.

2.1.3. p-jet spaces

Given a scheme X of finite type over R we introduced in [3] a sequence of formal (p-adic) schemes over R, called the *p-jet spaces of* X, which we denoted by $J^r(X)$, $r \geqslant 0$. In case X is affine, $X = \mathit{Spec}\, R[x]/(f)$, with x a tuple of indeterminates and f a tuple of polynomials, we have

$$J^r(X) = \mathit{Spf}\, R\big[x, x', \ldots, x^{(r)}\big]\hat{}\big/\big(f, \delta f, \ldots, \delta^r f\big)$$

where $x', \ldots, x^{(r)}$ are new tuples of variables and $R[x, x', \ldots, x^{(r)}]\hat{}$ is a prolongation sequence via $\delta x = x'$, $\delta x' = x''$, ... For X not necessarily affine we set $\mathcal{O}^r(X) := \mathcal{O}(J^r(X))$; these rings form a prolongation sequence. If X is affine the prolongation sequence $(\mathcal{O}^r(X))_{r\geqslant 0}$ has the following universality property: if $(S^r)_{r\geqslant 0}$ is any prolongation sequence over R of p-adically complete rings S^r and $u : \mathcal{O}(X) \to S^0$ is any R-algebra homomorphism then there exists a unique morphism of prolongation sequences over R, $u^r : \mathcal{O}^r(X) \to S^r$, such that u^0 induces u. By this universality property, for X not necessarily affine, each element of $\mathcal{O}^r(X)$ naturally defines a function $X(R) \to R$. Such functions are called in [3,6] *δ-functions of order r*. If X is smooth then any element of $\mathcal{O}^r(X)$ is uniquely determined by the induced δ-function $X(R) \to R$. We set $\mathcal{O}^\infty(X) := \lim_\to \mathcal{O}^r(X)$. If X/R is smooth with $\overline{X}$ connected then the schemes $\overline{J^r(X)}$ are smooth varieties over k. Moreover $\mathcal{O}^r(X)$, $\mathcal{O}^\infty(X)$ are integral domains, and p is a prime element in these rings. In addition, $\mathcal{O}^\infty(X)$ is p-adically separated and topologically δ-generated by $\mathcal{O}(X)$ (and hence also by $\mathcal{O}^0(X) = \mathcal{O}(X)\hat{}$). If $X \to Y$ is an étale morphism then $J^r(X) \simeq J^r(Y) \widehat{\times}_{\hat{Y}} \hat{X}$.

Recall from [6, Proposition 3.45], that if X/R is smooth then for any R-derivation $\partial : \mathcal{O}^0(X) \to \mathcal{O}^0(X)$ there exists a (necessarily unique) complete sequence of conjugates $\partial_j : \mathcal{O}^\infty(X) \to \mathcal{O}^\infty(X)$ of ∂. Moreover $\partial_j \mathcal{O}^s(X) \subset \mathcal{O}^s(X)$ for all $j, s \geqslant 0$ and $\partial_j \mathcal{O}^s(X) = 0$ for $s < j$.

2.2. The axiomatic theory

In this section we develop the one prime version of the theory of this paper in the axiomatic setting of an arbitrary curve equipped with a line bundle. In the next section we will specialize our discussion to the case of modular (respectively Shimura) curves and their natural bundles of modular forms. The main result of this axiomatic section is Theorem 2.9. This Theorem will later be strengthened, in the concrete setting of modular curves; cf. Theorem 2.30. The strengthened version will morally say that the δ-Igusa curve is a connected pro-étale $\mathbb{Z}_p^\times$-cover of the modular curve.

2.2.1. Framed curves

We start with the following data:

$$X, \quad L, \tag{2.1}$$

where X is a smooth affine curve over R with connected reduction mod p, $\overline{X}$, and L is an invertible sheaf on X which we identify with its module of global sections. Consider the scheme

$$V := Spec\left(\bigoplus_{n\in\mathbb{Z}} L^{\otimes n}\right).$$

For any R-algebra B the set of B-points $V(B)$ naturally identifies with the set of pairs (P,ξ) where $P \in X(B)$ and ξ is a basis of the pull back of L to $Spec\,B$ by P. In particular $V \to X$ is a $\mathbb{G}_m$-torsor with respect to the action $B^\times \times V(B) \to V(B)$ given by

$$\big(\lambda, (P,\xi)\big) \mapsto \lambda \cdot (P,\xi) := \big(P, \lambda^{-1}\xi\big). \tag{2.2}$$

Set

$$\begin{aligned} S &:= S_X := \mathcal{O}(X), \\ M &:= M_X := \mathcal{O}(V) = \bigoplus_{n\in\mathbb{Z}} L^{\otimes n}. \end{aligned} \tag{2.3}$$

We assume in what follows that we are given one more piece of data, namely either an R-point of X:

$$P \in X(R) \tag{2.4}$$

or an open embedding $X \subset X^*$ into a smooth curve X^* over R, with connected reduction mod p, $\overline{X^*}$, plus an R-point of the reduced closed subscheme $X^* \setminus X$ of X^*,

$$P \in \big(X^* \setminus X\big)(R). \tag{2.5}$$

In the first case (2.4) we set $S_{for} = R[\![t]\!]$, where t is a variable. In the second case (2.5) we set $S_{for} := R((q)) := R[\![q]\!][q^{-1}]$, where q is a variable. Assume we are given an isomorphism between $Spf\,R[\![t]\!]$ (respectively $R[\![q]\!]$) and the completion of X (respectively X^*) along the image of P. Then, in both cases we have an induced homomorphism $E : S \to S_{for}$ which is injective, inducing an injective map $\overline{E} : \overline{S} \to \overline{S_{for}}$. We shall finally assume that we are given yet another piece of data, namely an extension of E to a homomorphism

$$E : M \to S_{for}. \tag{2.6}$$

It is convenient to collect all the above data in one definition as follows:

Definition 2.1. A *framed curve* is a tuple X, L, P, E where X, L are as in (2.1), P is either as in (2.4) or as in (2.5), and E is as in (2.6). For simplicity we also say that X is a framed curve with frame L, P, E. We say that X is *Serre–Tate-framed*, respectively *Fourier-framed*, according as P is as in (2.4) or as in (2.5). Accordingly E is referred to as a *Serre–Tate expansion map* respectively a *Fourier expansion map*.

2.2.2. The rings S^r, M^r

Assume we are given a framed curve $X = Spec\, S$. We may consider the following rings:

$$\begin{aligned}
S^r &:= S^r_X := \mathcal{O}^r(X), \quad r \geqslant 0,\\
M^r &:= M^r_X := \mathcal{O}^r(V), \quad r \geqslant 0,\\
S^\infty &:= \varinjlim S^r,\\
M^\infty &:= \varinjlim M^r.
\end{aligned} \tag{2.7}$$

An element $f \in M^r$ is said to be of weight $w \in W$ if, and only if, the induced δ-function $f : V(R) \to R$ satisfies

$$f(\lambda \cdot a) = \lambda^w f(a)$$

for all $\lambda \in R^\times$, $a \in V(R)$, where $(\lambda, a) \mapsto \lambda \cdot a$ is the natural action $R^\times \times V(R) \to V(R)$; cf. (2.2). We denote by $M^r(w) = M^r_X(w)$ the R-module of all elements of $M^r = M^r_X$ of weight w.

If L is trivial on X and x is a basis of L then we have identifications

$$\begin{aligned}
M &= S\big[x, x^{-1}\big],\\
M^r &= S^r\big[x, x^{-1}, x', \dots, x^{(r)}\big]\hat{},\\
M^r(w) &= S^r \cdot x^w \subset M^r.
\end{aligned}$$

We may also consider the element $x^{-1} \otimes x \in M \otimes_S M$ which we refer to as the *tautological element of* $M \otimes_S M$. Clearly $x^{-1} \otimes x$ do not depend on the choice of the basis x of L.

By [6, Proposition 3.14], the rings $\overline{S^r}$ are integral domains, and the maps $\overline{S^r} \to \overline{S^{r+1}}$ are injective. In particular the rings S^r are integral domains and the maps $S^r \to S^{r+1}$ are injective with torsion free cokernels. The analogous statements hold for M^r. So, in particular, $\overline{S^\infty}$ and $\overline{M^\infty}$ are integral domains.

Let $t', t'', \dots$ and $q', q'', \dots$ be new variables and consider the prolongation sequence $(S^r_{for})_{r \geqslant 0}$,

$$S^r_{for} = R[\![t]\!]\big[t', \dots, t^{(r)}\big]\hat{},$$

respectively

$$S^r_{for} = R((q))\hat{}\big[q', \dots, q^{(r)}\big]\hat{}.$$

We set

$$S^\infty_{for} := \varinjlim S^r_{for}.$$

Then the expansion maps induce, by universality, morphisms of prolongation sequences,

$$E^r : M^r \to S^r_{for}; \tag{2.8}$$

the maps E^r will be referred to as *δ-expansion maps* for M^r. They induce a *δ-expansion map*

$$E^\infty : M^\infty \to S^\infty_{for}. \tag{2.9}$$

We have the following *δ-expansion principle* for S^r:

Proposition 2.2. *The induced map*

$$\overline{E^r} : \overline{S^r} \to \overline{S^r_{for}}$$

is injective. In particular, $E^r : S^r \to S^r_{for}$, and hence the δ-expansion maps

$$E^r : M^r(w) \to S^r_{for}$$

are injective, with torsion free cokernel.

(In this paper the words "torsion free", without the specification "as an A-module", will always mean "torsion free as a $\mathbb{Z}$-module".)

Proof. In case (2.4) this is [6, Proposition 4.43]. Case (2.5) follows easily from case (2.4) applied to X^* instead of X. $\square$

2.2.3. The rings $S^r_\heartsuit$

Next, for a framed curve $X = Spec\ S$, we define

$$S^r_\heartsuit := Im\big(E^r : M^r \to S^r_{for}\big), \tag{2.10}$$
$$S^\infty_\heartsuit := \varinjlim S^r_\heartsuit = Im\big(E^\infty : M^\infty \to S^\infty_{for}\big).$$

The ring $S^\infty_\heartsuit$ will later morally play the role of "coordinate ring of the δ-Igusa curve".

The following is trivial to check (using the definitions and Proposition 2.2):

Proposition 2.3.

(1) *The homomorphisms $\overline{S^r} \to \overline{S^r_\heartsuit}$, $\overline{S^\infty} \to \overline{S^\infty_\heartsuit}$ are injective. In particular the homomorphisms $S^r \to S^r_\heartsuit$, $S^\infty \to S^\infty_\heartsuit$ are injective with torsion free cokernel.*

(2) *The homomorphisms $S^r_\heartsuit \to S^{r+1}_\heartsuit$ are injective.*

Remark 2.4. The ring $\overline{S^\infty_\heartsuit}$ is not a priori an integral domain and the map $\overline{S^\infty_\heartsuit} \to \overline{S^\infty_{for}}$ is not a priori injective. The ring $\overline{S^\infty_\heartsuit}$, however, has a natural quotient which is an integral domain, namely:

$$\widetilde{S^\infty_\heartsuit} := Im\big(\overline{M^\infty} \to \overline{S^\infty_{for}}\big). \tag{2.11}$$

This ring is going to play a role in what follows. We will prove later that, in the concrete setting of modular curves the map $\overline{S^\infty_\heartsuit} \to \overline{S^\infty_{for}}$ is injective (δ-expansion principle) hence $\overline{S^\infty_\heartsuit}$ is an integral domain and the surjection $\overline{S^\infty_\heartsuit} \to \widetilde{S^\infty_\heartsuit}$ is an isomorphism. Cf. Theorem 2.30.

Definition 2.5. A framed curve is called *ordinary* if there exists an element $f \in M^1(\phi - 1)$ which is invertible in the ring M^1, such that $E^1(f) = 1$.

Note that if an f as above exists then, by Proposition 2.2, f is necessarily unique. The terminology *ordinary* will be justified later in our applications to modular and Shimura curves.

In what follows we will analyze in some detail the structure of the rings $S^\infty_\heartsuit$ and the various rings constructed from it.

Definition 2.6. Let A be a k-algebra where k is a field. Let $A \subset B$ a ring extension, and Γ a profinite abelian group acting on B by A-automorphisms. We say that B is a *Γ-extension* of A if one can write A and B as filtered unions of finitely generated k-subalgebras, $A = \bigcup A_i$, $B = \bigcup B_i$, indexed by some partially ordered set, with $A_i \subset B_i$, and one can write Γ as an inverse limit of finite abelian groups, $\Gamma = \varprojlim \Gamma_i$, such that the Γ-action on B is induced by a system of compatible Γ_i-actions on B_i and

$$B_i^{\Gamma_i} = A_i$$

for all i. (Then, of course, we also have $B^{\Gamma} = A$.) If in addition one can choose the above data such that each A_i is smooth over k and each B_i is étale over A_i we say that B is an *ind-étale Γ-extension* of A.

Here are a couple of easy facts about this concept:

Lemma 2.7.

(1) *Assume B is a Γ-extension of A and $C := B/I$ is a quotient of B by an ideal I. Then C is integral over A.*

(2) *Assume B is an ind-étale Γ-extension of A and let I be a prime ideal of B such that $I \cap A = 0$. Then $C := B/I$ is an ind-étale Γ'-extension of A where Γ' is a closed subgroup of Γ.*

Proof. Assertion (1) is clear. Let's prove assertion (2). Using the notation in Definition 2.6 set $Y_i = \mathit{Spec}\, B_i$, $V_i := \mathit{Spec}\, A_i$, $Z_i := \mathit{Spec}\, C_i$, $C_i := B_i/B_i \cap I$. Let $\Gamma_i' := \{\gamma \in \Gamma_i;\ \gamma Z_i = Z_i\}$. By Lemma 2.8 below C_i is étale over A_i and $C_i^{\Gamma_i'} = A_i$ so one can take $\Gamma' := \varprojlim \Gamma_i'$ acting on $C = \varprojlim C_i$. □

We have used the following "well-known" lemma (whose proof will be "recalled" for convenience):

Lemma 2.8. *Let V be a smooth affine variety over a field k, let $Y \to V$ be a finite étale map, and let G be a finite abelian group acting on Y such that $Y/G = V$. Let $Z \subset Y$ be a subvariety that dominates V and let $G' = \{\gamma \in G;\ \gamma Z = Z\}$. Then Z is a connected component of Y (hence is étale over V) and $Z/G' = V$.*

Proof. Since V is smooth the connected components $Z_1, \dots, Z_n$ of Y are irreducible so Z is a connected component of Y, say $Z = Z_1$. Since V is connected G acts transitively on the set $\{Z_1, \dots, Z_n\}$ hence the stabilizers in G of the various Z_is are conjugate in G, hence they are equal, because G is abelian. So

$$\mathcal{O}(V) = \mathcal{O}(Y)^G = \big(\mathcal{O}(Z_1) \times \cdots \times \mathcal{O}(Z_n)\big)^G = \big(\mathcal{O}(Z)^{G'} \times \cdots \times \mathcal{O}(Z)^{G'}\big)^{G/G'} \tag{2.12}$$

where $\mathcal{O}(Z_i)^{G'} \simeq \mathcal{O}(Z)^{G'}$ via *any* $\gamma \in G$ such that $\gamma Z = Z_i$ and G/G' acts on the product via the corresponding permutation representation. Since the last ring in (2.12) contains $\mathcal{O}(Z)^{G'}$ embedded diagonally it follows that $\mathcal{O}(Z)^{G'} = \mathcal{O}(V)$. □

Here is the main result of this section.

Theorem 2.9. *Let $X = \mathit{Spec}\, S$ be an ordinary framed curve. Then the ring $\overline{S^{\infty}_{\heartsuit}}$ is a quotient of an ind-étale $\mathbb{Z}_p^{\times}$-extension of $\overline{S^{\infty}}$.*

Recall the ring $\widetilde{S^{\infty}_{\heartsuit}}$; cf. (2.11). By Proposition 2.7 we get:

Corollary 2.10.

(1) *$\overline{S^{\infty}_{\heartsuit}}$ is an integral extension of $\overline{S^{\infty}}$.*

(2) *$\widetilde{S^{\infty}_{\heartsuit}}$ is an ind-étale Γ'-extension of $\overline{S^{\infty}}$, where Γ' is a closed subgroup of $\Gamma := \mathbb{Z}_p^{\times}$.*

For a refinement of this result in the setting of modular curves see Theorem 2.30.

For the proof of Theorem 2.9 we need a series of Lemmas. For the first two lemmas we let A be a δ-ring and we consider the prolongation sequence $B^r = A[z, z^{-1}, z', \dots, z^{(r)}]\hat{}$. We then denote by $O(r)$ any element of B^r.

Lemma 2.11. *Let $\varphi \in A$. Then, for any $n \geqslant 1$, we have*

$$\delta^n\left(\frac{z^\phi}{z} - \varphi\right) = z^{-p^n}\left(z^{(n)}\right)^p - z^{p^{n+1}-2p^n} z^{(n)} + O(n-1) + pO(n+1).$$

Proof. For $\varphi = 0$ this is [6, Lemma 5.19]. Assume now φ arbitrary. One checks by induction that

$$\delta^n(z - \varphi) = \delta^n z + U + pV,$$

where $U = O(n-1)$, $V = O(n)$. Replacing z by $\frac{z^\phi}{z}$ we get

$$\delta^n\left(\frac{z^\phi}{z} - \varphi\right) = \delta^n\left(\frac{z^\phi}{z}\right) + U\left(\frac{z^\phi}{z}, \dots, \delta^{n-1}\left(\frac{z^\phi}{z}\right)\right) + pV\left(\frac{z^\phi}{z}, \dots, \delta^n\left(\frac{z^\phi}{z}\right)\right),$$

and we conclude by the case $f = 0$ of the lemma. □

Lemma 2.12. *Let $\lambda = 1 + p^n a$, $a \in \mathbb{Z}$. Then*

$$\delta^n(\lambda z) = z^{(n)} + az^{p^n} + pO(n).$$

Proof. An easy exercise. See also [6, p. 79].

It is also convenient to formulate the following:

Lemma 2.13. *Let Q be a ring of characteristic p and consider the Q-algebra $Q' := Q[u]/(u^p - u - G)$ where $G \in Q$. Consider the action of $\mathbb{Z}/p\mathbb{Z} = \{\bar{a};\ a = 0, \dots, p-1\}$ on $Q[u]$ defined by $\bar{a} \cdot u = u + \bar{a}$ and consider the induced $\mathbb{Z}/p\mathbb{Z}$-action on Q'. Then any $\mathbb{Z}/p\mathbb{Z}$-invariant element of Q' is in Q.*

Proof. Let $c \in Q'$ be the class of u. Then Q' is a free Q-module with basis $1, c, \dots, c^{p-1}$. Assume $\sum_{i=0}^{p-1} \lambda_i c^i \in Q'$ is $\mathbb{Z}/p\mathbb{Z}$-invariant, where $\lambda_i \in Q$. We want to show that $\lambda_i = 0$ for $i \geqslant 1$. We may assume $\lambda_0 = 0$. Assume there is a $s \geqslant 1$ such that $\lambda_s \neq 0$ and let s be maximal with this property. Then

$$\lambda_s(c+1)^s + \lambda_{s-1}(c+1)^{s-1} + \cdots + = \lambda_s c^s + \lambda_{s-1}c^{s-1} + \cdots.$$

Picking out the coefficient of c^{s-1} we get $s\lambda_s = 0$ hence $\lambda_s = 0$, a contradiction. □

Proof of Theorem 2.9. For $r \geqslant 1$ set

$$N^r := \frac{M^r}{(f-1, \delta(f-1), \dots, \delta^{r-1}(f-1))}.$$

Note that

$$E^i\left(\delta^{i-1}(f-1)\right) = \delta^{i-1}\left(E^1(f-1)\right) = \delta^{i-1}(0) = 0.$$

So there are surjective homomorphisms $N^r \to S^r_{\heartsuit}$, hence surjective homomorphisms $\overline{N^r} \to \overline{S^r_{\heartsuit}}$, hence a surjective homomorphism

$$\varinjlim \overline{N^r} \to \varinjlim \overline{S^r_{\heartsuit}} = \overline{S^{\infty}_{\heartsuit}}. \tag{2.13}$$

Now let $X = \bigcup_\alpha X_\alpha$, $X_\alpha = Spec\, S_\alpha$, be an affine open covering such that L is trivial on each X_α. Let x_α be a basis of L on X_α and let $z_\alpha = x_\alpha^{-1}$. Set

$$S^r_\alpha := S^r_{X_\alpha} = \big(S^r \otimes_S S_\alpha\big)\hat{},$$
$$M^r_\alpha := M^r_{X_\alpha} = \big(M^r \otimes_S S_\alpha\big)\hat{}.$$

Then we have an identification

$$M^r_\alpha = S^r_\alpha\big[z_\alpha, z_\alpha^{-1}, z'_\alpha, \ldots, z_\alpha^{(r)}\big]\hat{}.$$

Write $f = \varphi_\alpha x_\alpha^{\phi-1}$, with $\varphi_\alpha \in S^1_\alpha$. Since f and x_α are invertible in M^1_α it follows that φ_α is invertible in M^1_α, hence in S^1_α. Set $N^r_\alpha = (N^r \otimes_S S_\alpha)\hat{}$; hence

$$N^r_\alpha := \frac{S^r_\alpha[z_\alpha, z_\alpha^{-1}, z'_\alpha, \ldots, z_\alpha^{(r)}]\hat{}}{(\frac{z_\alpha^\phi}{z_\alpha} - \varphi_\alpha, \delta(\frac{z_\alpha^\phi}{z_\alpha} - \varphi_\alpha), \ldots, \delta^{r-1}(\frac{z_\alpha^\phi}{z_\alpha} - \varphi_\alpha))}.$$

For $i \geqslant 1$ set $u_{i,\alpha} := \frac{z_\alpha^{(i)}}{z_\alpha^{p^i}}$. Also, for $r \geqslant 1$, set

$$Q_\alpha^{r,0} := \frac{\overline{S^r_\alpha}[z_\alpha, z_\alpha^{-1}]}{(z_\alpha^{p-1} - \overline{\varphi_\alpha})} = \frac{\overline{S^r_\alpha}[z_\alpha]}{(z_\alpha^{p-1} - \overline{\varphi_\alpha})}. \tag{2.14}$$

(The latter equality is true because $\varphi_\alpha \in (S^1_\alpha)^\times$.) Then, by Lemma 2.11 we have $\overline{N^1_\alpha} = Q_\alpha^{1,0}[u_{1,\alpha}]$ and

$$\overline{N^r_\alpha} = \frac{Q_\alpha^{r,0}[u_{1,\alpha}, \ldots, u_{r,\alpha}]}{(u_{1,\alpha}^p - u_{1,\alpha} - G_0, \ldots, u_{r-1,\alpha}^p - u_{r-1,\alpha} - G_{r-2})}, \quad r \geqslant 2,$$

where $G_0 \in Q_\alpha^{r,0}$, and

$$G_i \in Q_\alpha^{r,i} := \frac{Q_\alpha^{r,0}[u_{1,\alpha}, \ldots, u_{i,\alpha}]}{(u_{1,\alpha}^p - u_{1,\alpha} - G_0, \ldots, u_{i,\alpha}^p - u_{i,\alpha} - G_{i-1})}, \quad i \geqslant 1.$$

Clearly the schemes $Spec\, Q_\alpha^{r,i}$, for various αs naturally glue to give a scheme $Spec\, Q^{r,i}$; so $Q^{r,i} \otimes_{\overline{S}} \overline{S_\alpha} = Q_\alpha^{r,i}$ for all α. Note that we have

$$Q_\alpha^{r,i} = \frac{Q_\alpha^{r,i-1}[u_{i,\alpha}]}{(u_{i,\alpha}^p - u_{i,\alpha} - G_{i-1})} \tag{2.15}$$

and natural inclusions

$$Q_\alpha^{r,0} \subset Q_\alpha^{r,1} \subset \cdots \subset Q_\alpha^{r,r-1} \subset \overline{N^r_\alpha} = Q_\alpha^{r,r-1}[u_{r,\alpha}]. \tag{2.16}$$

So we have natural homomorphisms

$$\cdots \to Q_\alpha^{r,r-1} \to \overline{N_\alpha^r} \to Q_\alpha^{r+1,r} \to \overline{N_\alpha^{r+1}} \to \cdots$$

which shows that, for each α,

$$\Bigl(\varinjlim_r \overline{N^r}\Bigr) \otimes_{\overline{S}} \overline{S_\alpha} = \varinjlim_r \overline{N_\alpha^r} = \varinjlim_r Q_\alpha^{r,r-1} = \Bigl(\varinjlim_r Q^{r,r-1}\Bigr) \otimes_{\overline{S}} \overline{S_\alpha}.$$

These isomorphisms glue together to give an isomorphism

$$\varinjlim \overline{N^r} = \varinjlim Q^{r,r-1}.$$

We are left to proving that $\varinjlim Q^{r,r-1}$ is an ind-étale $\mathbb{Z}_p^\times$-extension of $\overline{S^\infty} = \varinjlim \overline{S^r}$.

Start by noting that the maps $Q_\alpha^{r,r-1} \to Q_\alpha^{r+1,r}$ are injective. Also $\overline{S_\alpha^r} \to Q_\alpha^{r,r-1}$ are injective and étale; cf. (2.14) and (2.15). Now the group $\Gamma = \mathbb{Z}_p^\times$ acts on M_α^r via the rule $\gamma \cdot z_\alpha^{(i)} = \delta^i(\gamma z_\alpha)$ for $\gamma \in \Gamma$. This induces a Γ-action on N_α^r and hence a Γ-action on $\overline{N_\alpha^r}$. The latter factors through an action of $\Gamma_r := (\mathbb{Z}/p^{r+1}\mathbb{Z})^\times$. Moreover, for $i \leqslant r-1$, $Q_\alpha^{r,i}$ is Γ_r-stable and the Γ_r-action on $Q_\alpha^{r,i}$ factors through a Γ_i-action. For a fixed r we will prove by induction on $0 \leqslant i \leqslant r-1$ that

$$\bigl(Q_\alpha^{r,i}\bigr)^{\Gamma_i} = \overline{S_\alpha^r}. \tag{2.17}$$

This will end the proof of the theorem; indeed from the above we trivially get that the maps $Q^{r,r-1} \to Q^{r+1,r}$ are injective, the maps $\overline{S^r} \to Q^{r,r-1}$ are injective and étale, and, with respect to the induced action,

$$\bigl(Q^{r,r-1}\bigr)^{\Gamma_{r-1}} = \overline{S^r},$$

showing that $\varinjlim Q^{r,r-1}$ is an ind-étale $\mathbb{Z}_p^\times$-extension of $\overline{S^\infty} = \varinjlim \overline{S^r}$.

Let us check (2.17). For $i = 0$ we proceed as follows. Let $b \in Q_\alpha^{r,0}$ be the class of z_α and let $\Gamma_0 = \mathbb{F}_p^\times = \langle \zeta \rangle$, ζ a primitive root. Then $Q_\alpha^{r,0}$ is a free $\overline{S_\alpha^r}$-module with basis $1, b, b^2, \ldots, b^{p-2}$. If $\sum_{l=0}^{p-2} \lambda_l b^l$ is Γ_0-invariant (where $\lambda_l \in \overline{S_\alpha^r}$) then $\sum_{l=0}^{p-2} \lambda_l \zeta^l b^l = \sum_{l=0}^{p-2} \lambda_l b^l$. Since ζ is primitive we get $\lambda_1 = \cdots = \lambda_{p-2} = 0$, and the case $i = 0$ is proved.

Now assume $(Q_\alpha^{r,i-1})^{\Gamma_{i-1}} = \overline{S_\alpha^r}$ and let us prove (2.17). Recall Eq. (2.15) and consider the subgroup

$$\Delta_i := \{\gamma_0, \ldots, \gamma_p\} \subset \Gamma_i, \qquad \gamma_a = 1 + p^i a + p^{i+1}\mathbb{Z};$$

so Δ_i is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ via $\gamma_a \mapsto \bar{a}$. Note that Δ_i acts trivially on $Q_\alpha^{r,i-1}$. By Lemma 2.12 the Δ_i-action on $Q_\alpha^{r,i}$ corresponds to the $\mathbb{Z}/p\mathbb{Z}$-action induced by $\bar{a} \cdot u_i = u_i + \bar{a}$, so we are in the situation described in Lemma 2.13 and we may conclude by that Lemma plus the equality $(Q_\alpha^{r,i-1})^{\Gamma_{i-1}} = \overline{S_\alpha^r}$. This ends the proof of (2.17) and hence of the theorem. □

Remark 2.14. Assume that the following conditions are satisfied. (This is the case, as we shall see, in the applications to modular curves.)

(1) The element $\overline{\varphi} \in \overline{S^1}$ belongs to $\overline{S^0}$.
(2) The polynomial $z^{p-1} - \overline{\varphi}$ is irreducible in $Frac(\overline{S^0})[z]$.

(Here *Frac* means *field of fractions*.) Then the field

$$\frac{Frac(\overline{S^0})[z]}{(z^{p-1}-\overline{\varphi})} \tag{2.18}$$

is embedded into the ring $\overline{S^{\infty}_{\heartsuit}} \otimes_{\overline{S^0}} Frac(\overline{S^0})$.

2.2.4. The rings $M^r_{\heartsuit}$

For our applications to Igusa modular forms we need one more general construction. Given a framed curve $X = Spec\, S$ we define

$$M^r_{\heartsuit} := \left(S^r_{\heartsuit} \otimes_{S^r} M^r\right)\hat{},$$
$$M^{\infty}_{\heartsuit} := \varinjlim M^r_{\heartsuit}.$$

The inclusion $S^r_{\heartsuit} \subset S^r_{for}$ and the homomorphism $E^r : M^r \to S^r_{for}$ induce a homomorphism (still denoted by E^r and still referred to as *δ-expansion map*),

$$E^r : M^r_{\heartsuit} \to S^r_{for}.$$

Assume now L is trivial on X; if x is a basis of L then

$$M^r_{\heartsuit} = S^r_{\heartsuit}\left[x, x^{-1}, x', \ldots, x^{(r)}\right]\hat{}.$$

Define, in case L is trivial,

$$M^r_{\heartsuit}(w) := S^r_{\heartsuit} x^w \subset M^r_{\heartsuit};$$

the latter definition is independent of the choice of the basis x. (We will not need and hence we will not define the space $M^r_{\heartsuit}(w)$ in case L is not trivial.) Note also that the element

$$f^0 := E\left(x^{-1}\right) \otimes x = E\left(x^{-1}\right)x \in M^0_{\heartsuit}(1), \tag{2.19}$$

image of the tautological element $x^{-1} \otimes x \in M \otimes_S M$, does not depend on the choice of the basis x of L and has the property that

$$E^0\left(f^0\right) = 1.$$

We may refer to f^0 as the *tautological element* of $M^0_{\heartsuit}(1)$.

Lemma 2.15. *For any $g \in M^r(w)$ we have $(f^0)^{-w} g = E^r(g)$ in $M^r_{\heartsuit}(0) = S^r_{\heartsuit}$.*

Proof. If $g = \gamma x^w$, $\gamma \in S^r$, then the image of g in $M^r_{\heartsuit}$ equals $E^r(\gamma)x^w$ hence the image of $(f^0)^{-w} g$ in $M^r_{\heartsuit}$ equals

$$\left(E\left(x^{-1}\right)x\right)^{-w} E^r(\gamma)x^w = E^r\left(x^w\right)x^{-w} E^r(\gamma)x^w = E^r\left(\gamma x^w\right) = E^r(g). \qquad \square$$

The following is trivial to check:

Proposition 2.16. *Assume L is trivial on X. Then*:

(1) *The homomorphisms* $\overline{M^r} \to \overline{M^r_\heartsuit}$, $\overline{M^\infty} \to \overline{M^\infty_\heartsuit}$ *are inective. In particular the homomorphisms* $M^r \to M^r_\heartsuit$, $M^\infty \to M^\infty_\heartsuit$ *are injective with torsion free cokernel.*

(2) *The homomorphisms* $M^r_\heartsuit(w) \to S^r_{for}$ *are injective.*

(3) *The homomorphisms* $M^r_\heartsuit \to M^{r+1}_\heartsuit$ *are injective.*

2.3. *Application to δ-modular forms*

In what follows we shall apply the previous construction to certain framed curves arising from modular (respectively Shimura) curves.

2.3.1. *Review of modular curves*

Our reference here is [13, p. 112]. The discussion there involves the modular curve parameterizing elliptic curves with an embedding of μ_N rather than $\mathbb{Z}/N\mathbb{Z}$ as here. But, the two modular curves are isomorphic over $\mathbb{Z}[1/N, \zeta_N]$: see [13, p. 113].

Let $N > 3$ be an integer. Consider the $\mathbb{Z}[1/N, \zeta_N]$-scheme

$$Y_{\mathbb{Z}[1/N,\zeta_N]} := Y_1(N)_{\mathbb{Z}[1/N,\zeta_N]}$$

representing the functor taking a $\mathbb{Z}[1/N, \zeta_N]$-algebra B to the set of isomorphism classes of pairs (E, α) where E is an elliptic curve over B and α is a level $\Gamma_1(N)$-structure (i.e. $\alpha: (\mathbb{Z}/N\mathbb{Z})_B \subset E[N]$ is an inclusion). Inclusions as above are the same as inclusions $\mu_{N,B} \subset E[N]$ because we have fixed an Nth root of unity. Denote by $L_{\mathbb{Z}[1/N,\zeta_N]}$ the invertible sheaf on $Y_{\mathbb{Z}[1/N,\zeta_N]}$ obtained by taking the direct image of the sheaf of relative 1-forms on the universal elliptic curve over $Y_{\mathbb{Z}[1/N,\zeta_N]}$. Furthermore consider the $\mathbb{Z}[1/N, \zeta_N]$-scheme $X_1(N)_{\mathbb{Z}[1/N,\zeta_N]}$ defined by taking the Deligne–Rapoport compactification of $Y_{\mathbb{Z}[1/N,\zeta_N]}$: see [13, pp. 78–81] and still denote by $L_{\mathbb{Z}[1/N,\zeta_N]}$ the natural extension to $X_1(N)$ with the property that for any $\mathbb{Z}[1/N, \zeta_N]$-algebra B the B-module $M(B, \kappa, N)$ of *modular forms over B of level $\Gamma_1(N)$ and weight κ* (in the sense of [13, p. 111]), identifies with the space of sections $H^0(X_1(N)_B, L_B^{\otimes\kappa})$, where L_B is the corresponding sheaf obtained by pull-back. In particular the normalized Eisenstein forms E_4, $E_6.E_{p-1}$ belong to the spaces $M(\mathbb{Z}_p, 4, N)$, $M(\mathbb{Z}_p, 6, N)$, $M(\mathbb{Z}_p, p-1, N)$ respectively. The cusp ∞ on $X_1(N)_{\mathbb{Z}[1/N,\zeta_N]}$ is a $\mathbb{Z}[1/N, \zeta_N]$-point

$$P_{\mathbb{Z}[1/N,\zeta_N]} \in X_1(N)_{\mathbb{Z}[1/N]}\big(\mathbb{Z}[1/N, \zeta_N]\big).$$

Furthermore, for p not dividing N, we choose a homomorphism $\mathbb{Z}[1/N, \zeta_N] \to R = R_p = \hat{\mathbb{Z}}_p^{ur}$ and we denote by Y_R, L_R, P_R (or simply by Y, L, P) the objects over R obtained from $Y_{\mathbb{Z}[1/N]}$, $L_{\mathbb{Z}[1/N]}$, $P_{\mathbb{Z}[1/N,\zeta_N]}$ by base change. We denote by

$$Y_{ord} \subset Y = Y_R = Y_1(N)_R$$

the *ordinary locus* of Y i.e. the locus in Y where E_{p-1} is invertible. Finally we let $X = X_R$ be an arbitrary affine open set of $Y = Y_R$. (We will often assume in what follows that $X \subset Y_{ord}$ but at this stage we do not make this assumption.)

This provides us with data (2.1), (2.5), where $X^* = X_1(N)_R$. By [16] we also have at our disposal an expansion map $E : M \to S_{for}$ as in (2.6), the *Fourier expansion map*; recall from [16] that this map is obtained by interpreting sections of powers of L as modular forms and evaluating modular forms on the Tate curve $\mathrm{Tate}(q)/R((q))$ equipped with its standard 1-form and its standard immersion of $\mu_{N,R} \simeq (\mathbb{Z}/N\mathbb{Z})_R$. A tuple X, L, P, E as above is a Fourier-framed curve and X will be referred to as a *modular Fourier-framed curve*.

Alternatively we may keep the data X, L above as our data (2.1) but instead of P above we may choose a point $P \in X(R)$ represented by an elliptic curve E which is a canonical lift of an ordinary elliptic curve $\overline{E}$ over k. This provides us with data (2.4). Fix a $\mathbb{Z}_p$-basis of the Tate module $T_p(\overline{E})$. This basis defines an isomorphism between *Spf* $R[\![t]\!]$ and the completion of X along the image of P. The Serre–Tate parameter q corresponds to the value of $1 + t$. As in [19] (cf. also [6, p. 252]), we have at our disposal an expansion map $E : M \to S_{for}$ (2.6). A tuple X, L, P, E as above is a Serre–Tate-framed curve and X will be referred to as a *modular Serre–Tate-framed curve*.

In either case (Fourier and Serre–Tate) we have a natural homomorphism

$$\bigoplus_{\kappa \geqslant 0} M(R, \kappa, N) \to M = M_X.$$

2.3.2. Review of Shimura curves

The references here are [5,6,12].

Let D be a non-split indefinite quaternion algebra over $\mathbb{Q}$. Fix a maximal order $\mathcal{O}_D$ once and for all. Let $X^D(\mathcal{U})_{\mathbb{Z}[1/m]}$ be the Shimura curve attached to the pair $(D, \mathcal{U})$, where $\mathcal{U}$ is a sufficiently small compact subgroup of $(\mathcal{O}_D \otimes (\varprojlim \mathbb{Z}/n\mathbb{Z}))^\times$ such that $X^D(\mathcal{U})$ is connected and $m \in \mathbb{Z}_{>0}$ is an appropriate integer; see [12]. If D and $\mathcal{U}$ satisfy the conditions in [12]; then for some m the Shimura curve $X^D(\mathcal{U})_{\mathbb{Z}[1/m]}$ is a $\mathbb{Z}[1/m]$-scheme with geometrically integral fibers, such that for any $\mathbb{Z}[1/m]$-algebra B the set $X^D(\mathcal{U})_{\mathbb{Z}[1/m]}(B)$ is in bijection with the set of isomorphism classes of triples (E, i, α) where (E, i) is a false elliptic curve over B (i.e. E/B is an abelian scheme of relative dimension 2 and $i : \mathcal{O}_D \to End(E/B)$ is an injective ring homomorphism) and α is a level $\mathcal{U}$ structure. Let $Y_{\mathbb{Z}[1/m]}$ be an affine open subscheme of $X^D(\mathcal{U})_{\mathbb{Z}[1/m]}$. For any p not dividing m denote by $Y = Y_R$ the R-scheme obtained by base change from $Y_{\mathbb{Z}[1/m]}$, where $R = R_p = \hat{\mathbb{Z}}_p^{ur}$. We denote by $L = L_R$ the invertible sheaf on Y_R of *false* 1*-forms*; cf. [6, p. 230]. Finally let $X = X_R \subset Y_R$ be an arbitrary affine open set. The pair X, L provides us with the data (2.1).

Next, by the proof of Lemma 2.6 in [5], for any p sufficiently big, there exist infinitely many k-points $\overline{P} \in X^D(\mathcal{U})(k)$ whose associated triple $(\overline{E}, \bar{i}, \bar{\alpha})$ is such that

(1) $\overline{E}$ is ordinary, and
(2) if $\bar{\theta}$ is the unique principal polarization compatible with $\bar{i}$, then $(\overline{E}, \bar{\theta})$ is isomorphic to the polarized Jacobian of a genus-2 curve.

Choose a p and a point $\overline{P} \in X(k)$ as above. Let E be the canonical lift of $\overline{E}$. Since $End(E) \simeq End(\overline{E})$, the embedding $\bar{i} : \mathcal{O}_D \to End(\overline{E})$ induces an embedding $i : \mathcal{O}_D \to End(E)$. Also the level $\mathcal{U}$ structure $\bar{\alpha}$ lifts to a level $\mathcal{U}$ structure on (E, i). Let $P := (E, i, \alpha) \in X(R)$; this provides us with the data (2.4). Let $\overline{E}^\vee$ be the dual of $\overline{E}$. By Lemma 2.5 in [5], there exist $\mathbb{Z}_p$-bases of the Tate modules $T_p(\overline{E})$ and $T_p(\overline{E}^\vee)$, corresponding to each other under $\bar{\theta}$, such that any false elliptic curve over R lifting $(\overline{E}, \bar{i})$ has a diagonal Serre–Tate matrix $\mathrm{diag}(q, q^{disc(D)})$ with respect to these bases. Fix such bases. They define an isomorphism between $Spf\, R[[t]]$ and the completion of X along the image of P. The Serre–Tate parameter q corresponds to the value of $1 + t$. As in [6, p. 252], we have at our disposal an expansion map as in (2.6) (referred to in [6] as a *Serre–Tate expansion map*). A tuple X, L, P, E as above is a Serre–Tate-framed curve and X will be referred to as a *Shimura Serre–Tate-framed curve* or simply as a *Shimura framed curve* (for there is no Fourier side of the story in the Shimura curve case).

2.3.3. δ-modular functions and forms

In what follows we assume X is either a modular (Fourier or Serre–Tate) framed curve (cf. Section 2.3.1) or a Shimura framed curve (cf. Section 2.3.2).

Following [6] the elements of M^r will be called *δ-modular functions* (holomorphic on X) and the elements of $M^r(w)$ will be called *δ-modular forms of weight w* (holomorphic on X). The δ-expansion maps are called *δ-Fourier* respectively *δ-Serre–Tate* expansion maps.

Remark 2.17.

(1) Assume $X = Y = Y_1(N)_R$ and consider "test objects" of the form (E, α, ω, S^*) where $S^* = (S^n)$ is a prolongation sequence over R of p-adically complete rings, E/S^0 is an elliptic curve, α is a $\Gamma_1(N)$-

level structure on E, and ω is an invertible 1-form on E. Then giving an element of M^r_X is the same as giving a rule f that attaches to any test object as above an element

$$f\big(E,\alpha,\omega,S^*\big)\in S^r$$

which only depends on the isomorphism class of (E,α,ω) and is functorial in S^*. Moreover the elements of $M^r_X(w)$ correspond to rules f such that

$$f\big(E,\alpha,\lambda\omega,S^*\big)=\lambda^{-w}f\big(E,\alpha,\omega,S^*\big) \tag{2.20}$$

for $\lambda\in(S^0)^\times$. So the space $M^r_X(w)$ identifies with the space $M^r(\Gamma_1(N),R,w)$ in [7].

(2) Assume $X=Y_{ord}\subset Y$ is the locus in $Y=Y_1(N)_R$ where E_{p-1} is invertible. Then giving an element in M^r_X is the same as giving a rule f as above but only defined on test objects (E,α,ω,S^*) such that E has ordinary reduction mod p. Then the elements of $M^r_X(w)$ correspond to the rules (defined for elliptic curves with ordinary reduction) satisfying Eq. (2.20). So the space $M^r_X(w)$ identifies, in this case, with the space $M^r_{ord}(\Gamma_1(N),R,w)$ in [7].

Going back to the case of a general modular or Shimura framed curve we have, under certain additional assumptions, some remarkable forms f^1, f^∂, $f^\sharp$, respectively f^0, $f^\natural$, which we now review (respectively introduce).

2.3.4. Review of the forms f^1, f^∂

The references here are [4,1,5,6].

In the Fourier expansion case we set

$$\Psi:=\frac{1}{p}\log\frac{q^\phi}{q^p}:=\sum_{n\geqslant 1}(-1)^{n-1}n^{-1}p^{n-1}\left(\frac{q'}{q^p}\right)^n\in S^1_{for}=R((q))\hat{}\big[q'\big]\hat{}. \tag{2.21}$$

In the Serre–Tate expansion case we still denote by

$$\Psi\in S^1_{for}=R[\![t]\!]\big[t'\big]\hat{} \tag{2.22}$$

the image of Ψ in (2.21) via the homomorphism

$$R((q))\hat{}\big[q'\big]\hat{}\to R[\![t]\!]\big[t'\big]\hat{} \tag{2.23}$$

given by $q\mapsto t+1$, $q'\mapsto\delta(t+1)$. So Ψ in (2.22) is given by

$$\Psi=\frac{1}{p}(\phi-p)\log(1+t). \tag{2.24}$$

In the next two propositions $X=Spec\,S$ is either a modular framed curve or a Shimura framed curve; recall the δ-expansion maps $E^r:M^r\to S^r_{for}$, cf. (2.8).

Proposition 2.18. *(See [4,5].) There exists a unique form $f^1\in M^1(-1-\phi)$ such that*

$$E^1\big(f^1\big)=\Psi.$$

Proposition 2.19. *(See [1,5,6].) Assume the reduction mod p of X, $\overline{X}$, is contained in the ordinary locus of the modular (respectively Shimura) curve. Then there exists a unique form $f^{\partial} \in M^1(\phi - 1)$ which is invertible in the ring M^1 such that*

$$E^1\left(f^{\partial}\right) = 1.$$

Furthermore its reduction mod p, $\overline{f^{\partial}} \in \overline{M^1(\phi - 1)}$, coincides with the image of the Hasse invariant $\overline{H} \in \overline{M^0(p - 1)}$. In particular X is an ordinary framed curve in the sense of Definition 2.5.

So Theorem 2.9 and Corollary 2.10 hold for any X as in Proposition 2.19. For the definition of the Hasse invariant in the modular curve case see [16]. For the definition in the Shimura curve case see [6, p. 236]. Proposition 2.19 justifies the use of the terminology "ordinary" introduced in Definition 2.5. The condition that $\overline{X}$ be contained in the ordinary locus of the modular (respectively Shimura) curve is satisfied if we assume that a lift $H \in M^0(p - 1)$ of the Hasse invariant is invertible on X; in the case of modular curves (cf. Section 2.3.1) this is so if we assume that the Eisenstein form $E_{p-1} \in M(R, p - 1, N)$ is invertible on X, i.e. if $X \subset Y_{ord}$.

2.3.5. Igusa δ-modular functions and forms

Let X be either modular framed curve (cf. Section 2.3.1) or a Shimura framed curve (cf. Section 2.3.2). Assume $\overline{X}$ is contained in the ordinary locus of the modular (respectively Shimura) curve and assume L is trivial on X. The elements of $M^r_{\heartsuit}$ (respectively $M^r_{\heartsuit}(w)$) will be called *Igusa δ-modular functions* (respectively *Igusa δ-modular forms of weight w*). The "Igusa" terminology is justified by the fact that, due to Proposition 2.19, the assumptions in Remark 2.14 are satisfied in the modular curve case (of Section 2.3.1) and, in this case, the field (2.18) identifies with the field of rational functions on the Igusa curve [14, pp. 460–461].

2.3.6. The forms f^0 and $f^{\natural}$

Assume L is trivial on X. We may consider the Igusa δ-modular form $f^0 \in M^0_{\heartsuit}(1)$, cf. (2.19). So, by Proposition 2.16, $f^{\partial} = (f^0)^{\phi-1}$ in $M^1_{\heartsuit}$.

Corollary 2.20. $(\overline{f^0})^{p-1} = \overline{H}$ *in* $\overline{M^1_{\heartsuit}}$.

Proof. We have

$$\left(\overline{f^0}\right)^{p-1} = \overline{\left(f^0\right)^{\phi-1}} = \overline{f^{\partial}} = \overline{H}. \qquad \square$$

Also let us note that we have, at our disposal, a remarkable Igusa δ-modular form of weight 0:

$$f^{\natural} := \left(f^0\right)^{\phi+1} f^1 \in M^1_{\heartsuit}(0) = S^1_{\heartsuit}. \tag{2.25}$$

This form will play a key role in the several primes theory; cf. (3.12) below.

2.3.7. Review of the forms $f^{\sharp}$

The references here are [7,8].

First recall some standard definitions in the classical theory of complex modular forms [13]. Consider the space

$$S_m\left(\Gamma_1(N), \mathbb{C}\right) \subset M(\mathbb{C}, m, N)$$

of (classical) cusp forms $\sum a_n q^n$ of weight m on $\Gamma_1(N)$ over the complex field $\mathbb{C}$. On this space one has Hecke operators $T_m(n)$ acting, $n \geqslant 1$. An *eigenform* is a nonzero simultaneous eigenvector for all

$T_m(n)$, $n \geqslant 1$. An eigenform $f = \sum_{n\geqslant 1} a_n q^n$ is *normalized* if $a_1 = 1$; in this case $T_m(n)f = a_n f$ for all $n \geqslant 1$. One associates to any eigenform f its *system of eigenvalues* $l \mapsto a_l$, $(l, N) = 1$. A *newform* is a normalized eigenform whose system of eigenvalues does not come from a system of eigenvalues associated to an eigenform of level M with $M \mid N$, $M \neq N$. We denote by $S_m(\Gamma_0(N), \mathbb{C})$ the subspace of $S_m(\Gamma_1(N), \mathbb{C})$ of all cusp forms of weight m on $\Gamma_0(N)$. By Eichler–Shimura theory to each newform $f = \sum a_n q^n \in S_2(\Gamma_0(N), \mathbb{C})$ with $\mathbb{Z}$-Fourier coefficients there corresponds an elliptic curve E_f over $\mathbb{Q}$. We say that f is of CM type if E_f has CM.

Now we place ourselves in the context of a modular Fourier-framed curve.

Theorem 2.21. *(See [7].) Let $f = \sum a_n q^n \in S_2(\Gamma_0(N), \mathbb{C})$ be a newform with $\mathbb{Z}$-Fourier coefficients. Assume that f is not of CM type and p is sufficiently big. Then there is a unique form $f^\sharp \in M^2(0)$ with δ-Fourier expansion*:

$$E^2\big(f^\sharp\big) = \frac{1}{p}\sum_{n\geqslant 1}\frac{a_n}{n}\big(q^{n\phi^2} - a_p q^{n\phi} + pq^n\big) \in R((q))\hat{}\big[q', q''\big]\hat{}. \tag{2.26}$$

There is a corresponding result for the case when f is of CM type; cf. [7].

2.3.8. Canonical derivations on $S^\infty_\heartsuit$

In this section we let $X = Spec\, S$ be either a modular framed curve or a Shimura framed curve and we assume $\overline{X}$ is contained in the ordinary locus; we show that $S^\infty_\heartsuit$ (and other rings related to it) carry a complete system of conjugates of a classical derivation operator on modular forms. Recall the *Serre derivation operator* $\partial : M \to M$ introduced by Serre and Katz (cf. [16] for the modular curve case and also [6, p. 254], for the modular/Shimura case). Recall that $M = \bigoplus_{n\in\mathbb{Z}} L^{\otimes n}$ and $\partial(L^{\otimes n}) \subset L^{\otimes(n+2)}$. Similarly one can consider the *Euler derivation operator* $\mathcal{D} : M \to M$ defined by requiring that its restriction to $L^{\otimes n}$ is multiplication by n. Finally recall the Ramanujan form $P \in M^0(2)$; cf. [16] in the modular case and [6, Definition 8.33], in the modular/Shimura curve case. (We stress the fact that P is well defined because we are assuming that $\overline{X}$ is contained in the ordinary locus of the modular/Shimura curve.) Consider the derivation

$$\partial^* := \partial + P\mathcal{D} : M^0 \to M^0,$$

and its complete sequence of conjugates $\partial^*_j : M^\infty \to M^\infty$, $j \geqslant 0$.

Lemma 2.22. *The kernel of the δ-expansion map $E^\infty : M^\infty \to S^\infty_{for}$ is sent into itself by each of the derivations ∂^*_j.*

Proof. Assume first we are in the Serre–Tate case. It follows from [6, Proposition 8.42], that there exist derivations D_j on S^∞_{for} such that the difference $E^\infty \circ \partial^*_j - D_j \circ E^\infty$ vanishes on $\bigoplus_{w\in W(r)} M^r(w)$ for all r. It follows easily that the difference above vanishes on the whole of M^∞. But this implies the assertion of the lemma. A similar argument holds in the Fourier case; one uses [1, Proposition 4.2], instead. $\square$

Taking the derivations (still denoted by ∂^*_j) on $S^\infty_\heartsuit$ induced by $\partial^*_j : M^\infty \to M^\infty$ (cf. Lemma 2.22) we get:

Corollary 2.23. *There is a (necessarily unique) complete sequence of conjugates*

$$\partial^*_j : S^\infty_\heartsuit \to S^\infty_\heartsuit$$

of $\partial^* : M^0 \to M^0$. Moreover $\partial^*_j S^s_\heartsuit \subset S^s_\heartsuit$ for all $j, s \geqslant 0$ and $\partial^*_j S^s_\heartsuit = 0$ for $s < j$.

Uniqueness in the above statement follows from the fact that $S^{\infty}_{\heartsuit}$ is (clearly) p-adically separated and topologically δ-generated by M^0.

One can ask what is the effect of the derivations ∂_j^* when applied to the form $f^{\natural}$ in (2.25). Recall that this form was defined only in case L is trivial on X. Clearly $\partial_j^* f^{\natural} = 0$ for $j \geqslant 2$. For ∂_0^*, ∂_1^* we have:

Proposition 2.24.

$$\partial_0^* f^{\natural} = -1,$$
$$\partial_1^* f^{\natural} = 1.$$

Proof. Recall from [6, pp. 271–272], that

$$\partial_0^* f^1 = -\frac{1}{f^{\partial}},$$
$$\partial_1^* f^1 = f^{\partial}$$

in M^{∞}. By Lemma 2.15 it follows that

$$\partial_0^* f^{\natural} = \partial_0^*\big(E^1\big(f^1\big)\big) = E^1\big(\partial_0^* f^1\big) = E^1\left(-\frac{1}{f^{\partial}}\right) = -1.$$

The computation for $\partial_1^* f^{\natural}$ is similar. □

Now the derivations ∂_j^* in Corollary 2.23 induce k-derivations $\overline{\partial_j^*} : \overline{S^{\infty}_{\heartsuit}} \to \overline{S^{\infty}_{\heartsuit}}$.

Lemma 2.25. *The kernel of the natural map $\overline{E^{\infty}} : \overline{S^{\infty}_{\heartsuit}} \to \overline{S^{\infty}_{for}}$ is sent into itself by each of the derivations $\overline{\partial_j^*} : \overline{S^{\infty}_{\heartsuit}} \to \overline{S^{\infty}_{\heartsuit}}$.*

Proof. Let D_j be as in the proof of Lemma 2.22 and let $\overline{D_j} : \overline{S^{\infty}_{for}} \to \overline{S^{\infty}_{for}}$ be the induced derivations. So $\overline{E^{\infty}} \circ \overline{\partial_j^*} = \overline{D_j} \circ \overline{E^{\infty}}$ on $\overline{M^{\infty}}$ hence on $\overline{S^{\infty}_{\heartsuit}}$. This implies the statement of the lemma. □

Recall the ring $\widetilde{S^{\infty}_{\heartsuit}}$; cf. (2.11). Then we have:

Corollary 2.26. *The derivations $\overline{\partial_j^*} : \overline{S^{\infty}_{\heartsuit}} \to \overline{S^{\infty}_{\heartsuit}}$ induce derivations $\widetilde{\partial_j^*} : \widetilde{S^{\infty}_{\heartsuit}} \to \widetilde{S^{\infty}_{\heartsuit}}$.*

We want to further analyze the derivations in Corollary 2.26. It is convenient to give the following:

Definition 2.27. Let F be an algebra over a ring Λ. (In the applications Λ will be either R or $k = R/pR$.) We say that a sequence $\theta_0, \theta_1, \theta_2, \ldots \in Der_{\Lambda}(F, F)$ is *locally finite* if for any $x \in F$ there exists an integer j_0 such that for any $j \geqslant j_0$ we have $\theta_j x = 0$. Given such a locally finite sequence one has an F-linear map

$$F \times F \times F \times \cdots \to Der_{\Lambda}(F, F) \tag{2.27}$$

sending any vector $(c_0, c_1, c_2, \ldots)$ into the well defined derivation $\sum_{j \geqslant 0} c_j \theta_j$. A sequence of derivations is called a *pro-basis* of $Der_{\Lambda}(F, F)$ if it is locally finite and the map (2.27) is a bijection.

A basic example is the polynomial ring $F = \Lambda[T, T', T'', \ldots]$; in this case the sequence of derivations

$$\frac{\partial}{\partial T}, \frac{\partial}{\partial T'}, \frac{\partial}{\partial T''}, \ldots \in Der_{\Lambda}(F, F) \tag{2.28}$$

is a pro-basis of $Der_{\Lambda}(F, F)$.

Then we have the following:

Proposition 2.28. *Assume X is sufficiently small. (If X is a modular Fourier-framed curve it is enough to assume that the Eisenstein forms E_4, E_6, E_{p-1} are invertible in M.) Then the induced derivations*

$$\widetilde{\partial_0^*}, \widetilde{\partial_1^*}, \widetilde{\partial_2^*}, \ldots \in Der_k\big(\widetilde{S^{\infty}_{\heartsuit}}, \widetilde{S^{\infty}_{\heartsuit}}\big)$$

form a pro-basis of $Der_k(\widetilde{S^{\infty}_{\heartsuit}}, \widetilde{S^{\infty}_{\heartsuit}})$.

Proof. Shrinking X we may assume there is an étale coordinate T on X. (In the modular curve case, if E_4, E_6 are invertible in M then one can take $T = j$, the j-invariant.) By Corollary 2.10 and [6, Proposition 3.13], the derivations (2.28), for $\Lambda = k$, uniquely extend to derivations (still denoted by)

$$\frac{\partial}{\partial T}, \frac{\partial}{\partial T'}, \frac{\partial}{\partial T''}, \ldots \in Der_k\big(\widetilde{S^{\infty}_{\heartsuit}}, \widetilde{S^{\infty}_{\heartsuit}}\big).$$

The latter sequence is trivially checked to be a pro-basis of $Der_k(\widetilde{S^{\infty}_{\heartsuit}}, \widetilde{S^{\infty}_{\heartsuit}})$. Now express

$$\widetilde{\partial_j^*} = \sum_{s \geqslant 0} c_{js} \frac{\partial}{\partial T^{(s)}} \in Der_k\big(\widetilde{S^{\infty}_{\heartsuit}}, \widetilde{S^{\infty}_{\heartsuit}}\big),$$

with $c_{js} \in \widetilde{S^{\infty}_{\heartsuit}}$. Recall that $\partial_j^* T^{(s)} = 0$ for $s < j$. Hence $c_{js} = 0$ for $s < j$. Also

$$\widetilde{\partial_j^*} T^{(j)} = \overline{\phi^j\big(\partial^* T\big)} = \big(\overline{\partial^* T}\big)^{p^j} \in \widetilde{S^{\infty}_{\heartsuit}}.$$

Note that the image of $\overline{\partial^* T}$ in $\overline{S^{\infty}_{for}}$ is invertible; in the Serre–Tate expansion case this follows from [6, Propositions 8.34 and 8.42], while in the case of Fourier expansions this follows from the fact that

$$\partial j \in \mathbb{Z}_p^{\times} \cdot \frac{E_4^2 E_6}{\Delta}.$$

By shrinking X we may assume $\overline{\partial^* T}$ is invertible in $\widetilde{S^{\infty}_{\heartsuit}}$. (Again this is automatic in the modular case if E_4, E_6 are invertible in M.) In particular $c_{jj} \in (\widetilde{S^{\infty}_{\heartsuit}})^{\times}$ for all $j \geqslant 0$. Now for each $j \geqslant 0$ one can find inductively elements $\gamma_{js} \in \widetilde{S^{\infty}_{\heartsuit}}$ with $s \geqslant j$ such that

$$\frac{\partial}{\partial T^{(j)}} = \sum_{s \geqslant j} \gamma_{js} \widetilde{\partial_s^*} \in Der_k\big(\widetilde{S^{\infty}_{\heartsuit}}, \widetilde{S^{\infty}_{\heartsuit}}\big).$$

So any derivation

$$\theta = \sum_{j \geqslant 0} c_j \frac{\partial}{\partial T^{(j)}} \in Der_k\big(\widetilde{S^{\infty}_{\heartsuit}}, \widetilde{S^{\infty}_{\heartsuit}}\big)$$

with $c_j \in \widetilde{S_{\heartsuit}^{\infty}}$ can be written as

$$\theta = \sum_{s \geqslant 0} \left(\sum_{j=0}^{s} c_j \gamma_{js} \right) \widetilde{\partial_s^*}.$$

This representation is easily seen to be unique by evaluating successively at $T, T', T', \ldots$. This ends the proof of the proposition. □

Our next purpose is to explain the relation between our Igusa δ-modular functions and Katz's generalized p-adic modular functions [17,15]. This relation will play a role later, in the proof of Theorem 2.30 below.

We first need to review some of Katz's concepts.

2.3.9. Review of Katz generalized p-adic modular functions

The references here are [17,15].

Let B be a p-adically complete ring, $p \geqslant 5$, and let N be an integer coprime to p. Consider the functor

$$\{p\text{-adically complete } B\text{-algebras}\} \to \{\text{sets}\} \tag{2.29}$$

that attaches to any A the set of isomorphism classes of triples $(E/A, \varphi, \iota)$, where E is an elliptic curve over A, φ is a trivialization, and ι is an arithmetic level N structure. Recall that a *trivialization* is an isomorphism between the formal group of E and the formal group of the multiplicative group; an *arithmetic level N structure* is defined as an inclusion of flat group schemes over B of μ_N into $E[N]$. The functor (2.29) is representable by a p-adically complete ring $\mathbb{W}(B, N)$. The elements of this ring are called by Katz [17] *generalized p-adic modular forms*. Note that $\mathbb{W}(B, N) = \mathbb{W}(\mathbb{Z}_p, N) \widehat{\otimes} B$. Moreover there is a $\mathbb{Z}_p^{\times}$-action on $\mathbb{W}(B, N)$ coming from the action of $\mathbb{Z}_p^{\times}$ on the formal group of the multiplicative group. There is a natural *Fourier expansion map* $E : \mathbb{W}(B, N) \to B((q))\hat{}$ which is injective and has a flat cokernel over B. Also $\mathbb{W}(\mathbb{Z}_p, N)$ possesses a natural ring endomorphism *Frob* which reduces modulo p to the p-power Frobenius endomorphism of $\mathbb{W}(\mathbb{Z}_p, N) \otimes \mathbb{Z}/p\mathbb{Z}$. So if $R = \hat{\mathbb{Z}}_p^{ur}$, as usual, and if ϕ is the automorphism of R lifting Frobenius then *Frob* $\widehat{\otimes}\, \phi$ is a lift of Frobenius on

$$\mathbb{W} := \mathbb{W}(R, N) = \mathbb{W}(\mathbb{Z}_p, N) \widehat{\otimes} R$$

which we denote by ϕ_0. Moreover the homomorphism $\mathbb{W}(R, N) \to R((q))\hat{}$ commutes with the action of ϕ_0 where ϕ_0 on $R((q))\hat{}$ is defined by $\phi_0(\sum a_n q^n) := \sum \phi(a_n) q^{np}$.

For any $\mathbb{Z}[1/N, \zeta_N]$-algebra B the space $M(B, \kappa, N)$ of modular forms over B of weight κ and level $\Gamma_1(N)$ has an embedding

$$M(B, \kappa, N) \subset \mathbb{W}(B, N).$$

The space $M(B, \kappa, N)$ is stable under the $\mathbb{Z}_p^{\times}$-action on $\mathbb{W}(B, N)$ and $\lambda \in \mathbb{Z}_p^{\times}$ acts on $M(B, \kappa, N)$ via multiplication by λ^k. Recall that we denoted by $Y_{ord} \subset Y = Y_R = Y_1(N)_R$ the locus in Y where the Eisenstein form $E_{p-1} \in M(\mathbb{Z}_p, p-1, N)$ is invertible. Then, since E_{p-1} is invertible in $\mathbb{W}$ we get a homomorphism

$$M_{Y_{ord}} = \bigoplus_{k \in \mathbb{Z}} L_{Y_{ord}}^{\otimes k} \to \mathbb{W}.$$

More generally, if X is any affine open subset of Y_{ord} then one can find $g \in M_{Y_{ord}}$ of weight 0, $\bar{g} \neq 0$, and a homomorphism

$$M := M_X := \bigoplus_{\kappa \in \mathbb{Z}} L_X^{\otimes \kappa} \to \mathbb{W}_g = \mathbb{W}[1/g]. \tag{2.30}$$

(So if $X = Y_{ord}$ we may take $g = 1$.) Since g has weight 0, $\widehat{\mathbb{W}_g}$ has an induced $\mathbb{Z}_p^\times$-action and the homomorphism (2.30) is $\mathbb{Z}_p^\times$-equivariant if $\lambda \in \mathbb{Z}_p^\times$ acts on each $L^{\otimes k}$ via multiplication by λ^k.

Finally recall Katz's *ring of divided congruences* [15],

$$\mathbb{D} := \mathbb{D}(R, N) := \left\{ f \in \bigoplus_{\kappa \geqslant 0} M(R, \kappa, N) \otimes_R K;\ E(f) \in R[\![q]\!] \right\},$$

where $K := R[1/p]$. This ring naturally embeds into Katz's *ring of holomorphic generalized p-adic modular forms*,

$$\mathbb{V} := \mathbb{V}(R, N) = \left\{ f \in \mathbb{W}(R, N);\ E(f) \in R[\![q]\!] \right\},$$

and the image of $\mathbb{D}$ in $\mathbb{V}$ is p-adically dense. For simplicity we sometimes identify $\mathbb{D}$, $\mathbb{V}$, $\mathbb{W}$ with subrings of $R((q))\hat{}$; i.e. we view

$$\mathbb{D} \subset \mathbb{V} \subset \mathbb{W} \subset R((q))\hat{}.$$

We will need the following:

Lemma 2.29. *$\mathbb{D} + R[\Delta^{-1}]$ is p-adically dense in $\mathbb{W}$.*

Proof. It is enough to check that $\mathbb{V} + R[\Delta^{-1}]$ is p-adically dense in $\mathbb{W}$.

We first claim that for any $f \in \mathbb{W}$ there exists a sequence of polynomials $F_n \in R[t]$ in a variable t such that $F_{n+1} - F_n \in p^n R[t]$ for $n \geqslant 0$ and such that

$$f - F_n(\Delta^{-1}) \in p^n R((q))\hat{} + R[\![q]\!].$$

To check the claim we construct F_n by induction. We may take $F_0 = 0$. Now, assuming F_n was constructed, write

$$f - F_n(\Delta^{-1}) = p^n G + p^{n+1} H + S, \quad G \in \sum_{i=1}^{N} R q^{-i},\ H \in R((q))\hat{},\ S \in R[\![q]\!].$$

Since $\Delta^{-1} - q^{-1} \in R[\![q]\!]$ we can find a polynomial $\Gamma \in R[t]$ of degree $\leqslant N$ such that $G - \Gamma(\Delta^{-1}) \in R[\![q]\!]$. Then set $F_{n+1} := F_n + p^n \Gamma$ which ends the inductive step of our construction.

Now let F_n be as in our claim above and set $F := \lim F_n \in R[t]\hat{}$. Then clearly $f - F(\Delta^{-1}) \in R[\![q]\!] \cap \mathbb{W} = \mathbb{V}$. This implies that $\mathbb{V} + R[\Delta^{-1}]$ is p-adically dense in $\mathbb{W}$ and we are done. □

2.3.10. Link between Igusa δ-modular functions and Katz generalized p-adic modular functions

Since $\mathbb{W} := \mathbb{W}(R, N)$ is flat over R and comes with a lift of Frobenius, $\mathbb{W}$ has a natural structure of δ-ring. The same is true about $R((q))\hat{}$ and, moreover the Fourier expansion map $\mathbb{W} \to R((q))\hat{}$ is a δ-ring homomorphism. Note that δ acts on $R((q))\hat{}$ via the operator

$$\delta_0\left(\sum a_n q^n\right) := \frac{(\sum \phi(a_n) q^{np}) - (\sum a_n q^n)^p}{p}; \tag{2.31}$$

this is *not the same* as the restriction of $\delta : S^{\infty}_{for} \to S^{\infty}_{for}$ to $R((q))\hat{}$. More generally, $\widehat{\mathbb{W}_g}$ has a structure of δ-ring and the induced homomorphism

$$\widehat{\mathbb{W}_g} \to R((q))\hat{} \tag{2.32}$$

is a δ-ring homomorphism. (The latter is well defined because the image of g in $k((q))$ is non-zero, hence g is invertible in $R((q))\hat{}$.) Now the reduction mod p of (2.32) is injective (because the cokernel of $\mathbb{W} \to R((q))\hat{}$ is flat over R) so (2.32) itself is injective. We have an induced commutative diagram of δ-rings

$$\begin{tikzcd} M^{\infty} \arrow[r] \arrow[d] & S^{\infty}_{for} \arrow[d] \\ \widehat{\mathbb{W}_g} \arrow[r] & R((q))\hat{} \end{tikzcd}$$

where the right vertical arrow is induced by $q' \mapsto 0$, $q'' \mapsto 0$, etc. Note that $\mathbb{Z}_p^{\times}$ naturally acts on M^{∞} compatibly with δ (with $\lambda \in \mathbb{Z}_p^{\times}$ acting on $L^{\otimes n}$ via multiplication by λ^n) and the map $M^{\infty} \to \widehat{\mathbb{W}_g}$ is $\mathbb{Z}_p^{\times}$-equivariant. Now the kernel of the upper horizontal map is mapped to 0 in $R((q))\hat{}$, so is mapped into 0 in $\widehat{\mathbb{W}_g}$ (because the lower horizontal arrow is injective). Hence we have an induced homomorphism

$$S^{\infty}_{\heartsuit} \to \widehat{\mathbb{W}_g}. \tag{2.33}$$

By Proposition 2.18 the kernel of (2.33) contains the images of the elements

$$f^1, \quad \delta f^1, \quad \delta^2 f^1, \quad \ldots. \tag{2.34}$$

If L is trivial with basis x we set

$$f^{\flat} := f^1 x^{\phi+1} \in S^1 = M^1(0)$$

and then the kernel of (2.33) contains the images of the elements

$$f^{\flat}, \quad \delta f^{\flat}, \quad \delta^2 f^{\flat}, \quad \ldots. \tag{2.35}$$

In a similar vein note that the kernel of the map

$$M^{\infty} \to S^{\infty}_{\heartsuit} \subset S^{\infty}_{for} \tag{2.36}$$

contains

$$f^{\partial} - 1, \quad \delta\big(f^{\partial} - 1\big), \quad \delta^2\big(f^{\partial} - 1\big), \quad \ldots. \tag{2.37}$$

In the next section we will show, under an appropriate hypothesis, that the kernels of (2.33) and (2.36) are topologically generated by the elements (2.35) and (2.37) respectively.

2.3.11. Refinement of results in the modular case

Using the link between Igusa δ-modular functions and Katz generalized p-adic modular forms we can refine Theorem 2.9 in the modular case as follows:

Theorem 2.30. *Assume $X = Spec\, S$ is a modular Fourier-framed curve with E_{p-1} invertible on X. The following hold*:

(1) *The map $\overline{S^{\infty}_{\heartsuit}} \to \overline{S^{\infty}_{for}}$ is injective; in particular $\overline{S^{\infty}_{\heartsuit}}$ is an integral domain, and the map $\overline{S^{\infty}_{\heartsuit}} \to \widetilde{S^{\infty}_{\heartsuit}}$ is an isomorphism. Moreover the ring $\overline{S^{\infty}_{\heartsuit}}$ is an ind-étale $\mathbb{Z}_p^{\times}$-extension of $\overline{S^{\infty}}$.*

(2) *The kernel of $\overline{M^{\infty}} \to \overline{S^{\infty}_{\heartsuit}}$ is generated by*

$$\overline{f^{\partial}-1},\quad \overline{\delta\big(f^{\partial}-1\big)},\quad \overline{\delta^2\big(f^{\partial}-1\big)},\quad \ldots.$$

(3) *The kernel of $\overline{S^{\infty}_{\heartsuit}} \to \overline{\mathbb{W}}_g$ is generated by the images of*

$$\overline{f^1},\quad \overline{\delta f^1},\quad \overline{\delta^2 f^1},\quad \ldots.$$

(4) *The kernel of $\overline{M^{\infty}} \to \overline{\mathbb{W}}_g$ is generated by the elements*

$$\overline{f^{\partial}-1},\quad \overline{f^1},\quad \overline{\delta\big(f^{\partial}-1\big)},\quad \overline{\delta f^1},\quad \overline{\delta^2\big(f^{\partial}-1\big)},\quad \overline{\delta^2 f^1},\quad \ldots.$$

Corollary 2.31. *Assume $X = Spec\, S$ is a modular Fourier-framed curve with E_{p-1} invertible on X. The following hold*:

(1) *The inclusion $S^{\infty}_{\heartsuit} \subset S^{\infty}_{for}$ has torsion free cokernel.*

(2) *The kernel of $M^{\infty} \to S^{\infty}_{for}$ is the p-adic closure of the ideal generated by the elements*

$$f^{\partial}-1,\quad \delta\big(f^{\partial}-1\big),\quad \delta^2\big(f^{\partial}-1\big),\quad \ldots.$$

(3) *The kernel of $S^{\infty}_{\heartsuit} \to R((q))\hat{}$ is the p-adic closure of the ideal generated by the images of the elements*

$$f^1,\quad \delta f^1,\quad \delta^2 f^1,\quad \ldots.$$

(4) *The kernel of $M^{\infty} \to R((q))\hat{}$ is the p-adic closure of the ideal generated by the elements*

$$f^{\partial}-1,\quad f^1,\quad \delta\big(f^{\partial}-1\big),\quad \delta f^1,\quad \delta^2\big(f^{\partial}-1\big),\quad \delta^2 f^1,\quad \ldots.$$

Remark 2.32. Conclusion (1) in Corollary 2.31 should be viewed as a δ-*expansion principle*. Conclusions (2) and (4) should be viewed as δ-analogues of the theorem of Swinnerton–Dyer and Serre according to which the kernel of the Fourier expansion map

$$\bigoplus_{\kappa \geqslant 0} M(\mathbb{F}_p, \kappa, N) \to \mathbb{F}_p[\![q]\!]$$

is generated by $E_{p-1}-1$; cf. [14, p. 459].

To prove Theorem 2.30 we need the following:

Lemma 2.33. *Any ordinary k-point of $Y = Y_1(N)_R$ has an affine open neighborhood $X \subset Y$ such that E_{p-1} is invertible on X, L is trivial on X and the natural homomorphism*

$$\bar{S} \to \frac{\overline{S^r}}{(\overline{f^\flat}, \overline{\delta f^\flat}, \dots, \overline{\delta^{r-1} f^\flat})}$$

is an isomorphism.

Proof. Same argument as in [8, Lemma 4.66] (where the special case $r = 2$ was considered). □

Proof of Theorem 2.30. We are going to use the notation in the proof of Theorem 2.9. In particular recall the rings $Q^{r,r-1}$ which are finite étale extensions of $\overline{S^r}$, with

$$\left(Q^{r,r-1}\right)^{\Gamma_{r-1}} = \overline{S^r}. \tag{2.38}$$

Note that assertion (4) follows from assertions (2) and (3).

We claim that in order to prove assertions (1) and (2) it is enough to show that all the rings $Q^{r,r-1}$ are integral domains. Indeed if this is so then

$$Q^\infty := \varinjlim Q^{r,r-1}$$

is an integral domain. We have surjections

$$Q^\infty \to \overline{S^\infty_\heartsuit} \to \widetilde{S^\infty_\heartsuit}, \tag{2.39}$$

where the last ring is an integral domain. Let I be the kernel of the composition (2.39). Since the composition $\overline{S^\infty} \to Q^\infty \to \widetilde{S^\infty_\heartsuit}$ is injective (cf. Proposition 2.2), upon viewing $\overline{S^\infty}$ as a subring of Q^∞, it follows that $I \cap \overline{S^\infty} = 0$. Since Q^∞ is an integral domain and an integral extension of $\overline{S^\infty}$ it follows that $I = 0$. This forces the surjections in (2.39) to be isomorphisms, and so assertions (1) and (2) of the theorem follow.

Next note that since $\mathit{Spec}\, Q^{r,r-1}$ is étale and finite over $\mathit{Spec}\, \overline{S^r}$ and since the latter is smooth over k, it follows that $\mathit{Spec}\, Q^{r,r-1}$ is smooth over k so, in particular its connected components are irreducible and they are finite and étale over $\mathit{Spec}\ \overline{S^r}$. So in order to prove that $Q^{r,r-1}$ is an integral domain it is enough to prove that $\mathit{Spec}\, Q^{r,r-1}$ is connected.

Consequently in order to prove the theorem we need to prove connectivity of $\mathit{Spec}\, Q^{r,r-1}$ and assertion (3). We will prove these two facts simultaneously. To prove either of these facts it is enough to prove that these facts hold for each of the open sets of a given open cover of X. So we may assume, after shrinking X, that the conclusion of Lemma 2.33 holds for X, in particular L is trivial on the whole of X so $f^\flat$ is defined and f^1 and $f^\flat$ differ by a unit. Consider the scheme $\mathit{Spec}\, T^r$ defined by the Cartesian diagram

$$\begin{tikzcd}
\mathit{Spec}\, T^r \arrow[r] \arrow[d] & \mathit{Spec}\, Q^{r,r-1} \arrow[d] \\
\mathit{Spec}\, \bar{S} \arrow[r] & \mathit{Spec}\, \overline{S^r}
\end{tikzcd}$$

where the bottom horizontal arrow is defined by the surjection

$$\overline{S^r} \to \frac{\overline{S^r}}{(\overline{f^\flat}, \overline{\delta f^\flat}, \dots, \overline{\delta^{r-1} f^\flat})} = \bar{S},$$

cf. Lemma 2.33. The natural $\mathbb{Z}_p^\times$-equivariant homomorphism $M^\infty \to \widehat{\mathbb{W}_g}$ maps f^1, δf^1, $\delta^2 f^1, \ldots$ into 0; cf. Proposition 2.18. So this homomorphism also maps $f^\flat$, $\delta f^\flat$, $\delta^2 f^\flat, \ldots$ into 0. One the other hand this homomorphism also maps $f^\partial - 1$, $\delta(f^\partial - 1)$, $\delta^2(f^\partial - 1), \ldots$ into 0. So we get an induced $\mathbb{Z}_p^\times$-equivariant homomorphism $\overline{N^r} \to \overline{\mathbb{W}}_g$, hence (by restriction) we get a $\mathbb{Z}_p^\times$-equivariant homomorphism $Q^{r,r-1} \to \overline{\mathbb{W}}_g$, and hence we get an induced $\mathbb{Z}_p^\times$-equivariant homomorphism

$$T^r = \frac{Q^{r,r-1}}{(\overline{f^\flat}, \ldots, \overline{\delta^{r-1} f^\flat})} \to \overline{\mathbb{W}}_g.$$

Since $Spec\,\overline{\mathbb{W}}_g$ is irreducible the closure Z of the image of $Spec\,\overline{\mathbb{W}}_g \to Spec\,T^r$ is contained in one of the connected components of $Spec\,T^r$. Since Z dominates $Spec\,\overline{S}$ and since $Spec\,T^r$ is finite and étale over $Spec\,\overline{S}$, it follows that Z is a connected component of $Spec\,T^r$. Note that Z is a $\mathbb{Z}_p^\times$-invariant subset of $Spec\,T^r$, hence Γ_{r-1}-invariant. Recall that by (2.38) Γ_{r-1} acts transitively on the fibers of $Spec\,Q^{r,r-1} \to Spec\,\overline{S^r}$. Hence Γ_{r-1} acts transitively on the fibers of $Spec\,T^r \to Spec\,\overline{S}$. Since each connected component of $Spec\,T^r$ surjects onto $Spec\,\overline{S}$ and since Z is Γ_{r-1}-invariant it follows that $Spec\,T^r$ must be connected. Since $Spec\,T^r$ is smooth over k and connected it follows that T^r is an integral domain. Since $Spec\,T^r$ is connected it must coincide with Z hence $Spec\,\overline{\mathbb{W}}_g \to Spec\,T^r$ is dominant. Since T^r is an integral domain, $T^r \to \overline{\mathbb{W}}_g$ is injective. So $\varinjlim T^r \to \overline{\mathbb{W}}_g$ is injective. But

$$\varinjlim T^r = \varinjlim Q^{r,r-1} / \big(\overline{f^\flat}, \overline{\delta f^\flat}, \overline{\delta^2 f^\flat}, \ldots\big) = \overline{S^\infty_\heartsuit} / \big(\overline{f^\flat}, \overline{\delta f^\flat}, \overline{\delta^2 f^\flat}, \ldots\big).$$

This proves assertion (3).

On the other hand since each connected component of $Spec\,Q^{r,r-1}$ surjects onto $Spec\,\overline{S^r}$ and $Spec\,T^r$ is connected it follows that $Spec\,Q^{r,r-1}$ itself is connected. This ends the proof of the theorem. □

Corollary 2.34. *Assume $X = Spec\,S$ is a modular Fourier-framed curve with E_{p-1} invertible on X. Let $f(q) \in R((q))$ be contained in the image of the map $M \otimes_R K \to K((q))$. Then $f(q)$ is contained in the image of the map $M^\infty \to R((q))\hat{}$.*

Proof. Write $f(q) = E(\frac{G}{p^\nu}) = \frac{G(q)}{p^\nu}$, where $G \in M$. The image of G in $R((q)) \otimes \mathbb{Z}/p^\nu\mathbb{Z}$ is 0 so the image of G in $S^\infty_\heartsuit \otimes \mathbb{Z}/p^\nu\mathbb{Z}$ is in the kernel of $S^\infty_\heartsuit \otimes \mathbb{Z}/p^\nu\mathbb{Z} \to S^\infty_{for} \otimes \mathbb{Z}/p^\nu\mathbb{Z}$. But the latter morphism is injective; indeed this is trivially checked by induction on ν, using Theorem 2.30. It follows that the image of G in $S^\infty_\heartsuit \otimes \mathbb{Z}/p^\nu\mathbb{Z}$ is 0, hence the image of G in $S^\infty_\heartsuit$ belongs to $p^\nu S^\infty_\heartsuit$. Hence the image of G in $R((q))\hat{}$ belongs to $p^\nu \cdot Im(M^\infty \to R((q))\hat{})$. It follows that $f(q)$ belongs to the image of $M^\infty \to R((q))\hat{}$. □

Recall that we denoted by Y_{ord} the locus in $Y := Y_1(N)_R$ where E_{p-1} is invertible.

Corollary 2.35. *Consider the modular Fourier-framed curve $X = Spec\,S = Y_{ord}$. Then the image of $M^\infty \to R((q))\hat{}$ contains $\mathbb{D}$ and hence is p-adically dense in $\mathbb{W}$.*

Proof. By Corollary 2.34 the image of $M^\infty \to R((q))\hat{}$ contains the ring $\mathbb{D}$. But this image also contains the ring $R[\Delta^{-1}]$. We conclude by Lemma 2.29. □

Let us also note the following refinement of our results on conjugate derivations.

Proposition 2.36. *Assume $X = Spec\,S$ is a modular Fourier-framed curve and assume the Eisenstein forms E_4, E_6, E_{p-1} are invertible on X. Then the sequence*

$$\partial_0^*, \partial_1^*, \partial_2^*, \ldots \in Der_R\big(S^\infty_\heartsuit, S^\infty_\heartsuit\big)$$

is a pro-basis of $Der_R(S^\infty_\heartsuit, S^\infty_\heartsuit)$.

Proof. Assume $\sum_{j\geqslant 0} c_j \partial_j^* = 0$ with $c_j \in S_{\heartsuit}^{\infty}$ and let us prove that $c_j = 0$ for all j. Assume this is not the case. We may assume not all $\bar{c}_j$ are 0 in $\overline{S_{\heartsuit}^{\infty}}$. We get $\sum_j \bar{c}_j \overline{\partial_j^*} = 0$ in $Der_k(\overline{S_{\heartsuit}^{\infty}}, \overline{S_{\heartsuit}^{\infty}})$. By Proposition 2.28 and Theorem 2.30 we get $\bar{c}_j = 0$ for all j, a contradiction.

Now let $\partial \in Der_R(S_{\heartsuit}^{\infty}, S_{\heartsuit}^{\infty})$. Using Proposition 2.28 and Theorem 2.30 we can find elements $\gamma_j \in (S_{\heartsuit}^{\infty})\hat{}$, $j \geqslant 0$, such that

$$\partial = \sum_{j\geqslant 0} \gamma_j \partial_j^* : S_{\heartsuit}^{\infty} \to \left(S_{\heartsuit}^{\infty}\right)\hat{}.$$

Evaluating ∂j we get $\gamma_0 = 1$. Since the image of $\partial\delta j$ in $(S_{\heartsuit}^{\infty})\hat{}$ is in $S_{\heartsuit}^{\infty}$ and $\partial_1\delta j = \phi(\partial j) \in (S_{\heartsuit}^{\infty})^{\times}$ we get $\gamma_1 \in S_{\heartsuit}^{\infty}$. Since the image of $\partial\delta^2 j$ in $(S_{\heartsuit}^{\infty})\hat{}$ is in $S_{\heartsuit}^{\infty}$ and $\partial_2\delta^2 j = \phi^2(\partial j) \in (S_{\heartsuit}^{\infty})^{\times}$ we get $\gamma_2 \in S_{\heartsuit}^{\infty}$. Continuing in this way we get that $\gamma_j \in S_{\heartsuit}^{\infty}$ for all j which ends the proof of the proposition. □

3. The theory for several primes

In this section several primes will be involved. So it will be important to keep track of the prime p used in the one prime theory of the previous section by making p appear as an index for the various objects considered there. In particular the objects

$$\delta,\ \phi,\ R,\ J^r(X),\ S^r,\ S^r_{\heartsuit},\ M^r,\ M^r_{\heartsuit},\ S^r_{for},\ f^0,\ f^1,\ f^{\partial},\ f^{\sharp},\ f^{\natural},\ H,\ \Psi,\ \text{etc.}$$

introduced in the previous section in the case of the prime p will be denoted from now on by

$$\delta_p,\ \phi_p,\ R_p,\ J^r_p(X),\ S^r_p,\ S^r_{p\heartsuit},\ M^r_p,\ M^r_{p\heartsuit},\ S^r_{for,p},\ f^0_p,\ f^1_p,\ f^{\partial}_p,\ f^{\sharp}_p,\ f^{\natural}_p,\ H_p,\ \Psi_p,\ \text{etc.}$$

3.1. *Review of concepts and terminology from [9]*

For any two distinct rational primes p_1, p_2 consider the polynomial C_{p_k,p_l} in the ring $\mathbb{Z}[X_0, X_1, X_2]$ defined by

$$C_{p_1,p_2}(X_0, X_1, X_2) := \frac{C_{p_2}(X_0^{p_1}, p_1 X_1)}{p_1} - \frac{C_{p_1}(X_0^{p_2}, p_2 X_2)}{p_2} - \frac{\delta_{p_1} p_2}{p_2} X_2^{p_1} + \frac{\delta_{p_2} p_1}{p_1} X_1^{p_2}. \quad (3.1)$$

Let $\mathcal{P} = \{p_1, \dots, p_d\}$ be a finite set of primes in $\mathbb{Z}$. A *$\delta_{\mathcal{P}}$-ring* is a ring A equipped with p_k-derivations $\delta_{p_k} : A \to A$, $k = 1, \dots, d$, such that

$$\delta_{p_k}\delta_{p_l} a - \delta_{p_l}\delta_{p_k} a = C_{p_k,p_l}(a, \delta_{p_k} a, \delta_{p_l} a) \quad (3.2)$$

for all $a \in A$, $k, l = 1, \dots, d$. A *homomorphism of $\delta_{\mathcal{P}}$-rings* A and B is a homomorphism of rings $\varphi : A \to B$ that commutes with the p_k-derivations in A and B, respectively. If $\phi_{p_k}(x) = x^{p_k} + p_k \delta_{p_k} x$ is the homomorphism associated to δ_{p_k}, condition (3.2) implies that

$$\phi_{p_k}\phi_{p_l}(a) = \phi_{p_l}\phi_{p_k}(a) \quad (3.3)$$

for all $a \in A$. Conversely, if the commutation relations (3.3) hold, and the p_ks are non-zero divisors in A, then conditions (3.2) hold, and we have that $\phi_{p_k}\delta_{p_l} a = \delta_{p_l}\phi_{p_k} a$ for all $a \in A$. If A is a $\delta_{\mathcal{P}}$-ring then for all k, the p_k-adic completions $A^{\widehat{p_k}}$ are $\delta_{\mathcal{P}}$-rings in a natural way.

Note that, since

$$Aut(R_{p_k}) = Aut(R_{p_k}/p_k R_{p_k}) = \varprojlim \mathbb{Z}/n\mathbb{Z}$$

is commutative, it follows that to give a $\delta_{\mathcal{P}}$-ring structure on R_{p_k} is equivalent to giving automorphisms

$$\phi_{p_1}, \ldots, \phi_{p_{k-1}}, \phi_{p_{k+1}}, \ldots, \phi_{p_d}$$

of R_{p_k}; the automorphism ϕ_{p_k} is, of course, uniquely determined. From now on we shall fix, for each k a $\delta_{\mathcal{P}}$-ring structure on R_{p_k}.

For a relation between these concepts and the theory of lambda rings we refer to [2] and the references therein.

We let $\mathbb{Z}_{\geqslant 0} = \{0, 1, 2, 3, \ldots\}$, and let $\mathbb{Z}^d_{\geqslant 0}$ be given the product order. We let e_k be the element of $\mathbb{Z}^d_{\geqslant 0}$ all of whose components are zero except the k-th, which is 1. We set $e = \sum e_k = (1, \ldots, 1)$. For $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d_{\geqslant 0}$ we set $\mathcal{P}^i = p_1^{i_1} \ldots p_d^{i_d}$, $\delta^i_{\mathcal{P}} = \delta^{i_1}_{p_1} \ldots \delta^{i_d}_{\mathcal{P}}$, $\phi^i_{\mathcal{P}} = \phi_{\mathcal{P}^i} = \phi^{i_1}_{p_1} \ldots \phi^{i_d}_{p_d}$. We define $W_{\mathcal{P}} = \mathbb{Z}[\phi_{p_1}, \ldots, \phi_{p_d}]$, the ring of polynomials in the commutative variables $\phi_{p_1}, \ldots, \phi_{p_d}$; this ring has a natural order defined by the condition that a polynomial is $\geqslant 0$ if and only if its coefficients are $\geqslant 0$. We set $W_{\mathcal{P}}(r) = \sum_{i \leqslant r} \mathbb{Z}\phi^i_{\mathcal{P}}$. Define $deg : W_{\mathcal{P}} \to \mathbb{Z}$ by $deg(\sum c_n \phi_n) = \sum c_n$.

A *$\delta_{\mathcal{P}}$-prolongation system* $A^* = (A^r)$ is an inductive system of rings A^r indexed by $r \in \mathbb{Z}^d_{\geqslant 0}$, provided with transition maps $\varphi_{rr'} : A^r \to A^{r'}$ for any pair of indices r, r' such that $r \leqslant r'$, and equipped with p_k-derivations

$$\delta_{p_k} : A^r \to A^{r+e_k},$$

$k = 1, \ldots, d$, such that (3.2) holds for all k, l, and such that

$$\varphi_{r+e_k, r'+e_k} \circ \delta_{p_k} = \delta_{p_k} \circ \varphi_{rr'} : A^r \to A^{r'+e_k}$$

for all $r \leqslant r'$ and all k. A morphism of prolongation systems $A^* \to B^*$ is a system of ring homomorphisms $u^r : A^r \to B^r$ that commute with the φs and the δs of A^* and B^*, respectively.

Any $\delta_{\mathcal{P}}$-ring A induces a $\delta_{\mathcal{P}}$-prolongation system A^* where $A^r = A$ for all r and $\varphi =$identity. If A is a $\delta_{\mathcal{P}}$-ring and A^* is the associated $\delta_{\mathcal{P}}$-prolongation system, we say that a $\delta_{\mathcal{P}}$-prolongation system B^* is a *$\delta_{\mathcal{P}}$-prolongation system over A* if it is equipped with a morphism $A^* \to B^*$. We have a natural concept of morphism of $\delta_{\mathcal{P}}$-prolongation systems over A.

Consider the ring

$$\mathbb{Z}_{(\mathcal{P})} = \bigcap_{k=1}^{d} \mathbb{Z}_{(p_k)} \subset \mathbb{Q}.$$

For any affine scheme of finite type X over $\mathbb{Z}_{(\mathcal{P})}$ we considered in [9] a system of schemes of finite type, $\mathcal{J}^r_{\mathcal{P}}(X)$ over $\mathbb{Z}_{(\mathcal{P})}$, called the $\delta_{\mathcal{P}}$-jet spaces of X; if $X = Spec\, \mathbb{Z}_{(\mathcal{P})}[x]/(f)$ then

$$\mathcal{J}^r_{\mathcal{P}}(X) = Spec\, \mathbb{Z}_{(\mathcal{P})}\big[\delta^i_{\mathcal{P}} x;\ i \leqslant r\big] / \big(\delta^i_{\mathcal{P}} f;\ i \leqslant r\big).$$

Cf. also [2], where these spaces were introduced independently. The rings $\mathcal{O}(\mathcal{J}^r_{\mathcal{P}}(X))$ form then a $\delta_{\mathcal{P}}$-prolongation system.

3.2. Complements to [9]

3.2.1. Splitting of completions

We will need the following *splitting result* for the various p_k-adic completions of the $\delta_{\mathcal{P}}$-jet spaces; here X is an affine scheme of finite type over $\mathbb{Z}_{(\mathcal{P})}$.

Proposition 3.1. *We have a natural isomorphism of $\delta_{\mathcal{P}}$-prolongation systems*

$$\mathcal{O}\big(\mathcal{J}^r_{\mathcal{P}}(X)\big)^{\widehat{p_k}} \simeq \left(\bigotimes_{i\leqslant r-r_ke_k} \mathcal{O}\big(\mathcal{J}^{r_k}_{\{p_k\}}(X_i)^{\widehat{p_k}}\big)\right)^{\widehat{p_k}},$$

where, in the right-hand side, $\otimes = \otimes_{\mathbb{Z}_{p_k}}$, $X_i = X$ for all i, δ_{p_k} is induced by the operators

$$\delta_{p_k} : \mathcal{O}\big(\mathcal{J}^{r_k}_{\{p_k\}}(X_i)\big) \to \mathcal{O}\big(\mathcal{J}^{r_k+1}_{\{p_k\}}(X_i)\big),$$

and δ_{p_j} $(j \neq k)$ are defined by the automorphisms ϕ_{p_j} induced from

$$\mathcal{O}\big(\mathcal{J}^{r_k}_{\{p_k\}}(X_i)\big) \simeq \mathcal{O}\big(\mathcal{J}^{r_k}_{\{p_k\}}(X_{i+e_j})\big).$$

Proof. One shows that the right-hand side satisfies the universality property of the left-hand side; the universality property in question is explained in [9, Remark 2.20].

Corollary 3.2. *We have a natural isomorphism of $\delta_{\mathcal{P}}$-prolongation systems*

$$\big(\mathcal{O}\big(\mathcal{J}^r_{\mathcal{P}}(X)\big) \otimes_{\mathbb{Z}_{(\mathcal{P})}} R_{p_k}\big)^{\widehat{p_k}} \simeq \left(\bigotimes_{i\leqslant r-r_ke_k} \mathcal{O}\big(J^{r_k}_{p_k}(X_{i,k})\big)\right)^{\widehat{p_k}},$$

where, in the right-hand side, $\otimes = \otimes_{R_{p_k}}$ and $X_{i,k} = X_{R_{p_k}}$ for all i.

3.2.2. Analytic continuation

Let us explain, in an abstract setting, a concept that will play a key role in the concrete discussion of the next section. This concept is a generalization of a concept introduced in [9].

Assume we are given a set of primes $\mathcal{P} = \{p_1, \dots, p_d\}$ and consider the $\delta_{\mathcal{P}}$-prolongation system

$$\begin{aligned} S^r_{for,k} &:= R_{p_k}\left[\!\left[\delta^i_{\mathcal{P}}t;\ i\leqslant r,\ i_k = 0\right]\!\right]\left[\delta^i_{\mathcal{P}}t;\ i\leqslant r,\ i_k \geqslant 1\right]^{\widehat{p_k}} \\ &\simeq R_{p_k}[\![t_i;\ i\leqslant r - r_ke_k]\!]\left[\delta^j_{p_k}t_i;\ i\leqslant r - r_ke_k,\ 1\leqslant j\leqslant r_k\right]^{\widehat{p_k}}, \end{aligned} \tag{3.4}$$

where t_i correspond to $\phi^i_{\mathcal{P}}t$ for $i \leqslant r - r_ke_k$.

Assume now that:

(1) For each $k = 1, \dots, d$ we are given a $\delta_{\mathcal{P}}$-prolongation system

$$\big(A^r_k\big); \tag{3.5}$$

(2) For each k we are given a morphism of $\delta_{\mathcal{P}}$-prolongation systems

$$E^r_k : A^r_k \to S^r_{for,k}. \tag{3.6}$$

Finally consider the ring

$$S^r_{for,0} := \mathbb{Z}_{(\mathcal{P})}\left[\!\left[\delta^i_{\mathcal{P}}t;\ i\leqslant r\right]\!\right]. \tag{3.7}$$

For each k we shall view both $S^r_{for,k}$ and $S^r_{for,0}$ as subrings of $R_{p_k}[\![\delta^i_{\mathcal{P}}t; i\leqslant r]\!]$.

Definition 3.3. We say that a family

$$(f_1, \ldots, f_d) \in \prod_{k=1}^{d} A_k^r$$

can be *analytically continued* if there exists (a necessarily unique) $f_0 \in S^r_{for,0}$ such that for all $k = 1, \ldots, d$ we have

$$E_k^r(f_k) = f_0 \in R_{p_k} [\![\delta^i_{\mathcal{P}} t;\ i \leqslant r]\!].$$

We say that $(f_1, \ldots, f_d)$ has *expansion* f_0.

3.3. *Application to δ-modular forms*

In what follows we specialize our discussion to the case of differential modular forms.

3.3.1. The rings S_k^r, M_k^r

We assume $Y_{\mathbb{Z}[1/N]}$ and $Y_{\mathbb{Z}[1/m]}$ are as in Sections 2.3.1 and 2.3.2 respectively. In the first case we assume N is not divisible by any of the primes in $\mathcal{P}$, and in the second case we assume the primes in $\mathcal{P}$ are sufficiently big. We let $Y_{\mathbb{Z}_{(\mathcal{P})}}$ and $Y_{R_{p_k}}$ be the curves over $\mathbb{Z}_{(\mathcal{P})}$ and R_{p_k} respectively obtained via base change. As in Sections 2.3.1 and 2.3.2 we let $X_{R_{p_k}} \subset Y_{R_{p_k}}$ be open affine subsets. We assume that $\overline{X_{R_{p_k}}}$ is contained in the ordinary locus. We also assume, for simplicity, that $\overline{X_{R_{p_k}}}$ is principal in $\overline{Y_{R_{p_k}}}$ defined by a function $s_k \in \mathcal{O}(\overline{Y_{R_{p_k}}})$ and that $L_{R_{p_k}}$ is trivial on $X_{R_{p_k}}$ with basis x_k. (Although the choice of $X_{R_{p_k}}$ is arbitrary with the above properties, the theory is easily seen to be independent, in an "obvious sense" from these choices.)

Now define

$$S_k^r := \left(\left(\mathcal{O}\left(\mathcal{J}^r_{\mathcal{P}}(Y_{\mathbb{Z}_{(\mathcal{P})}})\right) \otimes_{\mathbb{Z}_{(\mathcal{P})}} R_{p_k}\right)\left[\phi^i_{\mathcal{P}} s_k^{-1};\ i \leqslant r\right]\right)^{\widehat{p_k}} = \left(\bigotimes_{i \leqslant r - r_k e_k} \mathcal{O}\left(J^{r_k}_{p_k}(X_{i,k})\right)\right)^{\widehat{p_k}}, \tag{3.8}$$

where $X_{i,k} = X_{R_{p_k}}$ for all i. (The last equality follows from Corollary 3.2.) Then (S_k^r) has a natural structure of δ_{p_k}-prolongation sequence. Actually, once we have fixed a $\delta_{\mathcal{P}}$-ring structure on R_{p_k}, (S_k^r) has a naturally induced structure of $\delta_{\mathcal{P}}$-prolongation system (extending the previous one). We also have a natural morphism of δ_{p_k}-prolongation sequences $S_{p_k}^{r_k} \to S_k^r$. Set

$$\begin{aligned} M_k^r &:= S_k^r\left[\delta^i_{\mathcal{P}} x_k, \phi^i_{\mathcal{P}} x_k^{-1};\ i \leqslant r\right]^{\widehat{p_k}} = \left(\bigotimes_{i \leqslant r - r_k e_k} \mathcal{O}\left(J^{r_k}_{p_k}(X_{i,k})\right)\left[\delta^j_{p_k} x_{i,k}, x_{i,k}^{-1};\ 0 \leqslant j \leqslant r_k\right]^{\widehat{p_k}}\right)^{\widehat{p_k}}, \\ M_k^r(w) &:= S_k^r x_k^w, \end{aligned} \tag{3.9}$$

where $x_{i,k}$ are new variables, "copies" of x_k. For each k, (M_k^r) is a $\delta_{\mathcal{P}}$-prolongation system and we have a natural morphism of δ_{p_k}-prolongation sequences $M_{p_k}^{r_k} \to M_k^r$. In particular we may consider the image of $f^1_{p_k} \in M^1_{p_k}(-1 - \phi_{p_k})$ in $M_k^{e_k}(-1 - \phi_{p_k})$ which we continue to denote by $f^1_{p_k}$. Similarly, if $f = \sum a_n q^n$ is a newform of weight 2 on $\Gamma_0(N)$ with $\mathbb{Z}$-Fourier coefficients and if the primes in $\mathcal{P}$ are sufficiently big then we may consider the image of $f^\sharp_{p_k} \in M^2_{p_k}(0)$ in $M_k^{2e_k}(0)$ which we continue to denote by $f^\sharp_{p_k}$.

Furthermore define

$$S_k^{\infty} := \varinjlim S_k^r,$$
$$M_k^{\infty} := \varinjlim S_k^r,$$
$$S_{for,k}^{\infty} := \varinjlim S_{for,k}^r.$$

3.3.2. $\delta_{\mathcal{P}}$-expansion maps

Next we will construct a natural morphism of $\delta_{\mathcal{P}}$-prolongation systems

$$E_k^r : M_k^r \to S_{for,k}^r,$$

called *δ-expansion maps*, as follows.

Assume first we are in the "Fourier case". Consideration of the Tate curve yields a homomorphism $\mathcal{O}(Y_{\mathbb{Z}_{(\mathcal{P})}}) \to R_{p_k}((q))^{\widehat{p_k}}$. Composing this with the homomorphism

$$R_{p_k}((q))^{\widehat{p_k}} \to R_{p_k}[\![t]\!], \quad q \mapsto t+1$$

we get a homomorphism $\mathcal{O}(Y_{\mathbb{Z}_{(\mathcal{P})}}) \to R_{p_k}[\![t]\!]$. By universality we obtain a homomorphism of $\delta_{\mathcal{P}}$-prolongation systems

$$\mathcal{O}\big(\mathcal{J}_{\mathcal{P}}^r(Y_{\mathbb{Z}_{(\mathcal{P})}})\big) \to S_{for,k}^r.$$

Using the fact that $E(s_k)$ is invertible in $R_{p_k}[\![t]\!]$ we get a homomorphism $S_k^r \to S_{for,k}^r$. Sending $x_k \mapsto E(x_k)$ we get the desired homomorphism $E_k^r : M_k^r \to S_{for,k}^r$. Note that this homomorphism extends, in the obvious sense, the homomorphism

$$M_{p_k}^r \xrightarrow{E^r} S_{for,p_k}^r \xrightarrow{\iota} R_{p_k}[\![t]\!]\big[\delta_{p_k} t, \ldots, \delta_{p_k}^r t\big]^{\widehat{p_k}},$$

where ι is induced by $q \mapsto t+1$. Consequently, by Proposition 2.18, we have

$$E_k^{e_k}\big(f_{p_k}^1\big) = \Psi_{p_k}. \tag{3.10}$$

Next we consider the "Serre–Tate" case. For this case we assume we are given a collection of points

$$(P_k)_{1 \leqslant k \leqslant d}, \tag{3.11}$$

$P_k \in X_{R_{p_k}}(R_{p_k})$, as in Sections 2.3.1 and 2.3.2 respectively. (We do not assume any compatibility between the points P_k. One of the most remarkable aspects of the theory is that its constructions and results are valid without assuming any compatibility between the P_ks. Also the theory is, in a sense easily made precise, independent of the choice of the P_ks.) Using the decompositions (3.4) and (3.8) plus the one prime theory we get induced maps $E_k^r : M_k^r \to S_{for,k}^r$. Once again (3.10) holds.

Note the following *$\delta_{\mathcal{P}}$-expansion principle*:

Proposition 3.4. *The homomorphism*

$$\overline{E_k^r} : \overline{S_k^r} \to \overline{S_{for,k}^r}$$

is injective. In particular the homomorphisms

$$E_k^r : M_k^r(w) \to S_{for,k}^r$$

are injective with torsion free cokernel.

Proof. This follows from the one prime situation (cf. Proposition 2.2) plus the "splittings" (3.4) and (3.8). □

3.3.3. The rings $S^r_{k\heartsuit}$, $M^r_{k\heartsuit}$

As in the case of one prime we define

$$S^r_{k\heartsuit} := Im\big(E^r_k : M^r_k \to S^r_{for,k}\big),$$
$$M^r_{k\heartsuit} := \big(S^r_{k\heartsuit} \otimes_{S^r_k} M^r_k\big)^{\widehat{p_k}},$$
$$S^\infty_{k\heartsuit} = \varinjlim S^r_{k\heartsuit},$$
$$M^\infty_{k\heartsuit} = \varinjlim M^r_{k\heartsuit}.$$

The inclusion $S^r_{k\heartsuit} \subset S^r_{for,k}$ and the homomorphism $E^r_k : M^r_k \to S^r_{for,k}$ induce a homomorphism (still denoted by E^r_k and still referred to as *$\delta_{\mathcal{P}}$-expansion map*),

$$E^r_k : M^r_{k\heartsuit} \to S^r_{for,k}.$$

Note that $S^r_{\heartsuit}$ is p_k-adically complete and if x_k is a basis of $L = L_{R_{p_k}}$ then

$$M^r_{k\heartsuit} = S^r_{k\heartsuit}\big[\delta^i_{\mathcal{P}} x_k, \phi^i_{\mathcal{P}} x_k^{-1};\ i \leqslant r\big]^{\widehat{p_k}}.$$

Note that $(S^r_{k\heartsuit})$, $(M^r_{k\heartsuit})$ have natural structures of $\delta_{\mathcal{P}}$-prolongation systems. Define

$$M^r_{k\heartsuit}(w) := S^r_{k\heartsuit} x^w_k \subset M^r_{k\heartsuit};$$

the latter definition is independent of the choice of the basis x_k. We have morphisms of δ_{p_k}-prolongation sequences $S^r_{p_k\heartsuit} \to S^r_{k\heartsuit}$, $M^{r_k}_{p_k\heartsuit} \to M^r_{k\heartsuit}$. In particular we may consider the image of $f^0_{p_k} \in M^0_{p_k\heartsuit}(1)$ into $M^0_{k\heartsuit}(1)$ which we will still denote by $f^0_{p_k}$. Clearly we have $E^0_k(f^0_{p_k}) = 1$.

Proposition 3.5.

(1) *The homomorphisms $\overline{S^r_k} \to \overline{S^r_{k\heartsuit}}$, $\overline{S^\infty_k} \to \overline{S^\infty_{k\heartsuit}}$, $\overline{M^r_k} \to \overline{M^r_{k\heartsuit}}$, $\overline{M^\infty_k} \to \overline{M^\infty_{k\heartsuit}}$ are injective. In particular the homomorphisms $S^r_k \to S^r_{k\heartsuit}$, $S^\infty_k \to S^\infty_{k\heartsuit}$, $M^r_k \to M^r_{k\heartsuit}$, $M^\infty_k \to M^\infty_{k\heartsuit}$ are injective with torsion free cokernel.*

(2) *The homomorphisms $M^r_{k\heartsuit}(w) \to S^r_{for,k}$ are injective.*

(3) *The homomorphisms $S^r_{k\heartsuit} \to S^{r+1}_{k\heartsuit}$, $M^r_{k\heartsuit} \to M^{r+1}_{k\heartsuit}$ are injective.*

Proof. Use Proposition 2.3 and the "splittings" (3.4) and (3.8). □

Remark 3.6. Again $\overline{S^\infty_{k\heartsuit}}$ is not a priori an integral domain; but it possesses a natural quotient which is an integral domain:

$$\widetilde{S^\infty_{k\heartsuit}} := Im\big(\overline{M^\infty_k} \to \overline{S^\infty_{for,k}}\big).$$

By the proof of Theorem 2.9 and the "splittings" (3.4) and (3.9) we get:

Theorem 3.7. *The ring $\widetilde{S^\infty_{k\heartsuit}}$ is a quotient of an ind-étale Γ-extension of $\overline{S^\infty_k}$, where Γ is a profinite abelian group.*

Corollary 3.8.

(1) $\overline{S^{\infty}_{k\heartsuit}}$ *is an integral extension of* $\overline{S^{\infty}_{k}}$.

(2) $\widetilde{S^{\infty}_{k\heartsuit}}$ *is an ind-étale* Γ'*-extension of* $\overline{S^{\infty}_{k}}$*, where* Γ' *is a closed subgroup of* Γ.

Theorem 3.7 can be morally viewed as saying that the "$\delta_{\mathcal{P}}$-Igusa curve" (of which the rings $S^{\infty}_{k\heartsuit}$ are an incarnation) is a "formal profinite cover" (embedding into an "abelian formal pro-étale cover") of the modular/Shimura curve (whose $\delta_{\mathcal{P}}$-geometric incarnation are the rings S^{∞}_{k}).

One can prove analogues, in our several prime setting here, of Theorem 2.30 and Propositions 2.24, 2.28. We are not going to need these analogues so we are not going to state them explicitly. Instead, we will concentrate, in what follows, on the purely "several prime concept" of analytic continuation.

In the following definition the rings M^r_k will play the role of the rings A^r_k in (3.5) and the $\delta_{\mathcal{P}}$-expansion maps $E^r_k : M^r_k \to S^r_{for,k}$ will play the role of the maps (3.6).

Definition 3.9. A *$\delta_{\mathcal{P}}$-modular form of weight $w \in W_{\mathcal{P}}$ and order $r \in \mathbb{Z}^d_{\geqslant 0}$* is a family

$$f = (f_1, \dots, f_d) \in \prod_{k=1}^{d} M^r_k$$

that can be analytically continued and such that $f_k \in M^r_k(w)$ for all k. We denote by $M^r_{\mathcal{P}}(w)$ the group of all such forms. There is a naturally induced *expansion map*

$$E^r : M^r_{\mathcal{P}}(w) \to S^r_{for,0}$$

which by Proposition 3.5 is injective.

Similarly we may take the rings $M^r_{k\heartsuit}$ to play the role of the rings A^r_k in (3.5) and the $\delta_{\mathcal{P}}$-expansion maps $E^r_k : M^r_{k\heartsuit} \to S^r_{for,k}$ to play the role of the maps (3.6).

Definition 3.10. An *Igusa $\delta_{\mathcal{P}}$-modular form of weight $w \in W_{\mathcal{P}}$ and order $r \in \mathbb{Z}^d_{\geqslant 0}$* is a family

$$f = (f_1, \dots, f_d) \in \prod_{k=1}^{d} M^r_{k\heartsuit}$$

that can be analytically continued and such that $f_k \in M^r_{k\heartsuit}(w)$ for all k. We denote by $M^r_{\mathcal{P}\heartsuit}(w)$ the group of all such forms. There is a naturally induced *expansion map*

$$E^r : M^r_{\mathcal{P}\heartsuit}(w) \to S^r_{for,0}$$

which by Proposition 3.5 is injective.

So $M^r_{\mathcal{P}}(w)$ is a $\mathbb{Z}_{(\mathcal{P})}$-submodule of $M^r_{\mathcal{P}\heartsuit}(w)$.

Remark 3.11. If, in the above definitions, we are in the Fourier expansion case then we view f as being "analytically continued along the section ∞". If we are in the Serre–Tate expansion case and if the collection of points (P_k) in (3.11) comes from a $\overline{\mathbb{Q}}$-point P of the modular/Shimura curve, then we view f as being "analytically continued along P". Here, we say that (P_k) *comes from* P if for each k the elliptic (respectively false elliptic) curve corresponding to P_k is isomorphic over $\overline{\mathbb{Q}}$ to the curve corresponding to P.

3.3.4. Basic examples: the forms f^0, f^e, f^{2e}

We may now introduce some of the fundamental objects of our "several primes" theory.

First recall that we have at our disposal forms $f^0_{p_k} \in M^0_{k\heartsuit}(1)$. Then we have the following obvious

Proposition 3.12. *The family*

$$f^0 := \left(f^0_{p_1}, \dots, f^0_{p_d}\right) \in \prod_{k=1}^d M^0_{k\heartsuit}$$

is an Igusa $\delta_{\mathcal{P}}$-modular form of weight 1 *and order* 0; *i.e.* $f^0 \in M^0_{\mathcal{P}\heartsuit}(1)$. *Moreover f^0 has expansion* $E^0(f^0) = 1$.

In particular, for any $w \in W_{\mathcal{P}}(r)$ we have

$$\left(f^0\right)^w = \left(\left(f^0_{p_1}\right)^w, \dots, \left(f^0_{p_d}\right)^w\right) \in M^r_{\mathcal{P}\heartsuit}(w).$$

Note that if the weight w is divisible in $W_{\mathcal{P}}$ by $(\phi_{p_1} - 1)\dots(\phi_{p_d} - 1)$ then $(f^0)^w$ is actually belongs to $M^r_{\mathcal{P}\heartsuit}(w)$.

Next we introduce a form of order $e := (1, \dots, 1)$ which we shall call f^e. Indeed recall the forms $f^\natural_{p_k}$ and set, for each $k = 1, \dots, d$,

$$f^e_k := (-1)^{d-1}\left(\prod_{l \in I_k}\left(1 - \frac{\phi_{p_l}}{p_l}\right)\right) f^\natural_{p_k} \in S^r_{k\heartsuit} = M^r_{k\heartsuit}(0), \tag{3.12}$$

where $I_k = \{1, \dots, d\} \setminus \{k\}$. Also set

$$f^e_0 := \frac{1}{p_1 \dots p_d}(\phi_{p_1} - p_1)\dots(\phi_{p_d} - p_d)\log(1+t) \in S^e_{for,0} = \mathbb{Z}_{(\mathcal{P})}\left[\!\left[\delta^i_{\mathcal{P}} t;\ i \leqslant e\right]\!\right]. \tag{3.13}$$

The fact that the above series belongs to $\mathbb{Z}_{(\mathcal{P})}[\![\delta^i_{\mathcal{P}} t;\ i \leqslant e]\!]$ (and not merely to $\mathbb{Q}[\![\delta^i_{\mathcal{P}} t;\ i \leqslant e]\!]$) is due to the fact that

$$\frac{1}{p_k}(\phi_{p_k} - p_k)\log(1+t) \in \mathbb{Z}_{p_k}[\![t]\!][\delta_{p_k} t]^{\widehat{p_k}}$$

for each k.

Theorem 3.13. *The family*

$$f^e := \left(f^e_1, \dots, f^e_d\right) \in \prod_{k=1}^d M^e_{k\heartsuit}$$

is an Igusa $\delta_{\mathcal{P}}$-modular form of weight 0 *and order* e; *i.e.* $f^e \in M^e_{\mathcal{P}\heartsuit}(0)$. *Moreover f^e has expansion f^e_0; i.e.* $E^e(f^e) = f^e_0$.

Proof. Recall that $E^0_k(f^0_{p_k}) = 1$. On the other hand, by (3.10),

$$E^e_k\left(\left(f^1_{p_k}\right)^{\phi_n}\right) = \Psi^{\phi_n}_{p_k} = \frac{1}{p_k}\phi_n(\phi_{p_k} - p_k)\log(1+t).$$

Consequently

$$E_k^e\big(f_k^e\big) = (-1)^{d-1}\left(\prod_{l\in I_k}\left(1-\frac{\phi_{p_l}}{p_l}\right)\right)\frac{1}{p_k}(\phi_{p_k}-p_k)\log(1+t) = f_0^r. \qquad \square$$

Remark 3.14. Using the forms f^0 and f^e we may construct, for any $w \in W_{\mathcal{P}}(r)$, with $r \geqslant e$, the Igusa $\delta_{\mathcal{P}}$-modular form

$$\big(f^0\big)^w\big(f^e\big)^{\phi_{\mathcal{P}}^{r-e}} \in M^r_{\mathcal{P}\heartsuit}(w). \qquad \square$$

Finally we introduce $\delta_{\mathcal{P}}$-modular forms of order $2e$ and weight 0 which will be called f^{2e}. Indeed assume $f = \sum a_n q^n$ is a newform of weight 2 on $\Gamma_0(N)$ with $\mathbb{Z}$-Fourier coefficients. Assume, for simplicity, that f is not of CM type. (There is an analogue of what follows for the CM type case.) Assume the primes in $\mathcal{P}$ are sufficiently big. Define

$$f_k^{2e} = \left(\prod_{l\in I_k}\left(1 - a_{p_l}\frac{\phi_{p_l}}{p_l} + p_l\left(\frac{\phi_{p_l}}{p_l}\right)^2\right)\right) f^{\sharp}_{p_k} \in M_k^{2e}(0),$$

where $I_k = \{1,\dots,d\}\setminus\{k\}$. Also set

$$f_0^{2e} = \frac{1}{p_1\dots p_d}\big(\phi_{p_1}^2 - a_{p_1}\phi_{p_1} + p_1\big)\dots\big(\phi_{p_d}^2 - a_{p_d}\phi_{p_d} + p_d\big)\sum_n \frac{a_n}{n}q^n \in S^{2e}_{for,0}. \qquad (3.14)$$

Exactly as in the case of Theorem 3.13 we get

Theorem 3.15. *The family*

$$f^{2e} := \big(f_1^{2e},\dots,f_d^{2e}\big) \in \prod_{k=1}^d M_k^{2e}$$

is a $\delta_{\mathcal{P}}$-modular form of weight 0 *and order* $2e$; *i.e.* $f^{2e} \in M^{2e}_{\mathcal{P}}(0)$. *Moreover* f^{2e} *has expansion* f_0^{2e}; *i.e.* $E^{2e}(f^{2e}) = f_0^{2e}$.

3.3.5. Isogeny covariant Igusa $\delta_{\mathcal{P}}$-modular forms

The following definition extends the concept of isogeny covariance introduced in [4,5]. For each $k = 1,\dots,d$ let us fix, once and for all an element $\gamma_k \in \mathbb{Z}_{p_k}$ such that $\gamma_k \not\equiv 0, 1 \bmod p_k$ and such that γ_k is not a root of unity; the theory that follows is, in a sense that can be made precise, independent of the choice of γ_k. Set

$$[\gamma_k](t) := (1+t)^{\gamma_k} - 1 \in \mathbb{Z}_{p_k}[\![t]\!].$$

Definition 3.16. A series

$$F = F\big(\dots,\delta^i_{\mathcal{P}}t,\dots\big) \in S^r_{for,k},$$

is called *isogeny covariant* of degree $\nu \in \mathbb{Z}$ if

$$F\big(\dots,\delta^i_{\mathcal{P}}\big([\gamma_k](t)\big),\dots\big) = \gamma_k^{\nu} F\big(\dots,\delta^i_{\mathcal{P}}t,\dots\big).$$

Let $w \in W_{\mathcal{P}}(r)$ have even degree $deg(w)$. An element $f_k \in M_k^r(w)$ (respectively an element $f_k \in M_{k\heartsuit}^r(w)$) is called *isogeny covariant* if its expansion $E_k^r(f_k) \in S_{for,k}^r$ is isogeny covariant of degree $-\frac{deg(w)}{2}$. An $\delta_{\mathcal{P}}$-modular form $f = (f_k) \in M_{\mathcal{P}}^r(w)$ (respectively an Igusa $\delta_{\mathcal{P}}$-modular form $f = (f_k) \in M_{\mathcal{P}\heartsuit}^r(w)$) is called *isogeny covariant* if f_k is isogeny covariant for all k. We denote by $I_{\mathcal{P}}^r(w)$ (respectively by $I_{\mathcal{P}\heartsuit}^r(w)$) the $\mathbb{Z}_{(\mathcal{P})}$-module of all isogeny covariant $\delta_{\mathcal{P}}$-modular forms in $M_{\mathcal{P}}^r(w)$ (respectively the $\mathbb{Z}_{(\mathcal{P})}$-module of all isogeny covariant Igusa $\delta_{\mathcal{P}}$-modular forms in $M_{\mathcal{P}\heartsuit}^r(w)$). Hence

$$I_{\mathcal{P}}^r(w) := I_{\mathcal{P}\heartsuit}^r(w) \cap M_{\mathcal{P}}^r(w) \subset M_{\mathcal{P}\heartsuit}^r(w).$$

Note that the forms $f_{p_k}^0 \in M_{k\heartsuit}^0(1)$ and $f_{p_k}^1 \in M_{k\heartsuit}^{e_k}(-1-\phi_{p_k})$ are isogeny covariant. So we have:

Corollary 3.17. *Assume $w \in W_{\mathcal{P}}(r)$, $deg(w) = -2$, $r \geqslant e$. Then the form $(f_{p_k}^0)^w (f_k^e)^{\phi_{\mathcal{P}}^{r-e}} \in M_{k\heartsuit}^r(w)$ is isogeny covariant. In other words the form $(f^0)^w (f^e)^{\phi_{\mathcal{P}}^{r-e}}$ is isogeny covariant; i.e. $(f^0)^w (f^e)^{\phi_{\mathcal{P}}^{r-e}} \in I_{\mathcal{P}\heartsuit}^r(w)$.*

Similarly we have:

Corollary 3.18. *Assume $w \in W_{\mathcal{P}}(r)$, $deg(w) = 0$. Then the form $(f^0)^w$ is isogeny covariant; i.e. $(f^0)^w \in I_{\mathcal{P}\heartsuit}^r(w)$. Moreover, if w is divisible by $(\phi_{p_1} - 1)\dots(\phi_{p_d} - 1)$ in $W_{\mathcal{P}}$ then $(f^0)^w \in I_{\mathcal{P}}^r(w)$.*

Remark 3.19. Isogeny covariance is a property which is stronger than the property of being a "$\delta_{\mathcal{P}}$-Hecke eigenform"; cf. [4] for a discussion of this in the case of one prime. The $\delta_{\mathcal{P}}$-modular form f^{2e} is not isogeny covariant. Nevertheless, by what was shown in [7] f^{2e} is, in an appropriate sense, a "$\delta_{\mathcal{P}}$-Hecke eigenform".

3.3.6. Structure of $I_{\mathcal{P}\heartsuit}^r(w)$ for $deg(w) = -2, 0$

Here is the main result of this second part of our paper. It is a structure theorem for the module of isogeny covariant Igusa $\delta_{\mathcal{P}}$-modular forms of any order and any weight w with $deg(w) = -2$.

Theorem 3.20. *Let $w \in W_{\mathcal{P}}(r)$ with $deg(w) = -2$, $r \geqslant e$. Then the $\mathbb{Z}_{(\mathcal{P})}$-module of isogeny covariant $\delta_{\mathcal{P}}$-modular forms $I_{\mathcal{P}\heartsuit}^r(w)$ is free of rank $r_1 \dots r_d$ with basis*

$$\left\{ \left(f^0\right)^w \left(f^e\right)^{\phi_{\mathcal{P}}^{s-e}};\ e \leqslant s \leqslant r \right\}.$$

We need a couple of lemmas.

Lemma 3.21. *Assume $F \in S_{for,k}^r$ is isogeny covariant of degree 1 and let $K_{p_k} = R_{p_k}[1/p_k]$. Then the image of F in $K_{p_k}[[\delta_{\mathcal{P}}^i t;\ i \leqslant r]]$ is a K_{p_k}-linear combination of the series*

$$\phi_{\mathcal{P}}^i\left(\log(1+t)\right), \quad i \leqslant r.$$

Proof. Same argument as in the proof of [5, Lemma 3.9]. □

Lemma 3.22. *Consider the action $\star$ of $\mathbb{Q}[\phi_{p_1}, \dots, \phi_{p_d}]$ on $\mathbb{Z}_{p_k}[[t]] \otimes \mathbb{Q}$ defined by $\phi_n \star t := t^n$. If a polynomial $\Lambda = \sum \lambda_n \phi_n \in \mathbb{Q}[\phi_{p_1}, \dots, \phi_{p_d}]$ satisfies*

$$\Lambda \star \log(1+t) \in \mathbb{Z}_{p_k}[[T]] \otimes \mathbb{Q}$$

for some $k \in \{1, \dots, s\}$, then Λ is divisible in the ring $\mathbb{Q}[\phi_{p_1}, \dots, \phi_{p_s}]$ by $\phi_{p_k} - p_k$.

Proof. This was proved in the course of the proof of Theorem 3.4 in [9]; cf. Claim 2 of that proof. □

Proof of Theorem 3.20. Let $f=(f_1,\dots,f_d)\in M^r_{\mathcal{P}\heartsuit}(w)$ be isogeny covariant and fix an index k. By Lemma 3.21 the expansion $E^r_k(f_k)$, viewed as an element of $K_{p_k}[\![\delta^i_{\mathcal{P}}t;\ i\leqslant r]\!]$, can be written as

$$E^e_k(f_k)=\left(\sum_{n|\mathcal{P}^r}c_n\phi_n\right)\log(1+t),$$

with $c_n\in K_{p_k}$. This series, being equal to the expansion f_0 of f also belongs to $\mathbb{Z}_{(\mathcal{P})}[\![\delta^i_{\mathcal{P}}t;\ i\leqslant r]\!]$. Setting $\delta^i_{\mathcal{P}}t=0$ for all $i\neq 0$ we get that

$$\left(\sum_{n|\mathcal{P}^r}c_n\phi_n\right)\star\log(1+t)\in\mathbb{Z}_{(\mathcal{P})}[\![t]\!].$$

One checks by induction that $c_n\in\mathbb{Q}$ for all n. Now by Lemma 3.22 it follows that

$$\sum_{n|\mathcal{P}^r}c_n\phi_n=\frac{1}{p_1\dots p_d}\left(\sum_{e\leqslant s\leqslant r}b_{\mathcal{P}^s}\phi_{\mathcal{P}^s}\right)(\phi_{p_1}-p_1)\dots(\phi_{p_d}-p_d)$$

with $b_{\mathcal{P}^s}\in\mathbb{Q}$. Hence, for some sufficiently divisible integer $N\in\mathbb{Z}$, the expansion of Nf equals the expansion of

$$g:=N\sum_{e\leqslant s\leqslant r}b_{\mathcal{P}^s}\left(f^0\right)^w\left(f^e\right)^{\phi^{s-e}_{\mathcal{P}}}.$$

By the injectivity of the expansion map $E^r:M^r_{\mathcal{P}\heartsuit}(w)\to S^r_{for,0}$ (cf. Definition 3.10) it follows that $Nf=g$. By induction (looking at the coefficient of t to the lowest power) we get $b_{\mathcal{P}^s}\in\mathbb{Z}_{(\mathcal{P})}$ for all s and we are done. $\square$

Similarly (and indeed with a simpler argument which we leave to the reader) we get the following structure theorem for the module of isogeny covariant Igusa $\delta_{\mathcal{P}}$-modular forms of any order and any weight w with $deg(w)=0$.

Theorem 3.23. *Let $w\in W_{\mathcal{P}}(r)$ with $deg(w)=0$. Then the $\mathbb{Z}_{(\mathcal{P})}$-module of isogeny covariant $\delta_{\mathcal{P}}$-modular forms $I^r_{\mathcal{P}\heartsuit}(w)$ is free of rank one with basis $(f^0)^w$.*

3.3.7. Vanishing of $I^e_{\mathcal{P}}(-2)$

"Linear arithmetic partial differential operators" exist, as we have seen, on the "$\delta_{\mathcal{P}}$-Igusa curves"; but we do not expect them to exist on the modular curve itself. In other words we expect that there are no non-zero isogeny covariant $\delta_{\mathcal{P}}$-modular forms of weight w with $deg(w)=-2$; i.e., for such ws, $I^r_{\mathcal{P}}(w)=0$. We can prove this in the modular curve case, in the "simplest case" $d=2$, $r=e$, $w=-2$:

Theorem 3.24. *Assume we are in the modular curve case and assume $d=2$. Then $I^e_{\mathcal{P}}(-2)=0$.*

Proof. Assume $I^e_{\mathcal{P}}(-2)$ contains a non-zero element f. By Theorem 3.20 we have $f=c(f^0)^{-2}f^e$ for some $c\in\mathbb{Z}_{(\mathcal{P})}$. In particular we have

$$-c\left(f^0_{p_1}\right)^{\phi_{p_1}-1}f^1_{p_1}+\frac{c}{p_2}\left(f^0_{p_1}\right)^{\phi_{p_2}+\phi_{p_1}\phi_{p_2}-2}\left(f^1_{p_1}\right)^{\phi_{p_2}}\in M^e_1.$$

The first term in the sum above is in M_1^e hence the second term must also be in M_1^e. Since by Proposition 3.5 $M_1^e \to M_{1\heartsuit}^e$ has torsion free cokernel it follows that

$$\left(f_{p_1}^0\right)^{\phi_{p_2}+\phi_{p_1}\phi_{p_2}-2}\left(f_{p_1}^1\right)^{\phi_{p_2}} \in M_1^e$$

and hence

$$G := \left(f_{p_1}^0\right)^{2\phi_{p_2}-2}\left(f_{p_1}^1\right)^{\phi_{p_2}} \in M_1^e.$$

Reducing modulo p_1 and raising to power $\frac{p_1-1}{2}$ we get, using Corollary 2.20, that

$$\overline{H}_{p_1}^{\phi_{p_2}-1}\left(\left(\overline{f_{p_1}^1}\right)^{\phi_{p_2}}\right)^{\frac{p_1-1}{2}} = \overline{G}^{\frac{p_1-1}{2}}$$

in $\overline{M_{1\heartsuit}^e}$ and hence in $\overline{M_1^e}$. Writing $G = gx^{\phi_{p_2}-\phi_{p_1 p_2}-2}$ with $g \in S_1^e$, $\overline{H}_{p_1} = \bar{h}_{p_1}x^{p_1-1}$ with $\bar{h}_{p_1} \in \overline{S_1^0}$, and $f_{p_1}^1 = \eta x^{-1-\phi_{p_1}}$ with $\eta \in S_1^{e_1}$, we get

$$\frac{\bar{h}_{p_1}^{\phi_{p_2}}}{\bar{h}_{p_1}} \cdot \left(\bar{\eta}^{\phi_{p_2}}\right)^{\frac{p_1-1}{2}} = \bar{g}^{\frac{p_1-1}{2}}$$

in $\overline{S_k^e}$. In view of the “splitting” (3.8) we can derive a contradiction if we check the following:

Lemma 3.25. *Assume $\overline{X}$ is the reduction mod p of the open set of the modular curve as in Section* 2.3.1 *and let $\overline{H} = hx^{p-1}$ be the Hasse invariant. Consider the* 2*-fold product $\overline{X}^2 = \overline{X} \times \overline{X}$ and the projections $\pi_1, \pi_2 : \overline{X}^2 \to \overline{X}$. Then there is no rational function $\bar{u}$ such that*

$$\frac{\pi_1^*\bar{h}}{\pi_2^*\bar{h}} = \bar{u}^{\frac{p-1}{2}}. \tag{3.15}$$

Proof. Assume (3.15) holds for some $\bar{u}$. Restricting to a horizontal divisor $\overline{X} \times \{\text{point}\}$ we get

$$\bar{h} = \bar{v}^{\frac{p-1}{2}}$$

for some rational function $\bar{v}$ on $\overline{X}$. Recall that $\overline{H}$ has simple zeroes at the supersingular points; cf. [18, 12.4.3]. Pick a supersingular point s, let x_s be a basis of the line bundle L in a neighborhood of s and write $\overline{H} = \bar{h}_s x_s^{p-1}$ with $\bar{h}_s$ having a simple zero at s. Then we get

$$\bar{h}_s = \left(\bar{v}\frac{x^2}{x_s^2}\right)^{\frac{p-1}{2}}.$$

This implies that $\bar{h}_s$ has a zero at s of order divisible by $\frac{p-1}{2}$, a contradiction. $\square$

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