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A stochastic linear–quadratic problem with Lévy processes and its application to finance

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Abstract

We study a Linear–Quadratic Regulation (LQR) problem with Lévy processes and establish the closeness property of the solution of the multi-dimensional Backward Stochastic Riccati Differential Equation (BSRDE) with Lévy processes. In particular, we consider multi-dimensional and one-dimensional BSRDEs with Teugel's martingales which are more general processes driven by Lévy processes. We show the existence and uniqueness of solutions to the one-dimensional regular and singular BSRDEs with Lévy processes by means of the closeness property of the BSRDE and obtain the optimal control for the non-homogeneous case. An application of the backward stochastic differential equation approach to a financial (portfolio selection) problem with full and partial observation cases is provided.

MSC: primary 93E20; secondary 60H10; 91B28

Keywords: Linear–quadratic regulators; Lévy process; Backward stochastic (Riccati) differential equation; Regular and singular case

1. Introduction

This paper examines the following Backward Stochastic Differential Equation (BSDE) called the Backward Stochastic Riccati Differential Equation (BSRDE) with Lévy processes:

$$\begin{cases} \mathrm{d}\Pi(t) = -G(t, \Pi, \beta_\Pi, \gamma_\Pi)\mathrm{d}t + \sum_{i=1}^{k} \beta_\Pi^i(t)\mathrm{d}w^i(t) + \sum_{i=1}^{l}\sum_{j=1}^{\infty} \gamma_\Pi^{ij}(t)\mathrm{d}H^{ij}(t) \\ \Pi(T) = M(T), \end{cases} \tag{1.1}$$

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where

$$G(t,\Pi,\beta_\Pi,\gamma_\Pi) = \Bigg(\Pi(t^-)A + A^*\Pi(t^-) + M + \sum_{i=1}^{k}\beta_\Pi^i C^i + \sum_{i=1}^{l}\sum_{j=1}^{\infty}\gamma_\Pi^{ij}E^{ij} + \sum_{i=1}^{k}(C^i)^*(\beta_\Pi^i + \Pi(t^-)C^i) + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(E^{ij})^*(\gamma_\Pi^{ij} + \Pi(t^-)E^{ij})\Bigg)(t) - F(t,\Pi,\beta_\Pi,\gamma_\Pi)$$

and

$$F(t,\Pi,\beta_\Pi,\gamma_\Pi) = L(t)^*K(t)^{-1}L(t),$$

$$L(t) := \Bigg(\sum_{i=1}^{d}(B^i)^*\Pi(t^-) + \sum_{i=1}^{k}(D^i)^*(\beta_\Pi^i + \Pi(t^-)C^i) + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*(\gamma_\Pi^{ij} + \Pi(t^-)E^{ij})\Bigg)(t)$$

$$K(t) := \Bigg(N + \sum_{i=1}^{k}(D^i)^*\Pi(t^-)D^i + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*\Pi(t^-)F^{ij}\Bigg)(t).$$

The coefficients of A, B^i, C^i, D^i, E^{ij} and F^{ij} for all $i, j, d, k, l \in \mathbf{N}$ in the BSRDE (1.1) are in $n \times n$, $n \times m$, $n \times n$, $n \times m$, $n \times n$ and $n \times m$, respectively. $w_i(t), i \in \{1, \dots, k\}, k < \infty$, is a one-dimensional Wiener process and $H^{ij}(t), i \in \{1, \dots, l\}, l < \infty, j \in \{1, \dots, \infty\}$, is a one-dimensional stochastic process called Teugel's martingale (see Nualart and Schoutens [17]) defined in Section 2. We call Eq. (1.1) a one-dimensional BSRDE for $\Pi \in \mathbf{R}$, and a multi-dimensional BSRDE for $\Pi \in \mathbf{R}^{n\times n}, n > 1$.

There are many studies on the existence of a unique solution to the BSRDE based on the nonlinear BSDE including Pardoux and Peng [20]. For $\gamma_\Pi^{ij}(t) = E^{ij} = F^{ij} = 0$, for all i, j (the case where $\gamma_\Pi^{ij} = E^{ij} = F^{ij} = 0$, for all i, j, implies that the BSRDE does not have Teugel's martingales and $\Pi(t^-)$ in G of (1.1) is equal to $\Pi(t)$), the BSRDE is of a well-known form, which includes a class of the BSRDEs given in other studies such as Bismut [4], Wonham [24], Chen et al. [5], Kohlmann and Zhou [13], Kohlmann and Tang [11], Kohlmann and Tang [9], Kohlmann and Tang [10] and so on. Kohlmann and Tang [12] survey recent studies on the BSRDEs.

If the coefficients A, B, C^i, D^i, M, N ($M \in \mathbf{R}^{n\times n}, N \in \mathbf{R}^{m\times m}$) in (1.1) are constant and $\beta_\Pi^i = 0$, for all i, $\gamma_\Pi^{ij} = E^{ij} = F^{ij} = 0$, for all i, j, then the BSRDE becomes a nonlinear matrix ordinary differential equation. The solution of such a class of BSRDEs is completely derived by Wonham [24]. Bismut [4] first considers the case where these coefficients are random variables under $\beta_\Pi^i = 0, i \in \{1, \dots, k_0\}$, $C^i = D^i = 0, i \in \{k_0 + 1, \dots, k\}$ and $\gamma_\Pi^{ij} = E^{ij} = F^{ij} = 0$, for all i, j, and then Peng [21] studies the existence and uniqueness of the solution to the multi-dimensional BSRDE. Kohlmann and Tang [10] make use of the result of Peng [21] and show the closeness property of the solutions of the BSRDE for the case where $\gamma_\Pi^{ij} = E^{ij} = F^{ij} = 0$, for all i, j, and prove the existence and uniqueness for the singular case ($N = 0$).

On the other hand, for the one-dimensional BSRDE (1.1), i.e. $n = 1$, Kohlmann and Tang [9] prove the existence and uniqueness for the regular case with $\gamma_{\Pi}^{ij} = E^{ij} = F^{ij} = 0$, for all i, j, by means of a technique developed by Kobylanski [7], and for the singular case by the regular approximation method [11].

It is essential for application to a financial problem which is to obtain an optimal portfolio strategy (an optimal control process) derived by solving the Linear–Quadratic Regulators (LQR) problem to confirm the existence and uniqueness of the solution to the one-dimensional singular BSRDE. There are studies of applications, called the BSDE approach, of the LQR problem to finance such as Kohlmann and Zhou [13], Kohlmann and Tang [11], Kohlmann and Tang [9], Bender and Kohlmann [2] and so on. The optimal portfolio strategy problem as an LQR problem is formulated as follows: First, define the controlled process x which is the process of the difference between a wealth process with risky assets driven by Wiener processes and a target random variable at maturity. Then, with the existence of a unique solution to the BSRDE whose form is determined by that of the controlled process, the optimal strategy u can be obtained by solving a minimization problem

$$\mathrm{P}^1: \quad \inf_{u \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^m)} J(0,T;u), \qquad x_0 \in R^n, \tag{1.2}$$

where

$$J(t,T;u) = \mathbb{E}\left[\int_t^T (x^* M x + u^* N u)(s)\mathrm{d}s + x(T)^* M(T) x(T) | \mathcal{F}_t\right]. \tag{1.3}$$

The objective in this paper is to study the (mainly one-dimensional) BSRDE with Lévy processes represented by (1.1). We will solve the minimization problem (1.2) for the case where the controlled process is given by

$$\begin{cases} \mathrm{d}x(t) = \left(Ax + \sum_{i=1}^{d} B^i u\right)(t)\mathrm{d}t + \sum_{i=1}^{k}(C^i x + D^i u)(t)\mathrm{d}w^i(t) \\ \quad + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(E^{ij}(t)x(t^-) + F^{ij}(t)u(t))\mathrm{d}H^{ij}(t) \\ x(0) = x_0. \end{cases} \tag{1.4}$$

Some questions to be answered arise when we solve the LQR problem (1.2) with the controlled process (1.4) driven by Lévy processes. (1) Can we have the existence and uniqueness of the solution to the one-dimensional singular BSRDE with Lévy processes (1.1)? If existence and uniqueness of the solution to such a BSRDE were to hold, we would be able to prove this for the singular case by making use of the existence and uniqueness of the solution to the one-dimensional *regular* BSRDE with Lévy processes (1.1) and the closeness for the solution. (2) Can we have the existence and uniqueness of the solution for the singular case? (3) Do we have the closeness property of the solution of the BSRDE with Lévy processes?

In order to answer these questions on the existence and uniqueness of solutions to one-dimensional regular and singular BSRDEs, we make use of the *closeness* property of the solution of the BSRDE established by Kohlmann and Tang [10] for the solution of the multi-dimensional regular BSRDE (1.1) with $\gamma_{\Pi}^{ij} = E^{ij} = F^{ij} = 0$, for all i, j. We first show the property for the multi-dimensional regular BSRDE with Lévy processes. For the *regular* case with one dimension and $\gamma_{\Pi}^{ij} = E^{ij} = F^{ij} \neq 0$, for all i, j, we will prove existence of the solution through an approximating BSDE of Kohlmann and Tang [11]. They prove the existence and uniqueness of solution to the one-dimensional regular BSRDE (1.1) with $\gamma_{\Pi}^{ij} = E^{ij} = F^{ij} = 0$, for all i, j, by a technique developed by Kobylanski [7]. They also prove this for the *singular* case. In this paper, we instead use the closeness property of the solution for the regular case with Lévy processes and will find it more convenient than the Kohlmann and Tang [11] approach for proving existence of the solution. For the singular case we use the approximation method of Kohlmann and Tang [11] and the closeness property again. Finally, we extend to the *non-homogeneous* BSRDE with Lévy processes and apply the LQR problem to a financial problem.

In summary, the contributions of this paper include:

1. The closeness property of the solutions of the multi-dimensional regular BSRDE with Lévy processes is established.
2. The existence and uniqueness of solutions to the one-dimensional regular and singular BSRDEs with Lévy processes is included.
3. The application of the LQR problem is implemental in solving a financial (i.e. portfolio selection) problem with risky asset having Lévy processes (we often call them jump risks).

The rest of the paper is organized as follows. In Section 2, we provide a list of notation and the results: the "existence and uniqueness" and the "comparison theorem" for the solution to the BSDE, with respect to Teugel's martingale processes. Section 3 presents the optimal control for the LQR problem P^1 (1.2) and shows the property of closeness for the multi-dimensional BSRDE with Lévy processes. In Section 4, we first prove the existence and uniqueness of the solution to the one-dimensional regular BSRDE with Lévy processes and subsequently prove them for the singular case by the closeness of the solution obtained in Section 3. Section 5 is devoted to the LQR problem in the non-homogeneous case and the application to the financial problem with full and partial observation cases.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space, where $\{\mathcal{F}_t; t \in [0, T], T < \infty\}$ satisfies the usual conditions, a right continuous increasing family of complete sub σ-algebra of $\mathcal{F}$. Let $\{W(t), t \in [0, T]\}$ be a standard Wiener process in $\mathbf{R}^n$ and $\{L(t), t \in [0, T]\}$ be an $\mathbf{R}^n$-valued Lévy process with a Lévy measure v such that $\int_{\mathbf{R}} (1 \wedge z) v(\mathrm{d}z) < \infty$ which is independent of the Wiener process $W(t)$.

Assume that $\mathcal{F}_t$ is the smallest σ-algebra generated by $W(t)$ and $L(t)$, i.e.

$$\mathcal{F}_t = \sigma(W(s), s \leqslant t) \vee \sigma(L(s), s \leqslant t) \vee \mathscr{N},$$

where $\mathscr{N}$ is the totality of the P-null set.

We denote by $H^i(t), i \geqslant 1$, Teugel's martingales associated with the Lévy process $L(t)$ (see, e.g., Bahlali et al. [1], Nualart and Schoutens [17], Nualart and Schoutens [16]). $H^i(t)$ is given by

$$H^i(t) = c_{i,i} Y_t^{(i)} + c_{i,i-1} Y_t^{(i-1)} + \cdots + c_{i,1} Y_t^{(1)},$$

where $Y_t^{(i)} = L_t^{(i)} - \mathbb{E}[L_t^{(i)}]$ for all $i \geqslant 1$ and $L_t^{(i)}$ are power-jump processes: $L_t^{(1)} = L_t$ and $L_t^{(i)} = \sum_{0<s\leqslant t} (\Delta L_s)^i$ for $i \geqslant 2$. This representation is obtained from the chaos decomposition in Nualart and Schoutens [17]. They furthermore prove the existence of a unique solution of the BSDE [16]. Løkka [14] prove the martingale representation (Clark–Ocone formula) of the Lévy process in different ways based on the chaos expansion.

We introduce the following notation:

- $|A| = \sqrt{\sum_{i,j} A_{ij}^2}$.
- A^*: the transpose of matrix A.
- $(\mathscr{U}, d)$: the separable metric space.
- $\mathcal{S}^n, \mathcal{S}_+^n, \hat{\mathcal{S}}_+^n$: the space of all $n \times n$ symmetric matrices, the space of all non-negative definite matrices and the space of all positive definite matrices, respectively.
- $\mathbf{C}([0, T]; H)$: the Banach space of H-valued continuous functions on $[0, T]$.
- $l^2(H)$: the space of H-valued $\{f_i\}_{i \geqslant 1}$ such that $\sum_{i=1}^{\infty} |f_i(s)|^2 < \infty$.
- $\mathbf{l}_{\mathcal{F}}^2(0, T; H)$: the Banach space of $l^2(H)$-valued $\mathcal{F}_t$-adapted stochastic processes f_i, endowed with the norm $\mathbb{E} \int_0^T \sum_{i=1}^{\infty} |f_i(s)|^2 \mathrm{d}s$.
- $\mathcal{L}_{\mathcal{F}}^2(0, T; H)$: the Banach space of H-valued $\mathcal{F}_t$-adapted square integrable stochastic processes with the norm.
- $\mathcal{L}_{\mathcal{F}}^{\infty}(0, T; H)$: the Banach space of H-valued $\mathcal{F}_t$-adapted, essentially bounded stochastic process with the norm: $\|f(t)\|_{\mathcal{L}_{\mathcal{F}}^{\infty}} = \mathrm{esssup}_{t \in [0,T], \omega \in \Omega} |f(t)|$.
- $\mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P}, H)$: the Banach space of H-valued norm square integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.
- $\mathbf{L}^{\infty}(\Omega, \mathcal{F}, \mathbb{P}, H)$: the Banach space of H-valued, essentially norm bounded random variables f on $(\Omega, \mathcal{F}, \mathbb{P})$ with the norm: $\mathrm{esssup}_{\omega \in \Omega} \max_{t \in [0,T]} |f(t, \omega)|$.
- U: the admissible set in $\mathcal{L}_{\mathcal{F}}^2(t, T; \mathbf{R}^n)$.

We will need the existence and uniqueness of the solution to the BSDE driven by Teugel's martingales in order to prove those for the one-dimensional regular BSRDE with Teugel's martingales. Bahlali et al. [1] prove the existence and uniqueness of the solution to the BSDE with Teugel's martingales which are more general than the stochastic process with Poisson random measures.

The following two lemmas concerning the *uniqueness and existence* and the *comparison theorem* for the BSDE driven by both the Wiener process and Teugel's martingales are obtained by Bahlali et al. [1] using the results of Pardoux and Peng [20].

Lemma 2.1 (*Existence and Uniqueness*). *Assume that*

(1) *a terminal value* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$,
(2) *a process* $f : [0, T] \times \Omega \times \mathbf{R} \times \mathbf{R} \times l^2 \to \mathbf{R}$ *such that* (i) f *is progressively measurable s.t.* $f(\cdot, 0, 0, 0)$ *is in the space of real-valued square integrable and* $\mathcal{F}_t$*-progressively measurable processes with the norm* $\|f\| = \mathbb{E} \int_0^T |f(s)|^2 \mathrm{d}s < \infty$ *and* (ii) *there exists a constant* $L > 0$ *s.t.*

$$|f(t, \omega, y, u, z) - f(t, \omega, y', u', z')| \leqslant L(|y - y'| - |u - u'| - \|z - z'\|).$$

Then, the BSDE

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_{s-}, U_s, Z_s) \mathrm{d}s + \int_t^T U_s \mathrm{d}w(s) + \int_t^T \sum_{i=1}^{\infty} Z_s^i \mathrm{d}H^i(s) \\ \xi \quad \text{is an } \mathcal{F}_T \text{ measurable square integrable random variable} \end{cases} \tag{2.1}$$

has a unique solution (y, u, z) *on a Banach space such that*

$$\|(y, u, z)\|^2 = \mathbb{E} \left(\sup_{0 \leqslant t \leqslant T} |y_t|^2 + \int_0^T |u_s|^2 \mathrm{d}s + \int_0^T \|z_s\|^2 \mathrm{d}s \right) < \infty.$$

Lemma 2.2 (*Comparison Theorem*). *Suppose that* $(f^k, \xi^k), k = \{1, 2\}$ *satisfies* (1) *and* (2) *in Lemma* 2.1. *Assume that*

$$f^1(t, y, u, z) \leqslant f^2(t, y, u, z), \quad \forall (y, u, z) \in \mathbf{R} \times \mathbf{R} \times l^2, \mathrm{d}\mathbb{P} \times \mathrm{d}t - \; a.s.; \xi^1 \leqslant \xi^2.$$

Then, $Y_t^1 \leqslant Y_t^2, t \in [0, T]$.

Remark 2.1. In Bahlali et al. [1] and Nualart and Schoutens [16], Y_{t-}, the left limit value before it jumps at t, is set to the drift coefficient of the BSDE. For the controlled process $x(t)$, we set the variable $x(t^-)$ to the coefficient of Lévy processes as in (1.4); see e.g. Øksendal and Sulem [19] for the BSDE with Poisson random measures and Tang and Li [23] for the Poisson point process.

3. LQR and optimal control

This section deals with the multi-dimensional regular BSRDE with Lévy processes (1.1). We first present an optimal control for (1.2), and show the closeness property of the solution of the multi-dimensional regular BSRDE with Lévy processes. The closeness property is important for proving the existence and uniqueness of solutions to the *one-dimensional* regular and singular BSRDEs with Lévy processes in the next section.

Consider the stochastic control problem P^1 (1.2) and the cost function (1.3) with M and N in $R^{n\times n}$ and $R^{m\times m}$, respectively. The controlled process $x(t)$ is the solution of the stochastic differential equation (1.4) with the dimensions of A, B^i, C^i, D^i, E^{ij} and F^{ij} for all i in $n \times n$, $n \times m$, $n \times n$, $n \times m$, $n \times n$ and $n \times m$, respectively.

Let us introduce the multi-dimensional BSRDE with Lévy processes (1.4). The following proposition provides the optimal feedback control for P^1 (1.2).

Proposition 3.1. *Let* (Π, β, γ) *be an* $\mathcal{F}_t$*-adapted solution of the BSRDE* (1.1) *and in* $\mathcal{L}^\infty_{\mathcal{F}}(0, T; \mathcal{S}^n_+) \cap \mathbf{L}^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbf{C}([0, T]; \mathcal{S}^n_+), \mathcal{L}^2_{\mathcal{F}}(0, T; \mathcal{S}^n)^k, \mathbf{l}^2_{\mathcal{F}}(0, T; \mathcal{S}^n)^l)$. *Assume that* $(N + \sum_{i=1}^k (D^i)^* \Pi(t^-) D^i + \sum_{i=1}^l \sum_{j=1}^\infty (F^{ij})^* \Pi(t^-) F^{ij})(t)$ *is uniformly positive. Then*

$$\begin{aligned} \hat{u}(t) = &- \left(N + \sum_{i=1}^k (D^i)^* \Pi(t^-) D^i + \sum_{i=1}^l \sum_{j=1}^\infty (F^{ij})^* \Pi(t^-) F^{ij} \right)^{-1} (t) \\ &\times \left(\sum_{i=1}^d (B^i)^* \Pi(t^-) + \sum_{i=1}^k (D^i)^* (\beta^i_\Pi + \Pi(t^-) C^i) \right. \\ &\left. + \sum_{i=1}^l \sum_{j=1}^\infty (F^{ij})^* (\gamma^{ij}_\Pi + \Pi(t^-) E^{ij}) \right) (t) x(t^-) \end{aligned} \tag{3.1}$$

is an optimal feedback control, and

$$x(t)^* \Pi(t) x(t) = \operatorname*{essinf}_{u \in \mathcal{L}^2_{\mathcal{F}}(t, T; \mathbf{R}^m)} J(t, T; u). \tag{3.2}$$

Proof. In order to obtain the optimal control, we use the *completion of squares* (see Kohlmann and Tang [11], Kohlmann and Zhou [13], Chen et al. [5]).

Assume that the stochastic process $\Pi(t) \in \mathcal{L}^\infty_{\mathcal{F}}(0, T; \mathcal{S}^n_+) \cap \mathbf{L}^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbf{C}(0, T; \mathcal{S}^n_+))$ has the decomposition

$$\mathrm{d}\Pi(t) = \Gamma(t)\mathrm{d}t + \sum_{i=1}^{k} \Lambda^i(t)\mathrm{d}w^i + \sum_{i=1}^{l}\sum_{j=1}^{\infty} \Theta^{ij}(t)\mathrm{d}H^{ij}(t). \tag{3.3}$$

Applying the Itô–Lévy formula, we obtain

$$\begin{aligned}
\mathrm{d}x(t)^*\Pi(t)x(t) = \Bigg[& x^*(t^-)\Bigg\{A^*\Pi(t^-) + \Gamma + \Pi(t^-)A \\
& + \sum_{i=1}^{k}\{(C^i)^*(\Lambda^i + \Pi(t^-)C^i) + \Lambda^i C^i\} \\
& + \sum_{i=1}^{l}\sum_{j=1}^{\infty}\{(E^{ij})^*(\Theta^{ij} + \Pi(t^-)E^{ij}) + \Theta^{ij}E^{ij}\}\Bigg\}(t)x(t^-) \\
& + 2u(t)^*\Bigg\{\sum_{i=1}^{d}(B^i)^*\Pi(t^-) + \sum_{i=1}^{k}(D^i)^*(\Lambda^i + \Pi(t^-)C^i) \\
& + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*(\Theta^{ij} + \Pi(t^-)E^{ij})\Bigg\}(t)x(t^-) \\
& + u(t)^*\Bigg\{\sum_{i=1}^{k}(D^i)^*\Pi(t^-)D^i + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*\Pi(t^-)F^{ij}\Bigg\}(t)u(t)\Bigg]\mathrm{d}t \\
& + \sum_{i=1}^{k}\{\cdots\}\mathrm{d}w^i(t) + \sum_{i=1}^{l}\sum_{j=1}^{\infty}\{\cdots\}\mathrm{d}H^{ij}(t).
\end{aligned} \tag{3.4}$$

The cost function (1.3) with (3.4) can be manipulated as follows:

$$\begin{aligned}
J(s,T;u(\cdot)) = \mathbb{E}\Bigg[\int_s^T \Bigg[& x^*(t^-)\Bigg\{A^*\Pi(t^-) + \Gamma + \Pi(t^-)A \\
& + \sum_{i=1}^{k}\{(C^i)^*(\Lambda^i + \Pi(t^-)C^i) + \Lambda^i C^i\} \\
& + \sum_{i=1}^{l}\sum_{j=1}^{\infty}\{(E^{ij})^*(\Theta^{ij} + \Pi(t^-)E^{ij}) + \Theta^{ij}E^{ij}\} \\
& + M - L^*K^{-1}L\Bigg\}(t)x(t^-) \\
& + (u + K^{-1}Lx(t^-))^*(t)K(t)(u + K^{-1}Lx(t^-))(t)\Bigg]\mathrm{d}t \\
& + \big(x(T)^*(M(T) - \Pi(T))x(T) + x_s^*\Pi(s)x_s\big)\,|\mathcal{F}_s\Bigg]
\end{aligned}$$

where $K(t)$ and $L(t)$ have the forms defined in the Introduction section with $\Lambda^i = \beta^i_\Pi$ and $\Theta^{ij} = \gamma^{ij}_\Pi$.

Substituting the BSRDE (1.1) and $M(T) = \Pi(T)$ for the cost function, we have

$$\begin{aligned} J(s,T;u(\cdot)) &= \mathbb{E}\left[\int_s^T (u + K^{-1}Lx(t^-))^* K(u + K^{-1}Lx(t^-))(t)\mathrm{d}t|\mathcal{F}_s\right] + x_s^* \Pi(s) x_s \\ &\geqslant x_s^* \Pi(s) x_s = J(s,T;\hat{u}(\cdot)). \end{aligned} \tag{3.5}$$

K is uniformly positive for $s \in [t, T]$. Thus, the optimal feedback control $\hat{u}(\cdot)$ is given by (3.1). □

Now we show the closeness property of the solution of the BSRDE (1.1). To this end, let us consider the following multi-dimensional regular BSRDE: for $\tilde{m} \in \mathbf{N}$,

$$\begin{cases} \mathrm{d}\Pi^{\tilde{m}}(t) = -G^{\tilde{m}}(t, \Pi^{\tilde{m}}, \beta_{\Pi}^{\tilde{m}}, \gamma_{\Pi}^{\tilde{m}})\mathrm{d}t \\ \quad + \sum_{i=1}^{k} \beta_{\Pi}^{i\ \tilde{m}}(t)\mathrm{d}w^i(t) + \sum_{i=1}^{l}\sum_{j=1}^{\infty} \gamma_{\Pi}^{ij\ \tilde{m}}(t)\mathrm{d}H^{ij}(t) \\ \Pi^{\tilde{m}}(T) = M^{\tilde{m}}(T), \end{cases} \tag{3.6}$$

where

$$\begin{aligned} G^{\tilde{m}}(t, \Pi^{\tilde{m}}, \beta_{\Pi}^{\tilde{m}}, \gamma_{\Pi}^{\tilde{m}}) = \Bigg(& \Pi^{\tilde{m}}(t^-)A^{\tilde{m}} + (A^{\tilde{m}})^*\Pi^{\tilde{m}}(t^-) + M^{\tilde{m}} + \sum_{i=1}^{k} \beta_{\Pi}^{i\ \tilde{m}} C^{i\ \tilde{m}} \\ &+ \sum_{i=1}^{l}\sum_{j=1}^{\infty} \gamma_{\Pi}^{ij\ \tilde{m}} E^{ij\ \tilde{m}} + \sum_{i=1}^{k} (C^{i\ \tilde{m}})^*(\beta_{\Pi}^{i\ \tilde{m}} + \Pi^{\tilde{m}}(t^-)C^{i\ \tilde{m}}) \\ &+ \sum_{i=1}^{l}\sum_{j=1}^{\infty} (E^{ij\ \tilde{m}})^*(\gamma_{\Pi}^{ij\ \tilde{m}} + \Pi^{\tilde{m}}(t^-)E^{ij\ \tilde{m}})\Bigg)(t) \\ &- F^{\tilde{m}}(t, \Pi^{\tilde{m}}, \beta_{\Pi}^{\tilde{m}}, \gamma_{\Pi}^{\tilde{m}}) \end{aligned}$$

and

$$\begin{aligned} F^{\tilde{m}}(t, \Pi^{\tilde{m}}, \beta_{\Pi}^{\tilde{m}}, \gamma_{\Pi}^{\tilde{m}}) &= L'(t)^* K'(t)^{-1} L'(t), \\ L'(t) &:= \Bigg(\sum_{i=1}^{d} B^{i\ \tilde{m}^*} \Pi^{\tilde{m}}(t^-) + \sum_{i=1}^{k} D^{i\ \tilde{m}^*}(\beta_{\Pi}^{i\ \tilde{m}} + \Pi^{\tilde{m}}(t^-)C^{i\ \tilde{m}}) \\ &\quad + \sum_{i=1}^{l}\sum_{j=1}^{\infty} F^{ij\ \tilde{m}^*}(\gamma_{\Pi}^{ij\ \tilde{m}} + \Pi^{\tilde{m}}(t^-)E^{ij\ \tilde{m}})\Bigg)(t), \\ K'(t) &:= \left(N^{\tilde{m}} + \sum_{i=1}^{k} D^{i\ \tilde{m}^*}\Pi^{\tilde{m}}(t^-)D^{i\ \tilde{m}} + \sum_{i=1}^{l}\sum_{j=1}^{\infty} F^{ij\ \tilde{m}^*}\Pi^{\tilde{m}}(t^-)F^{ij\ \tilde{m}}\right)(t). \end{aligned}$$

Assume that for $\tilde{m} \in \mathbf{N}$:

H1: For all i, $A^{\tilde{m}}$, $B^{i\ \tilde{m}}$, $C^{i\ \tilde{m}}$, $D^{i\ \tilde{m}}$, $M^{\tilde{m}}$ and $N^{\tilde{m}}$ are $\mathcal{F}_t$-progressively measurable and, as $k \to \infty$, converge uniformly in (t, ω) to A^{∞}, $B^{i\ \infty}$, $C^{i\ \infty}$, $D^{i\ \infty}$, M^{∞} and N^{∞}, respectively. $\sum_{j=1}^{\infty} E^{ij\ \tilde{m}}$, $\sum_{j=1}^{\infty} F^{ij\ \tilde{m}}$ are $\mathcal{F}_t$-progressively measurable and, as $k \to \infty$, converge uniformly in (t, ω) to $\sum_{j=1}^{\infty} E^{ij\ \infty}$, $\sum_{j=1}^{\infty} F^{ij\ \infty}$.

H2: $M^{\tilde{m}}(T)$ is an $\mathcal{F}_T$-measurable and non-negative random matrix variable and, as $\tilde{m} \to \infty$, converges uniformly in ω to $M^{\infty}(T)$.

H3: $M^{\tilde{m}}(t)$ is a.s. a.e. non-negative.

H4: There exist constants ϵ_1, ϵ_2 and ϵ_3 which are independent of $\tilde{m}$ such that

$$|A^{\tilde{m}}|, |B^{i\tilde{m}}|, |C^{i\tilde{m}}|, |D^{i\tilde{m}}|, \sum_{j=1}^{\infty}|E^{ij\tilde{m}}|, \sum_{j=1}^{\infty}|F^{ij\tilde{m}}|, |M^{\tilde{m}}|, |N^{\tilde{m}}| \leqslant \epsilon_1,$$

$$\sum_{j=1}^{\infty}|E^{ij\tilde{m}}|^2, \sum_{j=1}^{\infty}|F^{ij\tilde{m}}|^2 \leqslant \epsilon_2,$$

$$N^{\tilde{m}} \geqslant \epsilon_3 I_{m\times m}.$$

H5: The BSRDE (3.6) has an $\mathcal{F}_t$-adapted unique solution $(\Pi^{\tilde{m}}, \beta_{\Pi}^{\tilde{m}}, \gamma_{\Pi}^{\tilde{m}}) \in \mathcal{L}_{\mathcal{F}}^{\infty}(0,T;\mathcal{S}_+^n) \cap \mathbf{L}^{\infty}(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbf{C}([0,T];\mathcal{S}_+^n), \mathcal{L}_{\mathcal{F}}^2(0,T;\mathcal{S}^n)^k, \mathbf{l}_{\mathcal{F}}^2(0,T;\mathcal{S}^n)^l)$ for all $\tilde{m}$.

We denote the difference between $\phi^{\tilde{m}}$ and $\phi^{\tilde{n}}$ as $\phi^{\tilde{m}\tilde{n}}$ where $\phi = M, N, x, A, B, C, D, E, F$.

Lemma 3.1. *Assume* H1–H5. *Then there exist constants* K_i, $i = 0, 1, \ldots, 3$, *such that*

$$\mathbb{E}|\Pi^{\tilde{m}}(t)|^2 + K_0\mathbb{E}\int_t^T \sum_{i=1}^{k}|\beta_{\Pi}^{i\tilde{m}}(s)|^2 \mathrm{d}s + K_1\mathbb{E}\int_t^T \sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma_{\Pi}^{ij\tilde{m}}(s)|^2\mathrm{d}s$$
$$\leqslant |M^{\tilde{m}}(T)|^2 + K_2 + K_3\mathbb{E}\int_t^T |\Pi^{\tilde{m}}(s^-)|^2\mathrm{d}s. \tag{3.7}$$

Proof. Applying the Itô–Lévy formula to $|\Pi^{\tilde{m}}(t)|^2$, we have

$$\begin{aligned}\mathrm{d}|\Pi^{\tilde{m}}(t)|^2 = -\Bigg[&2\mathrm{Tr}(\Pi^{\tilde{m}}(t^-)M^{\tilde{m}}) + 4\mathrm{Tr}(\Pi^{\tilde{m}}(t^-)(A^{\tilde{m}})^*\Pi^{\tilde{m}}(t^-))\\ &\times \sum_{i=1}^{k}\left\{4\mathrm{Tr}(\Pi^{\tilde{m}}(t^-)\beta^{i\tilde{m}}C^{i\tilde{m}}) + 2\mathrm{Tr}(\Pi^{\tilde{m}}(t^-)(C^{i\tilde{m}})^*\Pi^{\tilde{m}}C^{i\tilde{m}})\right\}\\ &+ \sum_{i=1}^{l}\left\{4\mathrm{Tr}(\Pi^{\tilde{m}}(t^-)\sum_{j=1}^{\infty}\gamma^{ij\tilde{m}}E^{ij\tilde{m}}) + 2\mathrm{Tr}(E^{ij\tilde{m}})^*\Pi^{\tilde{m}}E^{ij\tilde{m}}\right\}\\ &- 2\mathrm{Tr}(\Pi^{\tilde{m}}(t^-)F^{\tilde{m}}(t,\Pi^{\tilde{m}},\beta^{\tilde{m}},\gamma^{\tilde{m}})) - \sum_{i=1}^{k}|\beta^{i\tilde{m}}|^2 - \sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma^{ij\tilde{m}}|^2\Bigg](t)\mathrm{d}t\\ &+2\sum_{i=1}^{k}|\mathrm{Tr}(\Pi^{\tilde{m}}(t^-)\beta^{i\tilde{m}}(t))\mathrm{d}w^i(t)\\ &+2\sum_{i=1}^{l}\sum_{j=1}^{\infty}\mathrm{Tr}(\Pi^{k}(t^-)\gamma^{ij\tilde{m}}(t))\mathrm{d}H^{ij}(t).\end{aligned} \tag{3.8}$$

By the positivity of $\mathrm{Tr}(\Pi^{\tilde{m}}(t^-)F^{\tilde{m}}(t,\Pi^{\tilde{m}},\beta^{\tilde{m}},\gamma^{\tilde{m}}))$ (since F has a squared form and $\Pi^k \in \mathcal{S}_+$), Young's inequality, $\mathrm{Tr}(AB) \leqslant |A||B|$ and $\mathrm{Tr}(A^2) = |A|^2$ for $A \in \mathcal{S}$, (3.8) yields

$$\mathrm{d}|\Pi^{\tilde{m}}(t)|^2 \geqslant \left\{\text{const.} + \epsilon_0|\Pi^{\tilde{m}}(t^-)|^2 + \epsilon_1\sum_{i=1}^{k}|\beta^{i\tilde{m}}|^2 + \epsilon_2\sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma^{ij\tilde{m}}|^2\right\}(t)\mathrm{d}t$$

$$+\left(\sum_{i=1}^{k}|\beta^{i\tilde{m}}|^2+\sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma^{ij\tilde{m}}|^2\right)(t)\mathrm{d}t+2\sum_{i=1}^{k}\mathrm{Tr}(\Pi^{\tilde{m}}(t^-)\beta^{i\tilde{m}}(t))\mathrm{d}w^i(t)$$
$$+2\sum_{i=1}^{l}\sum_{j=1}^{\infty}\mathrm{Tr}(\Pi^{\tilde{m}}(t^-)\gamma^{ij\tilde{m}}(t))\mathrm{d}H^{ij}(t). \tag{3.9}$$

Taking the integral of (3.9) from t to T and expectation, we can obtain inequality (3.7). □

Lemma 3.2 (*A Priori Estimates*). *Under conditions* H1–H5*, we have estimates*

$$0\leqslant \Pi^{\tilde{m}}(t)\leqslant \epsilon\mathbf{I}_{n\times n}, \tag{3.10}$$
$$\mathbb{E}\left(\int_t^T\left(\sum_{i=1}^{k}|\beta^{i\tilde{m}}(s)|^2+\sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma^{ij\tilde{m}}(s)|^2\right)\mathrm{d}s\right)^p\leqslant\epsilon\quad\forall p\geqslant 1. \tag{3.11}$$

Proof. By Lemma 3.1, we have

$$\mathbb{E}|\Pi^{\tilde{m}}(t)|^2\leqslant \text{const.}+K_3\mathbb{E}\int_t^T|\Pi^{\tilde{m}}(s^-)|^2\mathrm{d}s. \tag{3.12}$$

Applying Gronwall's inequality (see e.g. Protter [22]) to (3.12), for a positive constant depending on T, $\mathbb{E}|\Pi^{\tilde{m}}(t)|^2\leqslant C(T)<\infty$. Therefore, since $\Pi^{\tilde{m}}(t)\in\mathcal{S}_+^n$, we can obtain the estimate (3.10). From this estimate (3.10) and inequality (3.9),

$$|\Pi^{\tilde{m}}(t)|^2+\int_t^T\sum_{i=1}^{k}|\beta^{i\tilde{m}}(s)|^2\mathrm{d}s+\int_t^T\sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma^{ij\tilde{m}}(s)|^2\mathrm{d}s$$
$$\leqslant \text{const.}+K_3\int_t^T\left(\sum_{i=1}^{k}|\beta^{i\tilde{m}}|^2+\sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma^{ij\tilde{m}}|^2\right)(s)\mathrm{d}s$$
$$-2\int_t^T\sum_{i=1}^{k}\mathrm{Tr}(\Pi^{\tilde{m}}(s^-)\beta^{i\tilde{m}}(s))\mathrm{d}w^i(s)$$
$$-2\int_t^T\sum_{i=1}^{l}\sum_{j=1}^{\infty}\mathrm{Tr}(\Pi^{\tilde{m}}(s^-)\gamma^{ij\tilde{m}}(s))\mathrm{d}H^{ij}(s).$$

Thus,

$$\int_t^T\left(\sum_{i=1}^{k}|\beta^{i\tilde{m}}|^2+\sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma^{ij\tilde{m}}|^2\right)(s)\mathrm{d}s$$
$$\leqslant \text{const.}+K_3\int_t^T\left(\sum_{i=1}^{k}|\beta^{i\tilde{m}}|^2+\sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma^{ij\tilde{m}}|^2\right)(s)\mathrm{d}s$$
$$-2\int_t^T\sum_{i=1}^{k}\mathrm{Tr}(\Pi^{\tilde{m}}(s^-)\beta^{i\tilde{m}}(s))\mathrm{d}w^i(s)$$
$$-2\int_t^T\sum_{i=1}^{l}\sum_{j=1}^{\infty}\mathrm{Tr}(\Pi^{\tilde{m}}(s^-)\gamma^{ij\tilde{m}}(s))\mathrm{d}H^{ij}(s). \tag{3.13}$$

From (3.13), the Burkholder–Davis–Gundy inequalities, $\mathrm{Tr}(AB)^2 \leqslant |A|^2|B|^2$ and (3.10),

$$\begin{aligned}
&\mathbb{E}\left(\int_t^T \left(\sum_{i=1}^k |\beta^{i\tilde{m}}|^2 + \sum_{i=1}^l \sum_{j=1}^\infty |\gamma^{ij\tilde{m}}|^2\right)(s)\mathrm{d}s\right)^p \\
&\leqslant 3^p \Bigg[\mathrm{const.}^p + K'^p \mathbb{E}\left(\int_t^T \left(\sum_{i=1}^k |\beta^{i\tilde{m}}|^2 + \sum_{i=1}^l \sum_{j=1}^\infty |\gamma^{ij\tilde{m}}|^2\right)(s)\mathrm{d}s\right)^p \\
&+ 2^p \mathbb{E}\Bigg(\int_t^T \sum_{i=1}^k \mathrm{Tr}(\Pi^{\tilde{m}}(s^-)\beta^{i\tilde{m}}(s))\mathrm{d}w^i(s) \\
&+ \int_t^T \sum_{i=1}^l \sum_{j=1}^\infty \mathrm{Tr}(\Pi^{\tilde{m}}(s^-)\gamma^{ij\tilde{m}}(s))\mathrm{d}H^{ij}(s)\Bigg)^p\Bigg] \\
&\leqslant 3^p \Bigg[\mathrm{const.}^p + K'^p \mathbb{E}\left(\int_t^T \left(\sum_{i=1}^k |\beta^{i\tilde{m}}|^2 + \sum_{i=1}^l \sum_{j=1}^\infty |\gamma^{ij\tilde{m}}|^2\right)(s)\mathrm{d}s\right)^p \\
&+ 2^p \mathbb{E}\Bigg(\sup_{\tau\in[t,T]} \Bigg|\int_t^\tau \sum_{i=1}^k \mathrm{Tr}(\Pi^{\tilde{m}}(s^-)\beta^{i\tilde{m}}(s))\mathrm{d}w^i(s) \\
&+ \int_t^\tau \sum_{i=1}^l \sum_{j=1}^\infty \mathrm{Tr}(\Pi^{\tilde{m}}(s^-)\gamma^{ij\tilde{m}}(s))\mathrm{d}H^{ij}(s)\Bigg|\Bigg)^p\Bigg] \\
&\leqslant 3^p \Bigg[\mathrm{const.}^p + K'^p \mathbb{E}\left(\int_t^T \left(\sum_{i=1}^k |\beta^{i\tilde{m}}| + \sum_{i=1}^l \sum_{j=1}^\infty |\gamma^{ij\tilde{m}}|\right)(s)\mathrm{d}s\right)^p \\
&+ C_p \mathbb{E}\left(\int_t^T \sum_{i=1}^k |\Pi^{\tilde{m}}(s^-)|^2|\beta^{i\tilde{m}}(s)|^2\mathrm{d}s + \int_t^T \sum_{i=1}^l \sum_{j=1}^\infty |\Pi^{\tilde{m}}(s^-)|^2|\gamma^{ij\tilde{m}}(s)|^2\mathrm{d}s\right)^{\frac{p}{2}}\Bigg] \\
&\leqslant 3^p \Bigg[\mathrm{const.}^p + K'^p \mathbb{E}\left(\int_t^T \left(\sum_{i=1}^k |\beta^{i\tilde{m}}|^2 + \sum_{i=1}^l \sum_{j=1}^\infty |\gamma^{ij\tilde{m}}|^2\right)(s)\mathrm{d}s\right)^p \\
&+ C'_p \mathbb{E}\left(\int_t^T \sum_{i=1}^k |\beta^{i\tilde{m}}(s)|^2\mathrm{d}s + \int_t^T \sum_{i=1}^l \sum_{j=1}^\infty |\gamma^{ij\tilde{m}}(s)|^2\mathrm{d}s\right)^{\frac{p}{2}}\Bigg]. \qquad (3.14)
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\mathbb{E}\left(\int_t^T \left(\sum_{i=1}^k |\beta^{i\tilde{m}}(s)|^2 + \sum_{i=1}^l \sum_{j=1}^\infty |\gamma^{ij\tilde{m}}(s)|^2\right)\mathrm{d}s\right)^p \leqslant (1 - 3^p K'^p)^{-1} 3^p \\
&\times \left[\mathrm{const.}^p + C'_p \mathbb{E}\left(\int_t^T \sum_{i=1}^k |\beta^{i\tilde{m}}(s)|^2\mathrm{d}s + \int_t^T \sum_{i=1}^l \sum_{j=1}^\infty |\gamma^{ij\tilde{m}}(s)|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]. \qquad (3.15)
\end{aligned}$$

Thus the estimate (3.11) holds. □

Let us consider the optimal control problem for each $\tilde{m}$. For this purpose, define the following problem:

$$\mathrm{P}^2: \qquad \inf_{u\in\mathcal{L}^2(0,T;\mathrm{R}^m)} J^{\tilde{m}}(0,T;u), \qquad x_0 \in \mathrm{R}^n, \tag{3.16}$$

where

$$J^{\tilde{m}}(t,T;u) = \mathbb{E}\left[\int_t^T ((x^{\tilde{m}})^* M^{\tilde{m}} x^{\tilde{m}} + u^* N^{\tilde{m}} u)(s)\mathrm{d}s + (x^{\tilde{m}}(T))^* M^{\tilde{m}}(T) x^{\tilde{m}}(T)|\mathcal{F}_t\right], \tag{3.17}$$

and the controlled process $x^{\tilde{m}}(t)$ is the solution of the following stochastic differential equation:

$$\begin{cases} \mathrm{d}x(t) = \left(A^{\tilde{m}}x + \sum_{i=1}^{d} B^{i^{\tilde{m}}}u\right)(t)\mathrm{d}t + \sum_{i=1}^{k}(C^{i^{\tilde{m}}}x + D^{i^{\tilde{m}}}u)(t)\mathrm{d}w^i(t) \\ \quad + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(E^{ij^{\tilde{m}}}x(t^-) + F^{ij^{\tilde{m}}}u(t))(t)\mathrm{d}H^{ij}(t) \\ x(0) = x_0. \end{cases} \tag{3.18}$$

Since $\mathbb{E}[\int_t^T (\hat{u}(s)^* N^{\tilde{m}}(s)\hat{u}(s))\mathrm{d}s|\mathcal{F}_t] \leqslant K|x_t|^2$ by Proposition 3.1 and the a priori estimate of Lemma 3.2, the optimal control problem can be given by

$$\operatorname*{essinf}_{u\in U_{ad}(t,T)} J^{\tilde{m}}(t,T;u),$$

where

$$U_{ad}(t,T) = \left\{u \in \mathcal{L}^2(t,T;\mathrm{R}^m); \mathbb{E}\left[\int_t^T |u|^2 \mathrm{d}s|\mathcal{F}_t\right] \leqslant K|x_t|^2\right\}.$$

By Lemma 3.3 in Kohlmann and Tang [10], we have

$$\begin{aligned} |J^{\tilde{m}}(t,T;u) - J^{\tilde{n}}(t,T;u)| \leqslant{} & \epsilon_0 \operatorname*{esssup}_{\omega\in\Omega} |N^{\tilde{m}\tilde{n}}(t)|\mathbb{E}[|x^{\tilde{m}}(T)|^2|\mathcal{F}_t] \\ & + \epsilon_0 \mathbb{E}[|x^{\tilde{m}\tilde{n}}(T)|^2|\mathcal{F}_t]^{\frac{1}{2}} \mathbb{E}[|x^{\tilde{m}}(T)|^2 + 2|x^{\tilde{n}}(T)|^2|\mathcal{F}_t]^{\frac{1}{2}} \\ & + \epsilon_0 T \mathbb{E}[\sup_{s\in[t,T]} |x^{\tilde{m}\tilde{n}}(s)|^2|\mathcal{F}_t]^{\frac{1}{2}} \\ & \times \mathbb{E}[\sup_{s\in[t,T]} (2|x^{\tilde{m}}(s)|^2 + 2|x^{\tilde{n}}(s)|^2)|\mathcal{F}_t]^{\frac{1}{2}} \\ & + \epsilon_0 \operatorname*{esssup}_{\omega\in\Omega} \int_0^T |M^{\tilde{m}\tilde{n}}(s)|\mathrm{d}s \mathbb{E}[\sup_{s\in[t,T]} |x^{\tilde{m}}(s)|^2|\mathcal{F}_t] \\ & + \epsilon_1 |x_t|^2 \operatorname*{esssup}_{s\in[t,T],\omega\in\Omega} |M^{\tilde{m}\tilde{n}}(s)|, \end{aligned} \tag{3.19}$$

where ϵ_0, ϵ_1 are positive constants.

Using the following lemma, we can prove $\Pi(t) \in \mathcal{L}^{\infty}_{\mathcal{F}}(0,T;\mathcal{S}^n_+) \cap \mathbf{L}^{\infty}(\Omega,\mathcal{F}_T,\mathbb{P};\mathbf{C}([0,T];\mathcal{S}^n_+))$.

Lemma 3.3. *Let* H1–H5 *be satisfied and* $x^{\tilde{m}}(t)$ *be the solution of* (3.18). *Then we have for each* $\tilde{m},\tilde{n} \in \mathbf{N}$ *the following two inequalities:*

$$\mathbb{E}[\sup_{s\in[t,T]} |x^{\tilde{m}}(s)|^2|\mathcal{F}_t] \leqslant \varepsilon_0 |x_t|^2 \tag{3.20}$$

$$\begin{aligned}
&\mathbb{E}[\sup_{s\in[t,T]} |x^{\tilde{m}\tilde{n}}(s)|^2|\mathcal{F}_t] \\
&\quad\leqslant \varepsilon_0\varepsilon_1|x_t|^2 K\mathbb{E}\left[\int_t^T \left(\sum_{i=1}^d |B^{i^{\tilde{m}\tilde{n}}}| + \sum_{i=1}^k |D^{i^{\tilde{m}\tilde{n}}}|^2 + \sum_{i=1}^l\sum_{j=1}^\infty |F^{i^{\tilde{m}\tilde{n}}}|^2\right)(s)\mathrm{d}s|\mathcal{F}_t\right] \\
&\quad + \varepsilon_0\varepsilon_1|x_t|^2\mathbb{E}\left[\int_t^T \left(|A^{\tilde{m}\tilde{n}}| + \sum_{i=1}^k |C^{i^{\tilde{m}\tilde{n}}}|^2 + \sum_{i=1}^l\sum_{j=1}^\infty |E^{i^{\tilde{m}\tilde{n}}}|^2\right)(s)\mathrm{d}s|\mathcal{F}_t\right].
\end{aligned} \tag{3.21}$$

Proof. By the Itô–Lévy formula and Young's inequality,

$$\begin{aligned}
\mathbb{E}[|x^{\tilde{m}}(T)|^2|\mathcal{F}_t] &= |x(t)|^2 + \mathbb{E}\left[\int_t^T \left\{2\mathrm{Tr}(A^{\tilde{m}}x^{\tilde{m}}x^{\tilde{m}}(s^-)) + 2\sum_{i=1}^k \mathrm{Tr}(B^{i^{\tilde{m}}}ux^{\tilde{m}}(s^-))\right.\right. \\
&\quad + \sum_{i=1}^k \mathrm{Tr}(C^{i^{\tilde{m}}}x^{\tilde{m}}C^{i^{\tilde{m}}}x^{\tilde{m}}) + \sum_{i=1}^l\sum_{j=1}^\infty \mathrm{Tr}(E^{ij^{\tilde{m}}}x^{\tilde{m}}(s^-)E^{ij^{\tilde{m}}}x^{\tilde{m}}(s^-)) \\
&\quad + \sum_{i=1}^k \mathrm{Tr}(D^{i^{\tilde{m}}}uD^{i^{\tilde{m}}}u) + \sum_{i=1}^l\sum_{j=1}^\infty \mathrm{Tr}(F^{ij^{\tilde{m}}}uF^{ij^{\tilde{m}}}u) \\
&\quad \left.\left. + \sum_{i=1}^k \mathrm{Tr}(C^{i^{\tilde{m}}}x^{\tilde{m}}D^{i^{\tilde{m}}}u) + \sum_{i=1}^l\sum_{j=1}^\infty \mathrm{Tr}(E^{ij^{\tilde{m}}}x^{\tilde{m}}(s^-)F^{ij^{\tilde{m}}}u)\right\}(s)\mathrm{d}s|\mathcal{F}_t\right] \\
&\leqslant |x(t)|^2 + \varepsilon_0'\mathbb{E}[\sup_{s\in[t,T]}|x^{\tilde{m}}(s)|^2|\mathcal{F}_t] + \varepsilon_2'\mathbb{E}\left[\int_t^T |u(s)|^2\mathrm{d}s|\mathcal{F}_t\right].
\end{aligned}$$

Therefore (3.20) can be obtained by the Burkholder–Davis–Gundy inequalities and $u(s) \in U_{ad}$.

Similarly we have (3.21), since

$$\begin{aligned}
\mathbb{E}[|x^{\tilde{m}\tilde{n}}(T)|^2|\mathcal{F}_t] &\leqslant \varepsilon_0''\mathbb{E}[\sup_{s\in[t,T]}|x^{\tilde{m}\tilde{n}}(s)|^2|\mathcal{F}_t] \\
&\quad + \varepsilon_1''\mathbb{E}\left[\int_t^T \left(\sum_{i=1}^d |B^{i^{\tilde{m}\tilde{n}}}| + \sum_{i=1}^k |D^{i^{\tilde{m}\tilde{n}}}|^2 + \sum_{i=1}^l\sum_{j=1}^\infty |F^{i^{\tilde{m}\tilde{n}}}|^2\right)(s)|u(s)|^2\mathrm{d}s|\mathcal{F}_t\right] \\
&\quad + \varepsilon_2''\mathbb{E}\left[\int_t^T \left\{\left(|A^{\tilde{m}\tilde{n}}| + \sum_{i=1}^k |C^{i^{\tilde{m}\tilde{n}}}|^2\right)|x^{\tilde{m}\tilde{n}}|^2\right.\right. \\
&\quad \left.\left. + \sum_{i=1}^l\sum_{j=1}^\infty |E^{i^{\tilde{m}\tilde{n}}}|^2|x^{\tilde{m}\tilde{n}}(s^-)|^2\right\}(s)\mathrm{d}s|\mathcal{F}_t\right]. \quad \square
\end{aligned}$$

The following theorem is one of the main results in this paper and can be regarded as an extension of Theorem 2.1 in Kohlmann and Tang [10] in that the controlled processes additionally driven by the Lévy processes.

Theorem 3.1. *Assume that* H1–H5 *hold. Then there exists a triplet of processes* $(\Pi, \beta_\Pi, \gamma_\Pi)$ *with*

$$\Pi \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathcal{S}^n_+) \cap \mathbf{L}^\infty(\Omega,\mathcal{F}_T,\mathbb{P};\mathbf{C}([0,T];\mathcal{S}^n_+)),$$
$$\beta_\Pi \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathcal{S}^n)^k,$$
$$\gamma_\Pi \in \mathbf{l}^2_{\mathcal{F}}(0,T;\mathcal{S}^n)^l$$

such that $(\Pi^{\tilde m}, \beta_\Pi^{\tilde m}, \gamma_\Pi^{\tilde m})$ *strongly converges to* $(\Pi, \beta_\Pi, \gamma_\Pi)$.

Proof. Since $(x_t)^*\Pi^{\tilde m}(t)x_t = J^{\tilde m}(t,T;\hat u)$ by Proposition 3.1, using (3.19) with Lemma 3.3 we have

$$\begin{aligned}
&|(x_t)^*(\Pi^{\tilde m}(t) - \Pi^{\tilde n}(t))x_t| = |J^{\tilde m}(t,T;\hat u) - J^{\tilde n}(t,T;\hat u)|\\
&\quad\leqslant \epsilon_0\varepsilon_0 \operatorname*{esssup}_{\omega\in\Omega} |N^{\tilde m\tilde n}(t)||x_t|^2 + \{\epsilon_0(3\varepsilon_0|x_t|^2)^{\frac12} + \epsilon_0 T(4\varepsilon_0|x_t|^2)^{\frac12}\}\\
&\qquad\times\left\{\varepsilon_0\varepsilon_1|x_t|^2 K\mathbb{E}\left[\int_t^T\left(\sum_{i=1}^d |B^{i\,\tilde m\tilde n}| + \sum_{i=1}^k |D^{i\,\tilde m\tilde n}|^2 + \sum_{i=1}^l\sum_{j=1}^\infty |F^{i\,\tilde m\tilde n}|^2\right)(s)\mathrm{d}s|\mathcal{F}_t\right]\right.\\
&\qquad\left. +\ \varepsilon_0\varepsilon_1|x_t|^2\mathbb{E}\left[\int_t^T\left(|A^{\tilde m\tilde n}| + \sum_{i=1}^k |C^{i\,\tilde m\tilde n}|^2 + \sum_{i=1}^l\sum_{j=1}^\infty |E^{i\,\tilde m\tilde n}|^2\right)(s)\mathrm{d}s|\mathcal{F}_t\right]\right\}^{\frac12}\\
&\qquad + \epsilon_0\varepsilon_0|x_t|^2 \operatorname*{esssup}_{\omega\in\Omega}\int_t^T |M^{\tilde m\tilde n}(s)|\mathrm{d}s + \epsilon_1|x_t|^2 \operatorname*{esssup}_{s\in[t,T],\omega\in\Omega} |M^{\tilde m\tilde n}(s)|.
\end{aligned}$$

$\Pi^{\tilde m}(t)$ converges to $\Pi^\infty(t)$ strongly in $\mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathcal{S}^n_+) \cap \mathbf{L}^\infty(\Omega,\mathcal{F}_T,\mathbb{P};\mathbf{C}([0,T];\mathcal{S}^n_+))$, as $\tilde m \to \infty$.

By the Lipschitz condition, for $L > 0$ and $L' \geqslant (1+L) > 0$,

$$\begin{aligned}
&|-G^{\tilde m}(s,\Pi^{\tilde m},\beta_\Pi^{\tilde m},\gamma_\Pi^{\tilde m}) + G^l(s,\Pi^{\tilde n},\beta_\Pi^{\tilde n},\gamma_\Pi^{\tilde n})|\\
&\quad\leqslant L\left(|\Pi^{\tilde m} - \Pi^{\tilde n}|(s^-) + \sum_{i=1}^k |\beta_\Pi^{i\ \tilde m} - \beta_\Pi^{i\ \tilde n}|(s) + \sum_{i=1}^l\sum_{j=1}^\infty |\gamma_\Pi^{i\ \tilde m} - \gamma_\Pi^{i\ \tilde n}|(s)\right)\\
&\quad\leqslant L'\left(1 + |\Pi^{\tilde m} - \Pi^{\tilde n}|(s^-) + \sum_{i=1}^k |\beta_\Pi^{i\ \tilde m} - \beta_\Pi^{i\ \tilde n}|(s) + \sum_{i=1}^l\sum_{j=1}^\infty |\gamma_\Pi^{i\ \tilde m} - \gamma_\Pi^{i\ \tilde n}|(s)\right)^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&LL'^{-1}\mathbb{E}\left[\int_t^T\left(|\Pi^{\tilde m} - \Pi^{\tilde n}|(s^-) + \sum_{i=1}^k |\beta_\Pi^{i\ \tilde m} - \beta_\Pi^{i\ \tilde n}|(s)\right.\right.\\
&\qquad\left.\left. +\ \sum_{i=1}^l\sum_{j=1}^\infty |\gamma_\Pi^{i\ \tilde m} - \gamma_\Pi^{i\ \tilde n}|(s)\right)\mathrm{d}s|\mathcal{F}_t\right]
\end{aligned}$$

$$\leqslant \mathbb{E}\left[\int_t^T \left(1 + |\Pi^{\tilde{m}} - \Pi^{\tilde{n}}|(s^-) + \sum_{i=1}^{k} |{\beta_{\Pi}^{i}}^{\tilde{m}} - {\beta_{\Pi}^{i}}^{\tilde{n}}|(s) + \sum_{i=1}^{l}\sum_{j=1}^{\infty} |{\gamma_{\Pi}^{i}}^{\tilde{m}} - {\gamma_{\Pi}^{i}}^{\tilde{n}}|(s)\right)^2 \mathrm{d}s|\mathcal{F}_t\right]. \tag{3.22}$$

Using Theorem 3.2 in Bahlali et al. [1] and the Lipschitz condition, for $C, L' > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T \left(|\Pi^{\tilde{m}} - \Pi^{\tilde{n}}|^2(s^-) + \sum_{i=1}^{k}(|{\beta_{\Pi}^{i}}^{\tilde{m}}|^2 + |{\beta_{\Pi}^{i}}^{\tilde{n}}|^2 - 2c_1|{\beta_{\Pi}^{i}}^{\tilde{m}}||{\beta_{\Pi}^{i}}^{\tilde{n}}|)\right.\right.\\
&\quad \left.\left. + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(|{\gamma_{\Pi}^{i}}^{\tilde{m}}|^2 + |{\gamma_{\Pi}^{i}}^{\tilde{n}}(s)|^2 - 2c_2|{\gamma_{\Pi}^{i}}^{\tilde{m}}(s)||{\gamma_{\Pi}^{i}}^{\tilde{n}}|)(s)\right)\mathrm{d}s|\mathcal{F}_t\right]\\
&\leqslant \mathbb{E}\left[\int_t^T \left(|\Pi^{\tilde{m}} - \Pi^{\tilde{n}}|^2(s^-) + \sum_{i=1}^{k}|{\beta_{\Pi}^{i}}^{\tilde{m}} - {\beta_{\Pi}^{i}}^{\tilde{n}}|^2(s)\right.\right.\\
&\quad \left.\left. + \sum_{i=1}^{l}\sum_{j=1}^{\infty}|{\gamma_{\Pi}^{i}}^{\tilde{m}} - {\gamma_{\Pi}^{i}}^{\tilde{n}}|^2(s)\right)\mathrm{d}s|\mathcal{F}_t\right]\\
&\leqslant C\left[\mathbb{E}[|\Pi_t^{\tilde{m}} - \Pi_t^{\tilde{n}}|\mathcal{F}_t] + L'\mathbb{E}\left[\int_t^T \left(1 + |\Pi^{\tilde{m}} - \Pi^{\tilde{n}}|(s^-)\right.\right.\right.\\
&\quad \left.\left.\left. + \sum_{i=1}^{k}|{\beta_{\Pi}^{i}}^{\tilde{m}} - {\beta_{\Pi}^{i}}^{\tilde{n}}|(s) + \sum_{i=1}^{l}\sum_{j=1}^{\infty}|{\gamma_{\Pi}^{i}}^{\tilde{m}} - {\gamma_{\Pi}^{i}}^{\tilde{n}}|(s)\right)^2\mathrm{d}s|\mathcal{F}_t\right]\right],
\end{aligned}$$

where c_1 and c_2 are non-negative constants that satisfy

$$\begin{aligned}
&\frac{1}{2}\frac{\sum_{i=1}^{k}|{\beta_{\Pi}^{i}}^{\tilde{m}}|^2 + |{\beta_{\Pi}^{i}}^{\tilde{n}}|^2}{\sum_{i=1}^{k}|{\beta_{\Pi}^{i}}^{\tilde{m}}||{\beta_{\Pi}^{i}}^{\tilde{n}}|} > c_1\\
&\geqslant -\frac{1}{2}\frac{\sum_{i=1}^{k}|{\beta_{\Pi}^{i}}^{\tilde{m}} - {\beta_{\Pi}^{i}}^{\tilde{n}}|^2 - |{\beta_{\Pi}^{i}}^{\tilde{m}}|^2 - |{\beta_{\Pi}^{i}}^{\tilde{n}}|^2}{\sum_{i=1}^{k}|{\beta_{\Pi}^{i}}^{\tilde{m}}||{\beta_{\Pi}^{i}}^{\tilde{n}}|} \vee 0 \quad \text{a.s. a.e.}
\end{aligned}$$

and

$$\frac{1}{2}\frac{\sum_{i=1}^{l}\sum_{j=1}^{\infty}|{\gamma_{\Pi}^{i}}^{\tilde{m}}|^2 + |{\gamma_{\Pi}^{i}}^{\tilde{n}}|^2}{\sum_{i=1}^{l}\sum_{j=1}^{\infty}|{\gamma_{\Pi}^{i}}^{\tilde{m}}||{\gamma_{\Pi}^{i}}^{\tilde{n}}|} > c_2$$

$$\geqslant -\frac{1}{2}\frac{\sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma_{\Pi}^{i}{}^{\tilde{m}}-\gamma_{\Pi}^{i}{}^{\tilde{n}}|^2-|\gamma_{\Pi}^{i}{}^{\tilde{m}}|^2-|\gamma_{\Pi}^{i}{}^{\tilde{n}}|^2}{\sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma_{\Pi}^{i}{}^{\tilde{m}}||\gamma_{\Pi}^{i}{}^{\tilde{n}}|}\vee 0 \quad \text{a.s. a.e.}$$

Since $|\Pi_t^{\tilde{m}}-\Pi_t^{\tilde{n}}|^2\geqslant 0$,

$$\begin{aligned}
&C^{-1}L'^{-1}\mathbb{E}\Bigg[\int_t^T\Bigg(|\Pi^{\tilde{m}}-\Pi^{\tilde{n}}|^2(s^-)+\sum_{i=1}^{k}(|\beta_{\Pi}^{i}{}^{\tilde{m}}|^2+|\beta_{\Pi}^{i}{}^{\tilde{n}}|^2\\
&\quad -\ 2c_1|\beta_{\Pi}^{i}{}^{\tilde{m}}||\beta_{\Pi}^{i}{}^{\tilde{n}}|)(s)+\sum_{i=1}^{l}\sum_{j=1}^{\infty}(|\gamma_{\Pi}^{i}{}^{\tilde{m}}|^2+|\gamma_{\Pi}^{i}{}^{\tilde{n}}|^2-2c_2|\gamma_{\Pi}^{i}{}^{\tilde{m}}||\gamma_{\Pi}^{i}{}^{\tilde{n}}|)(s)\Bigg)\mathrm{d}s|\mathcal{F}_t\Bigg]\\
&\quad\leqslant L'^{-1}\mathbb{E}[|\Pi_t^{\tilde{m}}-\Pi_t^{\tilde{n}}|\mathcal{F}_t]+\mathbb{E}\Bigg[\int_t^T\Bigg(1+|\Pi^{\tilde{m}}-\Pi^{\tilde{n}}|(s^-)\\
&\quad +\ \sum_{i=1}^{k}|\beta_{\Pi}^{i}{}^{\tilde{m}}-\beta_{\Pi}^{i}{}^{\tilde{n}}|(s)+\sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma_{\Pi}^{i}{}^{\tilde{m}}-\gamma_{\Pi}^{i}{}^{\tilde{n}}|(s)\Bigg)^2\mathrm{d}s|\mathcal{F}_t\Bigg].
\end{aligned}\tag{3.23}$$

For sufficiently small $\varphi>0$, we can take a coefficient $\varrho>0$ such that

$$\begin{aligned}
0&\leqslant \varphi LL'^{-1}\mathbb{E}\Bigg[\int_t^T\Bigg(|\Pi^{\tilde{m}}-\Pi^{\tilde{n}}|(s^-)+\sum_{i=1}^{k}|\beta_{\Pi}^{i}{}^{\tilde{m}}-\beta_{\Pi}^{i}{}^{\tilde{n}}|(s)\\
&\quad +\ \sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma_{\Pi}^{i}{}^{\tilde{m}}-\gamma_{\Pi}^{i}{}^{\tilde{n}}|(s)\Bigg)\mathrm{d}s|\mathcal{F}_t\Bigg]\\
&\leqslant C^{-1}L'^{-1}\mathbb{E}\Bigg[\int_t^T\Bigg(|\Pi^{\tilde{m}}-\Pi^{\tilde{n}}|^2(s^-)+\sum_{i=1}^{k}(|\beta_{\Pi}^{i}{}^{\tilde{m}}|^2+|\beta_{\Pi}^{i}{}^{\tilde{n}}|^2\\
&\quad -2c_1|\beta_{\Pi}^{i}{}^{\tilde{m}}||\beta_{\Pi}^{i}{}^{\tilde{n}}|)(s)+\sum_{i=1}^{l}\sum_{j=1}^{\infty}(|\gamma_{\Pi}^{i}{}^{\tilde{m}}|^2+|\gamma_{\Pi}^{i}{}^{\tilde{m}}|^2\\
&\quad -\ 2c_2|\gamma_{\Pi}^{i}{}^{\tilde{m}}||\gamma_{\Pi}^{i}{}^{\tilde{n}}|)(s)\Bigg)\mathrm{d}s|\mathcal{F}_t\Bigg]-\varrho L'^{-1}\mathbb{E}[|\Pi_t^{\tilde{m}}-\Pi_t^{\tilde{n}}|\mathcal{F}_t]\\
&\leqslant \mathbb{E}\Bigg[\int_t^T\Bigg(1+|\Pi^{\tilde{m}}-\Pi^{\tilde{n}}|(s^-)+\sum_{i=1}^{k}|\beta_{\Pi}^{i}{}^{\tilde{m}}-\beta_{\Pi}^{i}{}^{\tilde{n}}|(s)\\
&\quad +\ \sum_{i=1}^{l}\sum_{j=1}^{\infty}|\gamma_{\Pi}^{i}{}^{\tilde{m}}-\gamma_{\Pi}^{i}{}^{\tilde{n}}|(s)\Bigg)^2\mathrm{d}s|\mathcal{F}_t\Bigg].
\end{aligned}\tag{3.24}$$

Since $2C^{-1}L'^{-1}\mathbb{E}[\int_t^T \sum_{i=1}^k c_1|\beta_\Pi^{i\ \tilde{m}}(s)||\beta_\Pi^{i\ \tilde{n}}(s)|\mathrm{d}s|\mathcal{F}_t] \geqslant 0$ and $2C^{-1}L'^{-1}\mathbb{E}[\int_t^T \sum_{i=1}^l \sum_{j=1}^\infty c_2|\gamma_\Pi^{i\ \tilde{m}}(s)||\gamma_\Pi^{i\ \tilde{n}}(s)|\mathrm{d}s|\mathcal{F}_t] \geqslant 0$, we have by subtracting (3.23) from (3.24)

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T \left(|\Pi^{\tilde{m}} - \Pi^{\tilde{n}}|(s^-) + \sum_{i=1}^k |\beta_\Pi^{i\ \tilde{m}} - \beta_\Pi^{i\ \tilde{n}}|(s) + \sum_{i=1}^l \sum_{j=1}^\infty |\gamma_\Pi^{i\ \tilde{m}} - \gamma_\Pi^{i\ \tilde{n}}|(s)\right) \mathrm{d}s|\mathcal{F}_t\right] \\
&\leqslant \varphi^{-1}C^{-1}L^{-1}\mathbb{E}\left[\int_t^T \left(|\Pi^{\tilde{m}} - \Pi^{\tilde{n}}|^2(s^-) + \sum_{i=1}^k (|\beta_\Pi^{i\ \tilde{m}}|^2 + |\beta_\Pi^{i\ \tilde{n}}|^2)(s)\right.\right. \\
&\quad \left.\left. + \sum_{i=1}^l \sum_{j=1}^\infty (|\gamma_\Pi^{i\ \tilde{m}}|^2 + |\gamma_\Pi^{i\ \tilde{n}}|^2)(s)\right) \mathrm{d}s|\mathcal{F}_t\right]. \qquad (3.25)
\end{aligned}$$

Applying the Itô–Lévy formula to $|\Pi^{\tilde{m}}(s) - \Pi^{\tilde{n}}(s)|^2$ and using the Lipschitz condition, (3.25) and Lemma 3.2 with $p = 1$, we have

$$\begin{aligned}
&\mathbb{E}[|\Pi^{\tilde{m}}(t) - \Pi^{\tilde{n}}(t)|^2|\mathcal{F}_t] + \mathbb{E}\left[\int_t^T \sum_{i=1}^k |\beta_\Pi^{i\ \tilde{m}} - \beta_\Pi^{i\ \tilde{n}}|^2(s)\mathrm{d}s|\mathcal{F}_t\right] \\
&\quad + \mathbb{E}\left[\int_t^T \sum_{i=1}^l \sum_{j=1}^\infty |\gamma_\Pi^{i\ \tilde{m}} - \gamma_\Pi^{i\ \tilde{n}}|^2(s)\mathrm{d}s|\mathcal{F}_t\right] \\
&\leqslant 2\mathbb{E}\left[\int_t^T \mathrm{Tr}((\Pi^{\tilde{m}} - \Pi^{\tilde{n}})(s^-)\right. \\
&\quad \left.\times (-G^{\tilde{m}}(s, \Pi^{\tilde{m}}, \beta_\Pi^{\tilde{m}}, \gamma_\Pi^{\tilde{m}}) + G^{\tilde{n}}(s, \Pi^{\tilde{n}}, \beta_\Pi^{\tilde{n}}, \gamma_\Pi^{\tilde{n}})))\mathrm{d}s|\mathcal{F}_t\right] \\
&\leqslant 2\varphi^{-1}C^{-1} \operatorname*{esssup}_{s,\omega} |\Pi^{\tilde{m}\tilde{n}}(t)|(\operatorname*{esssup}_{s,\omega} |\Pi^{\tilde{m}\tilde{n}}(s)|^2 + 2\epsilon(T - t)).
\end{aligned}$$

Thus, as $\tilde{m} \to \infty$, $\beta_\Pi^{\tilde{m}}$ and $\gamma_\Pi^{\tilde{m}}$ converge to β_Π^∞ and γ_Π^∞ strongly in $\mathcal{L}_\mathcal{F}^2(0, T; \mathcal{S}^n)^k$ and $\mathbf{l}_\mathcal{F}^2(0, T; \mathcal{S}^n)^l$, respectively. □

4. Singular case

In the previous section, we have shown the closeness property of the solution of the multi-dimensional regular BSRDE (1.1). This property implies that an $\mathcal{F}_t$-adapted unique limit solution $(\Pi, \beta_\Pi, \gamma_\Pi)$, which is the limit as $\tilde{m} \to \infty$ in the BSRDE (3.6), holds under H1–H5. Since our consideration in this section is the one-dimensional regular and singular BSRDE (1.1), here we assume that $n = 1$ in (1.1) and the corresponding controlled process has the same dimensions. We will prove the existence and uniqueness for this case by using the closeness property.

We consider a stochastic control problem P^1 for the singular case, i.e. $N = 0$, assuming

$$M(T) \geqslant \epsilon \qquad (4.1)$$

$$\sum_{i=1}^k (D^i(s))^* D^i(s) + \sum_{i=1}^l \sum_{j=1}^\infty (F^{ij}(s))^* F^{ij}(s) \geqslant \epsilon I_{m\times m} \quad \forall s \in [t, T], \qquad (4.2)$$

where ϵ is a positive constant. Before considering the singular case, let us introduce the *approximate regular* control problem;

$$\mathrm{P}_\alpha^3: \quad V_\alpha(t) := \operatorname*{esssup}_{u\in\mathcal{L}^2_{\mathcal{F}}(t,T;\mathbf{R}^m)} J^\alpha(t,T;u),$$

where

$$J^\alpha(t,T;u) = J(t,T;u) + \frac{1}{\alpha}\mathbb{E}\left[\int_t^T |u(s)|^2 \mathrm{d}s|\mathcal{F}_t\right],$$

$J(t,T;u)$ is (1.3) with the BSRDE (1.1) and $J^\alpha(t,T;u)$ is the cost function with the following *approximate regular* BSRDE $\Pi^\alpha(s)$:

$$\begin{cases} \mathrm{d}\Pi^\alpha(t) = -G^\alpha(t,\Pi^\alpha,\beta^\alpha_{\Pi^\alpha},\gamma^\alpha_{\Pi^\alpha})\mathrm{d}t \\ \quad +\sum_{i=1}^k {\beta^\alpha_{\Pi^\alpha}}^i(t)\mathrm{d}w^i(t) + \sum_{i=1}^l\sum_{j=1}^\infty {\gamma^\alpha_{\Pi^\alpha}}^{ij}(t)\mathrm{d}H^{ij}(t) \\ \Pi^\alpha(T) = M(T), \end{cases} \tag{4.3}$$

where

$$\begin{aligned} G^\alpha(t,\Pi^\alpha,\beta^\alpha_{\Pi^\alpha},\gamma^\alpha_{\Pi^\alpha}) = \Bigg(&\Pi^\alpha(t^-)A + A^*\Pi^\alpha(t^-) + M \\ &+ \sum_{i=1}^k {\beta^\alpha_{\Pi^\alpha}}^i C^i + \sum_{i=1}^l\sum_{j=1}^\infty {\gamma^\alpha_{\Pi^\alpha}}^i E^{ij} + \sum_{i=1}^k (C^i)^*({\beta^\alpha_{\Pi^\alpha}}^i + \Pi^\alpha(t^-)C^i) \\ &+ \sum_{i=1}^l\sum_{j=1}^\infty (E^{ij})^*({\gamma^\alpha_{\Pi^\alpha}}^{ij} + \Pi^\alpha(t^-)E^{ij})\Bigg)(t) - F^\alpha(t,\Pi^\alpha,\beta^\alpha_{\Pi^\alpha},\gamma^\alpha_{\Pi^\alpha}) \end{aligned}$$

and

$$\begin{aligned} F^\alpha(t,\Pi^\alpha,\beta^\alpha_{\Pi^\alpha},\gamma^\alpha_{\Pi^\alpha}) &= (L^\alpha)^*(K^\alpha)^{-1}L^\alpha \\ L^\alpha &:= \Bigg(\sum_{i=1}^d (B^i)^*\Pi^\alpha(t^-) + \sum_{i=1}^k (D^i)^*({\beta^\alpha_{\Pi^\alpha}}^i + \Pi^\alpha(t^-)(C^i)) \\ &\quad + \sum_{i=1}^l\sum_{j=1}^\infty (F^{ij})^*({\gamma^\alpha_{\Pi^\alpha}}^{ij} + \Pi^\alpha(t^-)E^{ij})\Bigg)(t) \\ K^\alpha &:= \left(\frac{1}{\alpha}I_{m\times m} + N + \sum_{i=1}^k (D^i)^*\Pi^\alpha(t^-)D^i + \sum_{i=1}^l\sum_{j=1}^\infty (F^{ij})^*\Pi^\alpha(t^-)F^{ij}\right)(t). \end{aligned}$$

Our objective, here, is to prove the following theorem.

Theorem 4.1 (*Singular Case*). *Assume that $N = 0$, $M(t) \geqslant 0$, and (4.1) and (4.2) hold. Then the BSRDE (4.3) with $N = 0, \alpha \to \infty$ has a unique $\mathcal{F}_t$-adapted solution with*

$$\begin{aligned} &\Pi \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathbf{R}_+) \cap \mathbf{L}^\infty(\Omega,\mathcal{F}_T,\mathbb{P};\mathbf{C}([0,T];\mathbf{R}_+)), \\ &\beta_\Pi \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbf{R})^k, \\ &\gamma_\Pi \in \mathbf{l}^2_{\mathcal{F}}(0,T;\mathbf{R})^l. \end{aligned}$$

If the approximate regular BSRDE (4.3) for each α has an $\mathcal{F}_t$-adapted unique solution for each α, there exists a unique solution to the BSRDE for the singular case by Theorem 3.1. Therefore, Theorem 4.1 holds when the one-dimensional regular BSRDE has an $\mathcal{F}_t$-adapted unique solution in required spaces of H5.

In Kohlmann and Tang [9], the one-dimensional regular BSRDE is studied. They use a technique developed by Kobylanski [7] to prove the existence of the solution to the regular BSRDE. Here, instead of the technique of [7], we use Theorem 3.1 to obtain the unique solution of the one-dimensional regular BSRDE with Lévy processes. It goes without saying that we can obtain the same result, i.e. the uniform convergence of $\Pi^{\tilde{m}}$, the strong convergence of $\beta_\Pi^{\tilde{m}}$ and that of $\gamma_\Pi^{\tilde{m}}$ for the one-dimensional regular BSRDE, independently of whether we use the approach based on Kobylanski or Theorem 3.1. However we will find that the closeness property is useful for proving existence of the solution to the one-dimensional BSRDE.

To this end, define an approximating BSRDE as

$$\begin{cases} \mathrm{d}\Pi^p(t) = -G_p(t, \Pi^p, \beta_\Pi^p, \gamma_\Pi^p)\mathrm{d}t + \sum_{i=1}^{k} \beta_\Pi^{p\,i} \mathrm{d}w^i(t) + \sum_{i=1}^{l}\sum_{j=1}^{\infty} \gamma_\Pi^{p\,ij}\mathrm{d}H^{ij}(t) \\ \Pi^p(T) = M(T) + \dfrac{1}{p+1}, \end{cases} \tag{4.4}$$

for $p \in \mathbf{N}$ where

$$\begin{aligned} G_p(t, \Pi, \beta_\Pi, \gamma_\Pi) = {} & \left\{\left(2A + \sum_{i=1}^{k}(C^i)^2 + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(E^{ij})^2\right)\Pi(t^-) + M \right.\\ & \left. + \ 2\sum_{i=1}^{k} C^i\beta_\Pi^i + 2\sum_{i=1}^{l}\sum_{j=1}^{\infty} E^{ij}\gamma_\Pi^{ij}\right\}(t) + F_p(t, \Pi, \beta_\Pi, \gamma_\Pi) \\ =: {} & \left(a\Pi(t^-) + M + \sum_{i=1}^{k} c^i\beta_\Pi^i + \sum_{i=1}^{l}\sum_{j=1}^{\infty} e^{ij}\gamma_\Pi^{ij}\right)(t) \\ & + F_p(t, \Pi, \beta_\Pi, \gamma_\Pi), \end{aligned}$$

and

$$\begin{aligned} F_p(t, \Pi, \beta_\Pi, \gamma_\Pi) := {} & \sup_{\tilde{\Pi}\in\mathbf{R}_+, \tilde{\beta_\Pi}\in(\mathbf{R})^k, \tilde{\gamma_\Pi}\in(\mathbf{R})^l} \left[- F(t, \tilde{\Pi}, \tilde{\beta_\Pi}, \tilde{\gamma_\Pi}) - p|\Pi - \tilde{\Pi}|(t^-) \right.\\ & \left. - \ p\left|\sum_{i=1}^{k}(\beta_\Pi^i - \tilde{\beta_\Pi^i})\right|(t) - p\left|\sum_{i=1}^{l}\sum_{j=1}^{\infty}(\gamma_\Pi^{ij} - \tilde{\gamma_\Pi^{ij}})\right|(t)\right]. \end{aligned}$$

Then, the following four assertions with respect to the BSRDE (4.4) with F_p are obtained:

(i) quadratic growth in $(\Pi, \beta_\Pi, \gamma_\Pi)$,
(ii) monotonicity in p,
(iii) the uniform Lipschitz property and
(iv) the strong convergences of $(\Pi^p, \beta_\Pi^p, \gamma_\Pi^p)$.

These assertions can be obtained by adaptation of the proof of Lepeltier and San Martin [15]. By (ii) monotonicity, $F_p(t, \Pi, \beta_\Pi, \gamma_\Pi)$ has the relation

$$0 = F_0 \geqslant F_1 \geqslant \cdots \geqslant F_p \geqslant F_{p+1} \geqslant \cdots \geqslant F, \quad F_p \downarrow F. \tag{4.5}$$

By the comparison theorem of Lemma 2.2 and (4.5), the solution of BSRDE (4.4) has the relation

$$\Pi_0 \geqslant \Pi_1 \geqslant \cdots \geqslant \Pi_p \geqslant \Pi_{p+1} \geqslant \cdots, \quad \text{a.s.a.e.} \tag{4.6}$$

Remark 4.1. In Kohlmann and Tang [9], existence of the solution to the one-dimensional regular BSRDE is proved after they show the positivity of Π^p for each $p \in \mathbf{N}$. Instead of using the procedure in Kohlmann and Tang [9] we prove the existence of the solution to the BSRDE with Lévy processes (4.4) by using Theorem 3.1. As to $F_p(t, \Pi, \beta_\Pi, \gamma_\Pi)$, when the positivity of $-F_p(t, \Pi, \beta_\Pi, \gamma_\Pi)$ for each p is satisfied, Theorem 3.1 holds.

Proposition 4.1 (*Regular Case*). *Assume that $n = 1$ in (1.1) and that $M(t) \geqslant 0, t \in [0, T]$, and $N(t) \geqslant \epsilon I_{m\times m}$ for some positive constant ϵ. Then, the one-dimensional BSRDE (1.1) has a unique $\mathcal{F}_t$-adapted solution $(\Pi, \beta_\Pi, \gamma_\Pi)$ with*

$$\begin{aligned}
&\Pi \in \mathcal{L}^\infty_{\mathcal{F}}(0, T; \mathbf{R}_+) \cap \mathbf{L}^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbf{C}([0, T]; \mathbf{R}_+)),\\
&\beta_\Pi \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbf{R})^k,\\
&\gamma_\Pi \in \mathbf{l}^2_{\mathcal{F}}(0, T; \mathbf{R})^l.
\end{aligned}$$

Proof. We prove the existence and uniqueness of the solution to the one-dimensional regular BSRDE with Lévy processes (1.1).

(i) Existence. Consider an approximating BSRDE (4.4). Since the approximating BSDE (4.4) with above assertions has an $\mathcal{F}_t$-adapted unique solution for each p by Lemma 2.1 in Section 2 obtained by Bahlali [1], we have the limit solution by Theorem 3.1 when $(\Pi^p, \beta^p_\Pi, \gamma^p_\Pi)$ is the unique solution in $(\mathcal{L}^\infty_{\mathcal{F}}(0, T; \mathbf{R}_+) \cap \mathbf{L}^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbf{C}([0, T]; \mathbf{R}_+)), \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbf{R})^k, \mathbf{l}^2_{\mathcal{F}}(0, T; \mathbf{R})^l)$. We can obtain that $(\Pi^p, \beta^p_\Pi, \gamma^p_\Pi)$ is the unique solution on the Banach space by Lemma 2.1, and therefore $(\beta^p_\Pi, \gamma^p_\Pi)$ is in $(\mathcal{L}^2_{\mathcal{F}}(0, T; \mathbf{R})^k, \mathbf{l}^2_{\mathcal{F}}(0, T; \mathbf{R})^l)$. Furthermore we already know that $\Pi^p(t)$ is in $\mathcal{L}^\infty_{\mathcal{F}}(0, T; \mathbf{R}_+) \cap \mathbf{L}^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbf{C}([0, T]; \mathbf{R}_+))$ by Lemma 3.3.

(ii) Uniqueness. Consider the one-dimensional regular BSRDE with Lévy processes (1.1). Assume that $(\Pi, \beta_\Pi, \gamma_\Pi)$ and $(\tilde{\Pi}, \tilde{\beta}_\Pi, \tilde{\gamma}_\Pi)$ are two $\mathcal{F}_t$-adapted solution triplets in $(\mathcal{L}^\infty_{\mathcal{F}}(0, T; \mathbf{R}_+) \cap \mathbf{L}^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbf{C}([0, T]; \mathbf{R}_+)), \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbf{R})^k, \mathbf{l}^2_{\mathcal{F}}(0, T; \mathbf{R})^l)$, respectively. Then, we have

$$\begin{cases}
\mathrm{d}\delta\Pi(t) = -\left(a\delta\Pi(t^-) + \displaystyle\sum_{i=1}^k c^i \delta\beta^i_\Pi + \sum_{i=1}^l \sum_{j=1}^\infty \mathrm{e}^{ij}\delta\gamma^{ij}_\Pi + \delta F\right)(t)\mathrm{d}t\\
\quad + \displaystyle\sum_{i=1}^k \delta\beta_\Pi{}^i(t)\mathrm{d}w^i(t) + \sum_{i=1}^l \sum_{j=1}^\infty \delta\gamma_\Pi{}^{ij}(t)\mathrm{d}H^{ij}(t)\\
\mathrm{d}\delta\Pi(T) = 0,
\end{cases}$$

where $a(t)$, $c^i(t)$ and $e^{ij}(t)$ are the same as in (4.4), $\delta\Pi := \Pi - \tilde{\Pi}$, $\delta\beta^i_\Pi := \beta^i_\Pi - \tilde{\beta}^i_\Pi$, $\delta\gamma_\Pi{}^{ij} := \gamma_\Pi{}^{ij} - \tilde{\gamma}^{ij}_{\tilde{\Pi}}$ and $\delta F := F(t, \Pi, \beta_\Pi, \gamma_\Pi) - F(t, \tilde{\Pi}, \tilde{\beta}_{\tilde{\Pi}}, \tilde{\gamma}_{\tilde{\Pi}})$.

By Proposition 3.1, $x(s)^*\Pi(s)x(s) = x(s)^*\tilde{\Pi}(s)x(s)$ (i.e. $\Pi(s) = \tilde{\Pi}(s)$) a.s. for all $(s, x) \in [0, T] \times \mathbf{R}$.

Applying the Itô–Lévy formula to $|\delta\Pi(t)|^2$ with $\delta\Pi(s) = 0\ \forall s \in [0, T]$, we have

$$\mathbb{E}|\delta\Pi(t)|^2 + \mathbb{E}\int_0^T \left(\sum_{i=1}^k |\delta\beta_\Pi{}^i|^2 + \sum_{i=1}^l \sum_{j=1}^\infty |\delta\gamma_\Pi{}^{ij}|^2\right)(s)\mathrm{d}s$$
$$= 2\mathbb{E}\int_0^T |\delta\Pi(s^-)| \left(a\delta\Pi(s^-) + \sum_{i=1}^k c^i\delta\beta_\Pi{}^i + \sum_{i=1}^l \sum_{j=1}^\infty e^{ij}\delta\gamma_\Pi{}^{ij} + \delta F\right)(s)\mathrm{d}s$$
$$= 0.$$

Hence $\beta_\Pi = \tilde{\beta}_{\tilde{\Pi}}$ and $\gamma_\Pi = \tilde{\gamma}_{\tilde{\Pi}}$. □

Now we prove the existence and uniqueness for the one-dimensional singular case.

Proof (*Theorem* 4.1). Consider the BSRDE (4.3) with $N = 0$. Assume (4.1) and (4.2). By Proposition 4.1, there exists an $\mathcal{F}_t$-adapted unique solution $(\Pi^\alpha, \beta_\Pi^\alpha, \gamma_\Pi^\alpha)$ for each α. Therefore existence of the solution to the BSRDE (4.3) is proved by Theorem 3.1. Uniqueness of the solution can be obtained by the same procedure as in Proposition 4.1. □

5. Application to an elaborated LQR problem

5.1. Non-homogeneous stochastic optimal control

Consider the following cost function:

$$J(0, T; u) = \mathbb{E}\left[\int_0^T (x^*Mx + u^*Nu + 2\phi^*x + 2\psi^*u)(t)\mathrm{d}t + x(T)^*M(T)x(T) + 2\phi(T)^*x(T)\right], \tag{5.1}$$

and let the controlled process $x(t)$ be a solution of the stochastic process

$$\begin{cases} \mathrm{d}x(t) = \left(Ax + \sum_{i=1}^d B^i u + f\right)(t)\mathrm{d}t + \sum_{i=1}^k (C^i x + D^i u + \Phi^i)(t)\mathrm{d}w^i(t) \\ \quad + \sum_{i=1}^l \sum_{j=1}^\infty (E^{ij}(t)x(t^-) + F^{ij}(t)u(t) + \Psi^{ij}(t))\mathrm{d}H^{ij}(t) \\ x(0) = x_0. \end{cases} \tag{5.2}$$

We first introduce the BSRDE (1.1) and the following BSDE:

$$\begin{cases} \mathrm{d}r(t) = \Xi(t)\mathrm{d}t + \sum_{i=1}^k \beta_r^i(t)\mathrm{d}w^i(t) + \sum_{i=1}^l \sum_{j=1}^\infty \gamma_r^{ij}(t)\mathrm{d}H^{ij}(t) \\ r(T) = \phi(T), \end{cases} \tag{5.3}$$

where

$$\Xi(t) = -\left[\phi + A^*r(t^-) + \Pi(t^-)f + \sum_{i=1}^k \beta_\Pi^i \Phi^i + \sum_{i=1}^l \sum_{j=1}^\infty \gamma_\Pi^{ij}\Psi^{ij}\right.$$

$$
\begin{aligned}
&+ \sum_{i=1}^{k}(C^i)^*(\beta_r^i + \Pi(t^-)\Phi^i) + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(E^{ij})^*(\gamma_r^{ij} + \Pi(t^-)\Psi^{ij}) \\
&- \Bigg(\Pi(t^-)\sum_{i=1}^{d} B^i + \sum_{i=1}^{k}(\beta_\Pi^i + (C^i)^*\Pi(t^-))D^i \\
&+ \sum_{i=1}^{l}\sum_{j=1}^{\infty}(\gamma_\Pi^{ij} + (E^{ij})^*\Pi(t^-))F^{ij}\Bigg) \\
&\times \left(N + \sum_{i=1}^{k}(D^i)^*\Pi(t^-)D^i + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*\Pi(t^-)F^{ij}\right)^{-1} \\
&\times \Bigg(\psi + \sum_{i=1}^{d}(B^i)^* r(t^-) + \sum_{i=1}^{k}(D^i)^*(\beta_r^i + \Pi(t^-)\Phi^i(t)) \\
&+ \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*(\gamma_r^{ij} + \Pi(t^-)\Psi^{ij})\Bigg)\Bigg](t).
\end{aligned}
$$

Theorem 5.1. *If Eqs.* (1.1) *and* (5.3) *admit the unique solutions* $(\Pi, \beta_\Pi, \gamma_\Pi)$ *and* (r, β_r, γ_r), *then the optimal control problem has an optimal feedback control in* $\mathcal{L}^2_{\mathcal{F}}(0, T; \mathbf{R}^m)$*:*

$$
\begin{aligned}
\hat{u}(t) = &- \left(N + \sum_{i=1}^{k}(D^i)^*\Pi(t^-)D^i + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*\Pi(t^-)F^{ij}\right)^{-1}(t) \\
&\times \Bigg\{\Bigg(\sum_{i=1}^{d}(B^i)^*\Pi(t^-) + \sum_{i=1}^{k}(D^i)^*(\beta_\Pi^i + \Pi(t^-)C^i) \\
&+ \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*(\gamma_\Pi^{ij} + \Pi(t^-)E^{ij})\Bigg)\hat{x}(t^-) \\
&+ \Bigg(\psi + \sum_{i=1}^{d}(B^i)^* r(t^-) + \sum_{i=1}^{k}(D^i)^*(\beta_r^i + \Pi(t^-)\Phi^i) \\
&+ \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*(\gamma_r^{ij} + \Pi(t^-)\Psi^i)\Bigg)\Bigg\}(t).
\end{aligned} \tag{5.4}
$$

Proof. Having used *completion of squares* technique in order to obtain the optimal control of Proposition 3.1 in Section 3, we consider here the *non-homogeneous* case by means of the procedure in [3] with some modification. Though this procedure takes more time than the completion of squares technique to obtain an optimal control, the BSRDE (1.1) and the BSDE (5.3) can be derived explicitly.

Let the value function be (5.1) and the controlled process be (5.2). We denote two control processes as follows:

$$v(t) := \Lambda(t)x(t) + \lambda(t) + \mu(t), \tag{5.5}$$

$$u(t) := \Lambda(t)x(t) + \lambda(t), \tag{5.6}$$

where $v, u \in \mathbf{R}^n$, $\Lambda \in \mathbf{R}^{n\times n}$ and $\mu \in \mathbf{R}^n$. Denote the controlled processes with these control processes by x and y which are the solutions of the following stochastic differential equation:

$$\begin{cases} \mathrm{d}x(t) = \left\{\left(A + \sum_{i=1}^{d} B^i \Lambda\right) x + \sum_{i=1}^{d} B^i(\lambda + \mu) + f\right\}(t)\mathrm{d}t \\ \quad + \sum_{i=1}^{k}\left\{(C^i + D^i\Lambda)x + D^i(\lambda+\mu) + \Phi^i\right\}(t)\mathrm{d}w^i(t) \\ \quad + \sum_{i=1}^{l}\sum_{j=1}^{\infty}\left\{(E^{ij} + F^{ij}\Lambda(t))x(t^-) + F^{ij}(\lambda(t)+\mu(t)) + \Psi^i\right\}(t)\mathrm{d}H^{ij}(t) \\ x(0) = x, \end{cases}$$

$$\begin{cases} \mathrm{d}y(t) = \left\{\left(A + \sum_{i=1}^{d} B^i \Lambda\right) y + \sum_{i=1}^{d} B^i\lambda + f\right\}(t)\mathrm{d}t \\ \quad + \sum_{i=1}^{k}\left\{(C^i + D^i\Lambda)y + D^i\lambda + \Phi^i\right\}(t)\mathrm{d}w^i(t) \\ \quad + \sum_{i=1}^{l}\sum_{j=1}^{\infty}\left\{(E^{ij} + F^{ij}\Lambda(t))y(t^-) + F^{ij}\lambda(t) + \Psi^i\right\}(t)\mathrm{d}H^{ij}(t) \\ y(0) = x. \end{cases}$$

Defining $\bar{x}(t) := x(t) - y(t)$, then

$$\begin{cases} \mathrm{d}\bar{x}(t) = \left\{\left(A + \sum_{i=1}^{d} B^i \Lambda\right) \bar{x} + \sum_{i=1}^{d} B^i\mu\right\}(t)\mathrm{d}t \\ \quad + \sum_{i=1}^{k}\left\{(C^i + D^i\Lambda)\bar{x} + D^i\mu\right\}(t)\mathrm{d}w^i(t) \\ \quad + \sum_{i=1}^{l}\sum_{j=1}^{\infty}\left\{(E^{ij} + F^{ij}\Lambda(t))\bar{x}(t^-) + F^{ij}\mu(t)\right\}(t)\mathrm{d}H^{ij}(t) \\ \bar{x}(0) = 0. \end{cases}$$

Define a stochastic process $p(t) := \Pi(t)y(t) + r(t)$ where $p(t) \in \mathbf{R}^n$, $\Pi(t) \in \mathbf{R}^{n\times n}$ and $r(t) \in \mathbf{R}^n$. Assume that $\Pi(t)$ and $r(t)$ are the solutions of the following stochastic differential equations:

$$\mathrm{d}\Pi(t) = \dot{\Pi}(t)\mathrm{d}t + \sum_{i=1}^{k}\beta_{\Pi}^i(t)\mathrm{d}w^i(t) + \sum_{i=1}^{l}\sum_{j=1}^{\infty}\gamma_{\Pi}^{ij}\mathrm{d}H^{ij}(t), \tag{5.7}$$

$$\mathrm{d}r(t) = \dot{r}(t)\mathrm{d}t + \sum_{i=1}^{k}\beta_{r}^i(t)\mathrm{d}w^i(t) + \sum_{i=1}^{l}\sum_{j=1}^{\infty}\gamma_{r}^{ij}\mathrm{d}H^{ij}(t), \tag{5.8}$$

where $\dot{\Pi}$ and $\dot{r}$ are the time derivatives of Π and r, respectively. It should be noted that these equations correspond to the BSRDE (1.1) and the BSDE (5.3), respectively.

Substituting (5.5) and (5.6) into (5.1), we have

$$\begin{aligned} J(0,T;v) = J(0,T;u) &+ \mathbb{E}\int_0^T ((\bar{x})^* M x + (\Lambda\bar{x} + \mu)^* N(\Lambda\bar{x}+\mu))(s)\mathrm{d}s \\ &+ \mathbb{E}\bar{x}(T)^* M(T)\bar{x}(T) + \text{term}, \end{aligned}$$

since $\mathbb{E}[(\bar{x}(T))^* \Pi(T) y(T) + (\bar{x}(T))^* \phi(T)] = \mathbb{E}[(\bar{x}(T))^* p(T)]$, and

$$\begin{aligned}
\text{term} := {} & 2\mathbb{E}\left[\int_0^T \left\{(\bar{x})^* M y + (\Lambda\bar{x} + \mu)^* N(s)(\Lambda y + \lambda) + \phi^* \bar{x} \psi^* (\Lambda \bar{x} + \mu)\right\}(s)\mathrm{d}s\right] \\
& + 2\mathbb{E}[(\bar{x}(T))^* M(T) y(T) + (\bar{x}(T))^* \phi(T)] \\
= {} & 2\mathbb{E}\left[\int_0^T \left\{(\bar{x})^* M y + (\Lambda\bar{x} + \mu)^* N(\Lambda y + \lambda)\right.\right. \\
& \left.\left. + \ \phi^* \bar{x} \psi^* (\Lambda \bar{x} + \mu)\right\}(s)\mathrm{d}s\right] + 2\mathbb{E}\left[\int_0^T d((\bar{x}(s))^* p(s))\right],
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}((\bar{x}(t))^* p(t)) = {} & \left[(\bar{x})^* \left(A^* + \sum_{i=1}^d \Lambda^* (B^i)^*\right) p(t^-) + \sum_{i=1}^d \mu^* (B^i)^* p(t^-)\right. \\
& + \bar{x}^* (\dot{\Pi} y(t^-) + \dot{r}) + \bar{x}^* \Pi(t^-) \left\{\left(A + \sum_{i=1}^d B^i \Lambda\right) y + \sum_{i=1}^d B^i \lambda + f\right\} \\
& + (\bar{x}(t^-))^* \sum_{i=1}^k \beta_\Pi^i \{(C^i + D^i \Lambda) y + D^i \lambda + \Phi^{ij}\} \\
& + (\bar{x}(t^-))^* \sum_{i=1}^l \sum_{j=1}^\infty \gamma_\Pi^{ij} \{(E^{ij} + F^{ij} \Lambda) y(t^-) + F^{ij} \lambda + \Psi^{ij}\} \\
& + \sum_{i=1}^k [\bar{x}^* ((C^i)^* + \Lambda^* (D^i)^*)(\beta_\Pi^i y(t^-) + \beta_r^i) \\
& + \mu^* (D^i)^* (\beta_\Pi^i y(t^-) + \beta_r^i) + \bar{x}^* ((C^i)^* + \Lambda^* (D^i)^*) \Pi(t^-) \\
& \times \{(C^i + D^i \Lambda) y + D^i \lambda + \Phi^i\} \\
& + \mu^* (D^i) \Pi(t^-) \{(C^i + D^i \Lambda) y + D^i \lambda + \Phi^i\}] \\
& + \sum_{i=1}^l \sum_{j=1}^\infty [(\bar{x}(t^-))^* ((E^{ij})^* + \Lambda^* (F^{ij})^*)(\gamma_\Pi^{ij} y(t^-) + \gamma_r^{ij}) \\
& + \mu^* (F^{ij})^* (\gamma_\Pi^{ij} y(t^-) + \gamma_r^{ij}) + (\bar{x}(t^-))^* ((E^{ij})^* + \Lambda^* (F^{ij})^*) \Pi(t^-) \\
& \times \{((E^{ij})^* + (F^{ij}) \Lambda) y(t^-) + F^{ij} \lambda + \Phi^{ij}\} \\
& \left. + \ \mu^* F^{ij} \Pi(t^-) \{(E^{ij} + F^{ij} \Lambda) y(t^-) + F^{ij} \lambda + \Psi^{ij}\}]\right](t)\mathrm{d}t \\
& + \sum_{i=1}^k \{\cdots\} \mathrm{d}w^i(t) + \sum_{i=1}^l \sum_{j=1}^\infty \{\cdots\} \mathrm{d}H^{ij}(t).
\end{aligned}$$

With term $= 0$, the cost function $J(0, T; v)$ should be minimized. Therefore, to satisfy that term $= 0$, the drift term of term is manipulated as follows:

$$\bar{x}(t^-)^* \{(\text{term1}) y(t^-) + (\text{term2})\} + \mu(t)^* \{(\text{term3}) y(t^-) + \text{term4}\},$$

where

$$\begin{aligned}
\text{term1} \ =\ & \Bigg\{ M + \Lambda^* N \Lambda + \left(A^* + \Lambda^* \sum_{i=1}^{d} (B^i)^* \right) \Pi(t^-) \\
& + \dot{\Pi} + \Pi(t^-) \left(A + \sum_{i=1}^{d} B^i \Lambda \right) + \sum_{i=1}^{k} \beta^i (C^i + D^i \Lambda) \\
& + \sum_{i=1}^{k} ((C^i)^* + \Lambda^* (D^i)^*) \{\beta^i_{\Pi} + \Pi(t^-)(C^i + D^i \Lambda)\} \\
& + \sum_{i=1}^{l} \sum_{j=1}^{\infty} \gamma^{ij}_{\Pi} (E^{ij} + F^{ij} \Lambda) + \sum_{i=1}^{l} \sum_{j=1}^{\infty} ((E^{ij})^* + \Lambda^* (F^{ij})^*) \\
& \times \ \{\gamma^{ij}_{\Pi} + \Pi(t^-)(E^{ij} + F^{ij} \Lambda)\} \Bigg\} (t) = 0, \qquad (5.9)
\end{aligned}$$

$$\begin{aligned}
\text{term2} \ =\ & \Bigg\{ \Lambda^* N \lambda + \phi + \Lambda^* \psi + \left(A^* + \Lambda^* \sum_{i=1}^{d} (B^i)^* \right) r(t^-) + \dot{r} \\
& + \Pi(t^-) \left(\sum_{i=1}^{d} B^i \lambda + f \right) + \sum_{i=1}^{k} \beta^i (D^i \lambda + \Phi^i) + \sum_{i=1}^{l} \sum_{j=1}^{\infty} \gamma^{ij}_{\Pi} (F^{ij} \lambda + \Psi^{ij}) \\
& + \sum_{i=1}^{k} ((C^i)^* + \Lambda^* (D^i)^*) \{\beta^i_r + \Pi(t^-)(D^i \lambda + \Phi^i)\} \\
& + \sum_{i=1}^{l} \sum_{j=1}^{\infty} ((E^{ij})^* + \Lambda^* (F^{ij})^*) \times \{\gamma^{ij}_r + \Pi(t^-)(F^{ij} \lambda + \Psi^{ij})\} \Bigg\} (t) = 0, \qquad (5.10)
\end{aligned}$$

$$\begin{aligned}
\text{term3} \ =\ & \Bigg\{ N \Lambda + \sum_{i=1}^{d} (B^i)^* \Pi(t^-) + \sum_{i=1}^{k} (D^i)^* \{\beta^i_{\Pi} + \Pi(t^-)(C^i + D^i \Lambda)\} \\
& + \sum_{i=1}^{l} \sum_{j=1}^{\infty} (F^{ij})^* \{\gamma^{ij}_{\Pi} + \Pi(t^-)(E^{ij} + F^{ij} \Lambda)\} \Bigg\} (t) = 0, \qquad (5.11)
\end{aligned}$$

$$\begin{aligned}
\text{term4} \ =\ & \Bigg\{ N \lambda + \psi + \sum_{i=1}^{d} (B^i)^* r(t^-) + \sum_{i=1}^{k} (D^i)^* \{\beta^i_r + \Pi(t^-)(D^i \lambda + \Phi^i)\} \\
& + \sum_{i=1}^{l} \sum_{j=1}^{\infty} (F^{ij})^* \{\gamma^{ij}_r + \Pi(t^-)(F^{ij} \lambda + \Psi^{ij})\} \Bigg\} (t) = 0. \qquad (5.12)
\end{aligned}$$

From (5.11) and (5.12), we have

$$\begin{aligned}
\Lambda(t) \ =\ & - \left(N + \sum_{i=1}^{k} (D^i)^* \Pi(t^-) D^i + \sum_{i=1}^{l} \sum_{j=1}^{\infty} (F^{ij})^* \Pi(t^-) F^{ij} \right)^{-1} (t) \\
& \times \left(\sum_{i=1}^{d} (B^i)^* \Pi(t^-) + \sum_{i=1}^{k} (D^i)^* (\beta^i_r + \Pi(t^-) C^i) \right. \\
& \left. + \sum_{i=1}^{l} \sum_{j=1}^{\infty} (F^{ij})^* (\gamma^{ij}_{\Pi} + \Pi(t^-) E^{ij}) \right) (t),
\end{aligned}$$

$$\lambda(t) = -\left(N + \sum_{i=1}^{k}(D^i)^*\Pi(t^-)D^i + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*\Pi(t^-)E^{ij}\right)^{-1}(t) \times \left(\psi + \sum_{i=1}^{d}(B^i)^*r(t^-) + \sum_{i=1}^{k}(D^i)^*(\beta_r^i + \Pi(t^-)\Phi^i) + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*(\gamma_r^{ij} + \Pi(t^-)\Psi^{ij})\right)(t).$$

Substituting $\Lambda(t)/\lambda(t)$ into (5.9) and (5.10), $\dot{\Pi}(t)/\dot{r}(t)$ can be derived. From (5.7) and (5.8) with $\dot{\Pi}(t)$ and $\dot{r}(t)$, the BSRDE (5.7) and the associated BSDE (5.8) identify with (1.1) and (5.3), respectively. Thus, assuming that the BSRDE (1.1) and the associated BSDE (5.3) hold, then we have the optimal control (5.4) on substituting $\Lambda(t)$ and $\lambda(t)$ into (5.6). □

Now we consider the one-dimensional non-homogeneous case with partial observation. A partial observation problem is that of finding an optimal control for which the controller has less information than the full information $\mathcal{F}_t$. In particular, the partial observation problem is useful for constructing an economic model in which there are information gaps among economic agents, e.g. Øksendal [18], Kohlmann and Xiong [8].

Assume that the control process u is $\mathcal{H}_t$-adapted where

$$\mathcal{H}_t \subseteq \mathcal{F}_t \quad \text{for all } t \in [0, T],$$

and the admissible set for this control is denoted by $U_{\mathcal{H}} = \mathcal{L}^2_{\mathcal{H}}(t, T; \mathbf{R})$.

Then we have the following theorem for the optimal feedback control for the partial observation case.

Theorem 5.2. *Consider the cost function (5.1) and the controlled process (5.2) with $n = 1$. Assume that* H1–H5 *hold. If Eqs. (1.1) and (5.3) admit the unique solutions $(\Pi, \beta_\Pi, \gamma_\Pi)$ and (r, β_r, γ_r), then the optimal feedback control in $U_{\mathcal{H}}$ for the partial observation case is given by*

$$\hat{u}(t) = -\mathbb{E}\left[\left(N + \sum_{i=1}^{k}(D^i)^*\Pi(t^-)D^i + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*\Pi(t^-)E^{ij}\right)(t)|\mathcal{H}_t\right]^{-1} \times \mathbb{E}\left[\left\{\left(\sum_{i=1}^{d}B^i\Pi(t^-) + \sum_{i=1}^{k}(D^i)^*(\beta_\Pi^i + \Pi(t^-)C^i) + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*(\gamma_\Pi^{ij} + \Pi(t^-)E^{ij})\right)\hat{x}(t^-) + \left(\psi + \sum_{i=1}^{d}(B^i)^*r(t^-) + \sum_{i=1}^{k}(D^i)^*(\beta_r^i + \Pi(t^-)\Phi^i) + \sum_{i=1}^{l}\sum_{j=1}^{\infty}(F^{ij})^*(\gamma_r^{ij} + \Pi(t^-)\Psi^i)\right)\right\}(t)|\mathcal{H}_t\right]. \tag{5.13}$$

Proof. We can obtain the feedback control for the partial observation case by following the proof of Theorem 3.1 in [6], which studies a linear–quadratic problem with jump diffusions including the partial observation case.

Since (1.1) and (5.3) admit unique solutions, the cost function (5.1) becomes

$$J(0,T;u) \geqslant \mathbb{E}\left[\int_0^T K(t)(u(t)+K(t)^{-1}R(t))^2 \mathrm{d}t + \Pi(0)x_0^2 + 2r(0)x_0\right]$$
$$= \int_0^T \tilde{\mathbb{E}}[(u(t)+K(t)^{-1}R(t))^2]\mathrm{d}t + \Pi(0)x_0^2 + 2r(0)x_0$$
$$= \int_0^T \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[(u(t)+K(t)^{-1}R(t))^2|\mathcal{H}_t]]\mathrm{d}t + \Pi(0)x_0^2 + 2r(0)x_0,$$

where

$$K(t) = \left(N + \sum_{i=1}^k (D^i)^* \Pi(t^-) D^i + \sum_{i=1}^l \sum_{j=1}^\infty (F^{ij})^* \Pi(t^-) E^{ij}\right)(t)$$
$$R(t) = \left\{\left(\sum_{i=1}^d B^i \Pi(t^-) + \sum_{i=1}^k (D^i)^*(\beta_\Pi^i + \Pi(t^-)C^i) + \sum_{i=1}^l \sum_{j=1}^\infty (F^{ij})^*(\gamma_\Pi^{ij} + \Pi(t^-)E^{ij})\right)\hat{x}(t^-) + \left(\psi + \sum_{i=1}^d (B^i)^* r(t^-) + \sum_{i=1}^k (D^i)^*(\beta_r^i + \Pi(t^-)\Phi^i) + \sum_{i=1}^l \sum_{j=1}^\infty (F^{ij})^*(\gamma_r^{ij} + \Pi(t^-)\Psi^i)\right)\right\}(t)$$

and

$$\left.\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{1}{K(t)}.$$

When $\hat{u}(t) = -\tilde{\mathbb{E}}[K(t)^{-1}R(t)|\mathcal{H}_t]$, we can minimize the value of $\tilde{\mathbb{E}}[(K(t)(u(t)+R(t)))^2|\mathcal{H}_t]$. Therefore, by the Kallianpur–Striebel formula (e.g. Lemma 4.1.2 of [3])

$$\hat{u}(t) = -\tilde{\mathbb{E}}[K(t)^{-1}R(t)|\mathcal{H}_t] = -\frac{\mathbb{E}[R(t)|\mathcal{H}_t]}{\mathbb{E}[K(t)|\mathcal{H}_t]}.$$

Thus we have the optimal feedback for the partial observation case (5.13). □

5.2. *Application to a financial problem*

An application of the stochastic control (BSDE approach) to a financial problem pricing a contingent claim, which is a target random variable at maturity, is studied in Yong and Zhou [25], Kohlmann and Zhou [13] and so on. The approach in Kohlmann and Zhou [13] is to find an optimal control of the following stochastic control problem:

$$V(t,\hat{u}) = \min_u J(y,u) = \min_u \frac{1}{2}\mathbb{E}[|y(T)|^2],$$

where $y(t) = x(t) - \mathbb{E}[\xi|\mathcal{F}_t]$ is the controlled process of the difference between the portfolio process $x(t)$ and the contingent claim ξ taking expectation with the martingale representation.

The optimal control $\hat{u} \in \mathbf{R}^m$ for this problem is

$$\hat{u}_j(t) = -P(t)^{-1}B_j(t)^*[P(t)y(t)^* + \phi(t)] - P(t)^{-1}\beta_j(t) + z_j(t), \quad j = 1,\ldots,m,$$

where $(P(t), \phi(t), \beta(t)) \in \mathbf{C}([0, T]; \hat{\mathcal{S}}^n_+) \times \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbf{R}^n) \times \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbf{R}^{n\times m})$ is a solution of the *Stochastic* Riccati Equation (SRE) and the associated BSDE, which are subclasses of BSRDE (1.1) and (5.3), respectively. We rewrite this optimal control with the *approximate price* $p(t) := -P^{-1}(t)\phi(t)$,

$$\hat{u}_j(t) = -P(t)^{-1} B_j(t)^* P(t)[x(t)^* - p(t)] + q_j(t), \quad j = 1, \dots, m,$$

where $(p(t), q(t))$ is a solution pair for the following BSDE:

$$\begin{cases} \mathrm{d}p(t) = -\left[A(t)p(t) - \sum_{j=1}^{m} B_j(t)q_j(t) + f(t) \right] \mathrm{d}t + \sum_{j=1}^{m} q_j(t)\mathrm{d}W^j(t), \\ p(T) = \xi. \end{cases}$$

The advantage of this BSDE approach is that the option price can be obtained by the solution of the BSDE $p(t)$ with the optimal hedging strategy $\hat{u}$. Furthermore, from the financial point of view (in the *Black–Scholes model*), the optimal hedging strategy consists of two components: The *replicating portfolio* for the claim ξ and the *Merton portfolio* for the terminal utility function $u(x) = x^2$ (see Remark 5.2 in Kohlmann and Zhou [13]).

Our objective is to implement the BSDE approach with Lévy processes. Let $x(t) \in \mathbf{R}$ be a solution of the stochastic differential equation (5.2). We need the martingale representation and the unique solution of the BSDE for Lévy processes. Bahlali et al. [1] prove the representation theorem and unique solution for the more general Teugel's martingales. The martingale representation for Teugel's martingales is based on the chaos decomposition and is obtained in Nualart and Schoutens [17]. Moreover, Løkka [14] similarly obtained the Clark–Ocone formula by means of the chaos expansion (decomposition). Further, the unique solution for Teugel's martingales is proved in Nualart and Schoutens [16]. More generally, the unique solution for the BSDE decomposed explicitly into Brownian motion and Teugel's martingales is obtained in Bahlali et al. [1].

For our purpose, we introduce a financial market with one risk-free asset $P^0(t)$ and m risky assets $P^i(t)$, $i = 1, \dots, m$, whose prices at time $t \in [0, T]$ are given by the following SDEs:

$$\mathrm{d}P^0(t) = P^0(t) rf(t)\mathrm{d}t$$

$$\mathrm{d}P^i(t) = P^i(t^-)\mu^i(t)\mathrm{d}t + \sum_{j=1}^{m} P^i(t^-)\sigma^{ij}(t)\mathrm{d}w^j + \sum_{j=1}^{m}\sum_{k=1}^{\infty} P^i(t^-)\gamma^{ijk}(t)\mathrm{d}H^{jk}(t),$$

where $rf(t)$ is the risk-free rate, $\mu^i(t)$ is the rate of return on i-th asset and $w^j(t)$ and $H^{jk}(t)$ are one-dimensional Wiener process and Teugel's martingale on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$, and $\sigma^{ij}(t)$, $\gamma^{ijk}(t)$ are in $\mathbf{R}$. Assume that there exists a positive constant ϵ such that

$$\sum_{j=1}^{m} \sigma^j(t)\sigma^j(t)^* + \sum_{j=1}^{m}\sum_{k=1}^{\infty} \gamma^{jk}(t)\gamma^{jk}(t)^* =: \sigma(t)\sigma(t)^* + \sum_{k=1}^{\infty} \gamma^k(t)\gamma^k(t)^* \geqslant \epsilon I_{m\times m},$$

where $\sigma(t), \gamma^j(t) \in \mathbf{R}^{m\times m}$. Let ξ be a random variable in $\mathbf{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and have decomposition by the martingale representation theorem in Bahlali et al. [1]. Let $N^i(t)$ be i-th asset shares at t. Then by using $P^i(t)$, $i = 0, 1, \dots, m$, the portfolio process $x(t)$ is given by

$$\mathrm{d}x(t) = \sum_{i=0}^{m} N^i(t)\mathrm{d}P^i(t)$$

$$
\begin{aligned}
&= \left\{(rf)x + \sum_{i=1}^{m}(\mu^i - rf)N^i P^i(t^-)\right\}(t)\mathrm{d}t \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{m}(\sigma^{ij} N^i P^i(t^-))(t)\mathrm{d}w^j(t) \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{\infty}(\gamma^{ijk} N^i P^i(t^-))(t)\mathrm{d}H^{jk}(t) \\
&=: ((rf)x + \tilde{\mu}^*\pi)(t)\mathrm{d}t + \sum_{j=1}^{m}((\sigma^j)^*\pi)(t)\mathrm{d}w^j(t) \\
&\quad + \sum_{j=1}^{m}\sum_{k=1}^{\infty}\gamma^{jk}(t)^*\pi(t)\mathrm{d}H^{jk}(t),
\end{aligned} \tag{5.14}
$$

where $\tilde{\mu}(t) = (\mu^1(t) - rf(t), \ldots, \mu^m(t) - rf(t))^*$ and $\pi(t) = (N^1(t)P^1(t^-), \ldots, N^m(t)P^m(t^-))^*$.

Since the optimal control problem is to minimize the expected value of the difference between $x(T)$ and ξ, the control problem is given by

$$
\mathrm{P}^4: \quad V(0,\hat{u}) := \min_{u \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbf{R}^m)} \mathbb{E}|x(T) - \xi|^2. \tag{5.15}
$$

P^4 corresponds to the case where in the cost function (1.3), $M(t) = 0$, $0 \leqslant t < T$, $M(T) = 1$ are assumed. We make the following assumption.

Assumption 1. Consider the controlled process (5.2). Assume that in (5.2), $d = 1$, $k = m$, $l = m$, the coefficients are as follows:

$$
\begin{aligned}
&A(t) = rf(t), && B(t) = \tilde{\mu}(t)^*, && f(t) = 0, \\
&C^i(t) = 0, && D^i(t) = \sigma^i(t)^*, && \Phi^i(t) = 0, \\
&E^{ij}(t) = 0, && F^{ij}(t) = \gamma^{ij}(t)^*, && \Psi^{ij}(t) = 0, \\
&u(t) = \pi(t), && &&
\end{aligned}
$$

and $N(t) = 0$ for all t, $M(t) = 0$ for $0 \leqslant t < T$, $M(T) = 1$, where $N(t)$, $M(t)$ are in the cost function.

Under Assumption 1, the BSRDE (1.1) and the associated BSDE (5.3) become

$$
\begin{aligned}
\mathrm{d}\Pi(t) &= -(2\Pi(t^-)rf(t) - F(t,\Pi,\beta_\Pi,\gamma_\Pi))\mathrm{d}t \\
&\quad + \sum_{i=1}^{m}\beta^i_\Pi(t)\mathrm{d}w^i(t) + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma^{ij}_\Pi(t)\mathrm{d}H^{ij}(t),
\end{aligned} \tag{5.16}
$$

where

$$
F(t,\Pi,\beta_\Pi,\gamma_\Pi) = \left(\Pi(t^-)\tilde{\mu}^* + \sum_{i=1}^{m}\beta^i_\Pi\sigma^i + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma^{ij}_\Pi\gamma^{ij}\right)(t)
$$

$$\times \left(\sum_{i=1}^{m}(\sigma^i)^*\sigma^i + \sum_{i=1}^{m}\sum_{j=1}^{\infty}(\gamma^{ij})^*\gamma^{ij}\right)^{-1}(t)$$
$$\times \left(\tilde{\mu}\Pi(t^-) + \sum_{i=1}^{m}(\sigma^i)^*\beta_\Pi^i + \sum_{i=1}^{m}\sum_{j=1}^{\infty}(\gamma^{ij})^*\gamma_\Pi^{ij}\right)(t),$$

and

$$\begin{aligned} \mathrm{d}r(t) = &-\Bigg[\phi + (rf)r(t^-) - \left(\Pi(t^-)\tilde{\mu} + \sum_{i=1}^{m}\beta_\Pi^i(\sigma^i)^* + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma_\Pi^{ij}(\gamma^{ij})^*\right) \\ &\times \left(\sum_{i=1}^{m}\sigma^i\Pi(t^-)(\sigma^i)^* + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma^{ij}\Pi(t^-)(\gamma^{ij})^*\right)^{-1} \\ &\times \left(\tilde{\mu}r(t^-) + \sum_{i=1}^{m}\sigma^i\beta_r^i + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma^{ij}\gamma_r^{ij}\right)\Bigg](t)\mathrm{d}t \\ &+ \sum_{i=1}^{m}\beta_r^i(t)\mathrm{d}w^i(t) + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma_r^{ij}(t)\mathrm{d}H^{ij}(t). \end{aligned} \tag{5.17}$$

Theorem 5.3. *Let us consider a financial market as mentioned. Let us make Assumption 1, i.e. the controlled process (the portfolio process) is (5.14), the BSRDE is (5.16) and the associated BSDE is (5.17). Then the optimal control is given by*

$$\begin{aligned} \hat{\pi}(t) = &-\left(\sigma\sigma^* + \sum_{j=1}^{\infty}\gamma^j(\gamma^j)^*\right)^{-1}(t) \times \Bigg\{\left(\tilde{\mu} + \sum_{i=1}^{m}\sigma^i\beta_\Pi^i + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma^{ij}\gamma_\Pi^{ij}\right)\hat{x}(t^-) \\ &+ \Pi(t^-)^{-1}\left(\tilde{\mu}r(t^-) + \sum_{i=1}^{m}\sigma^i\beta_r^i + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma^{ij}\gamma_r^{ij}\right)\Bigg\}(t), \end{aligned} \tag{5.18}$$

and the approximate price process $p(t)$ is given by

$$\begin{aligned} \mathrm{d}p(t) = &-\Bigg\{\Pi^{-2}(rf)(2\Pi(t^-)r(t^-) + \Pi(t^-)^{-1}) \\ &- \Pi(t^-)^{-1}(r(t^-)(rf) + \phi) - \Pi(t)^{-2}\left(\sum_{i=1}^{m}\beta_\Pi^i\beta_r^i + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma_\Pi^{ij}\gamma_r^{ij}\right) \\ &- \vartheta_1\vartheta_2^{-1}\Bigg((\Pi^{-2}(rf) - \Pi(t^-)^{-2}r(t^-))\tilde{\mu} \\ &+ \Pi(t^-)^{-1}\sum_{i=1}^{m}\sigma^i\theta_1^i + \Pi(t^-)^{-1}\sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma^{ij}\theta_2^{ij}\Bigg)\Bigg\}(t)\mathrm{d}t \\ &+ \sum_{i=1}^{m}\theta_1(t)^i\mathrm{d}w^i(t) + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\theta_2(t)^{ij}\mathrm{d}H^{ij}(t) \end{aligned} \tag{5.19}$$

$$p(T) = \xi,$$

where

$$\vartheta_1(t) = \Pi(t^-)\tilde{\mu}(t) + \sum_{i=1}^{m} \beta_\Pi^i(t)\sigma^i(t)^* + \sum_{i=1}^{m}\sum_{j=1}^{\infty} \gamma_\Pi^{ij}(t)\gamma^{ij}(t)^*,$$

$$\vartheta_2(t) = \sum_{i=1}^{m} \sigma^i(t)\sigma^i(t)^* + \sum_{i=1}^{m}\sum_{j=1}^{\infty} \gamma^{ij}(t)\gamma^{ij}(t)^*,$$

$$\theta_1^i(t) = \Pi(t)^{-2}\beta_\Pi^i(t) - \Pi(t^-)^{-1}\beta_r^i(t),$$

$$\theta_2^{ij}(t) = \Pi(t)^{-2}\gamma_\Pi^{ij}(t) - \Pi(t^-)^{-1}\gamma_r^{ij}(t).$$

Proof. Since this control problem corresponds to the singular case, $(\Pi, \beta_\Pi, \gamma_\Pi)$ follows a unique triplet of the $\mathcal{F}_t$-adapted solution from Theorem 4.1. By Theorem 5.1, an optimal feedback control of this control problem is given by (5.18).

The approximate price $p(t)$ is $-\Pi(t)^{-1}r(t)$. Therefore, the dynamics of $p(t)$ can be written as

$$\mathrm{d}p(t) = \Pi(t)^{-2}r(t^-)\mathrm{d}\Pi(t) - \Pi(t^-)^{-1}\mathrm{d}r(t) + \Pi(t)^{-2}\mathrm{d}[\Pi, r](t).$$

Thus by substituting (5.16) and (5.17) into this equation, the approximate price becomes (5.19). □

Remark 5.1. The optimal control $\hat{u}(t)$ (5.18) consists of two hedging portfolios. To see this relationship, let us use the decomposition $\hat{\pi} =: \pi_0 + \pi_1$ where

$$\begin{aligned}\pi_0 &= -\left(\sigma\sigma^* + \sum_{j=1}^{\infty}\gamma^j(\gamma^j)^*\right)^{-1}(t) \\ &\quad\times\left(\tilde{\mu} + \sum_{i=1}^{m}\sigma^i\beta_\Pi^i + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma^{ij}\gamma_\Pi^{ij}\right)(t)(\hat{x} - p)(t^-) \\ \pi_1 &= -\left(\sigma\sigma^* + \sum_{j=1}^{\infty}\gamma^j(\gamma^j)^*\right)^{-1}(t)\Pi(t^-)^{-1} \\ &\quad\times\left\{\sum_{i=1}^{m}\sigma^i(\beta_\Pi^i - \beta_r^i r(t^-)) + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma^{ij}(\gamma_\Pi^{ij} - \gamma_r^{ij}r(t^-))\right\}(t).\end{aligned}$$

As we follow the interpretation of these hedging portfolios in Kohlmann and Tang [11], Kohlmann and Tang [9], π_1 is a generalized Merton type portfolio and π_0 is the perfect hedging portfolio for the contingent claim ξ. Each portfolio in our case takes jump risks into account.

Finally, we give the optimal control of the portfolio for the partial observation case. Let us make assumptions in Theorem 5.3, provided that the control process u is $\mathcal{H}_t$-adapted. Then the BSRDE for the singular case with $n = 1$ admits the unique solution by Theorem 4.1. By Theorem 5.2, we have the optimal feedback control for the partial observation case:

$$\hat{\pi}(t) = -\mathbb{E}\left[\left(\sigma\sigma^* + \sum_{j=1}^{\infty}\gamma^j(\gamma^j)^*\right)(t)\,\middle|\,\mathcal{H}_t\right]^{-1}$$

$$\times \mathbb{E}\left[\left\{\left(\tilde{\mu} + \sum_{i=1}^{m} \sigma^i \beta_{\Pi}^i + \sum_{i=1}^{m}\sum_{j=1}^{\infty} \gamma^{ij}\gamma_{\Pi}^{ij}\right)\hat{x}(t^-)\right.\right.$$
$$\left.\left. +\ \Pi(t^-)^{-1}\left(\tilde{\mu} r(t^-) + \sum_{i=1}^{m}\sigma^i\beta_r^i + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\gamma^{ij}\gamma_r^{ij}\right)\right\}(t)\,\middle|\,\mathcal{H}_t\right].$$

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