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When every finitely generated flat module is projective

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Abstract

We investigate the class of rings over which every finitely generated flat right module is projective.

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1. Introduction

A classical theorem of Bass states that every flat right module over a ring R is projective if and only if R is left perfect. It seems natural to ask, when, more generally, every finitely generated flat right module over R is projective. We refer to rings with this property as *right S-rings*, since the answer to this question was first given by Sakhajev. His results date back to the 70s (cf. [15]). The first proof in English, however, appeared only recently in Facchini, Herbera, and Sakhajev [6].

Examples of right S-rings are right noetherian rings, since over such rings every finitely generated right module is finitely presented and, over any ring, every finitely presented flat module is projective. It follows from another result of Bass, that every semiperfect ring is a right and left S-ring.

A crucial theorem in [6] says that a ring R is a right S-ring if and only if every sequence $A_1, A_2, \ldots$ of $n \times n$ matrices over R, such that $A_{i+1}A_i = A_i$ for every i, eventually consists

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of idempotents generating the same principal right ideal in the matrix ring R_n. We say the sequence *converges* in this case.

Using this characterization we refresh old and prove new results on right S-rings. For instance, the class of right S-rings is closed under Morita equivalence, under finite direct products, and under subrings. It follows from the latter that every right Ore domain (in fact, any right nonsingular ring of finite right Goldie dimension) is a right and left S-ring, and so is any free associative algebra over a field. Nevertheless there are domains that are neither right nor left S-rings. See Section 3 for all this.

From [6] it follows that we may assign to each sequence $A_1, A_2, \dots$ as above a projective right module P such that this sequence converges if and only if P is finitely generated. Using this we prove that the S-property can be lifted modulo any ideal contained in the prime radical. As a consequence, every ring with right Krull dimension is a right and left S-ring. Further, a triangular matrix ring R is a right S-ring if and only if each diagonal component of R is a right S-ring. See Propositions 5.8 and 5.9.

The most powerful reduction from matrices to elements is due to Vasconcelos [17]. We reformulate his result as follows: a commutative ring R is an S-ring if and only if every sequence $a_1, a_2, \dots$ of elements (as opposed to matrices) of R with $a_{i+1}a_i = a_i$ converges to an idempotent. Using this we prove, in Section 7, that every commutative ring of Goldie dimension one is an S-ring.

Endo [3] proved that a commutative ring is an S-ring if its localization with respect to the set of nonzero divisors is a semilocal ring, and verified the converse in some particular cases. We give an example showing that this converse is not true in general, see Example 7.8 below.

The main question that remains open is the symmetry of the concept of S-ring: is every right S-ring a left S-ring? (Cf. Question 3.9 below.) We give an affirmative answer in the cases of exchange rings, semihereditary rings, and semilocal rings, see Propositions 4.9, 4.10, and 6.4, respectively.

We thank Dolors Herbera for acquainting us with [6], to which our work—though largely independent—is tightly related. We found some overlap in the next, introductory, section unavoidable but do believe that our paper may serve as useful complementary reading. Last but not least, we owe thanks to the referee for his patience and a number of useful comments improving the presentation of the paper.

2. *a*-sequences

Let R be an associative ring with 1. A sequence $\langle a\rangle = \langle a_1, a_2, \dots\rangle$ of elements of R is said to be a *right a-sequence* if $a_{i+1}a_i = a_i$ for every $i = 1, 2 \dots$. A trivial instance of this is obtained when $e = e^2 \in R$ is an idempotent: then $\langle \bar{e}\rangle = \langle e, e, \dots\rangle$ is a right a-sequence. In particular, $\langle \bar{0}\rangle$ and $\langle \bar{1}\rangle$ are right a-sequences.

We say that two a-sequences $\langle a\rangle$ and $\langle b\rangle$ are *equivalent*, written $\langle a\rangle \sim \langle b\rangle$, if $a_i = b_i$ for all but finitely many i.

We collect some basic properties of a-sequences.

Lemma 2.1. *Let $\langle a\rangle$ be a right a-sequence.*

(1) *If a_i is right invertible, then $a_k = 1$ for every $k > i$; in particular $\langle a\rangle \sim \langle\bar{1}\rangle$.*
(2) *If $1 - a_i$ is left invertible, then $a_k = 0$ for every $k < i$. In particular this is the case if a_i is nilpotent, or $a_i \in \mathrm{Jac}(R)$.*

Proof. (1) Let $a_i b = 1$ for some $b \in R$. Multiplying $a_{i+1}a_i = a_i$ by b on the right, we obtain $a_{i+1} = 1$. Then $a_{i+2}a_{i+1} = a_{i+1}$ yields $a_{i+2} = 1$, and the assertion follows by induction.

(2) Writing $a_i a_{i-1} = a_{i-1}$ as $(1 - a_i)a_{i-1} = 0$ we conclude that $a_{i-1} = 0$. Then $a_{i-2} = a_{i-1}a_{i-2} = 0$, and the first part of the assertion follows by induction. For the second it remains to notice that if a_i is nilpotent, or $a_i \in \mathrm{Jac}(R)$, then $1 - a_i$ is invertible. □

Over domains or local rings a-sequences have a very simple form.

Lemma 2.2. *Let $\langle a\rangle \neq \langle\bar{0}\rangle$ be a right a-sequence over a ring R.*

(1) *If R is a domain, then $\langle a\rangle$ is of the form $\langle 0, \dots, 0, r, 1, 1, \dots\rangle$, where $0 \neq r \in R$;*
(2) *If R is local, then $\langle a\rangle$ is of the form $\langle 0, \dots, 0, r, s, 1, 1, \dots\rangle$, where $0 \neq r \in R$, and $sr = r$.*

Clearly any such sequence $\langle a\rangle$ is a right a-sequence.

Proof. (1) Let $r = a_i$ be the first nonzero element of $\langle a\rangle$. We rewrite the equality $a_{i+1}a_i = a_i$ as $(1 - a_{i+1})a_i = 0$, i.e., $(1 - a_{i+1})r = 0$. Since R is a domain, and $r \neq 0$, it follows that $a_{i+1} = 1$. But then by Lemma 2.1, $a_k = 1$ for every $k > i$.

(2) As above we have $(1 - a_{i+1})r = 0$. If $a_{i+1} \in \mathrm{Jac}(R)$, then $1 - a_{i+1}$ is invertible, hence $r = 0$, a contradiction. Otherwise, since R is local, $a_{i+1} = s$ is invertible, and $sr = r$. By Lemma 2.1 we obtain $a_k = 1$ for every $k > i + 1$. □

Next we show that every right a-sequence leads to an **a**scending chain of right ideals of the ring (whence the notation 'a').

Lemma 2.3. *Let $\langle a\rangle$ be a right a-sequence over a ring R. Then*

(1) *$a_k a_i = a_i$ for every $k > i$;*
(2) *$a_1R \subseteq a_2R \subseteq \cdots$ is an ascending chain of right ideals of R;*
(3) *if $a_k \in a_iR$ for some $k > i$, i.e., $a_iR = a_kR$, then a_k is an idempotent;*
(4) *if $e \in R$ is a central idempotent, then $\langle a\rangle e = \langle a_1e, a_2e, \dots\rangle$ is a right a-sequence.*

Proof. (1) (cf. [6, proof of Lemma 3.1]) By induction on $k - i \geqslant 1$. The initial step $k - i = 1$, i.e., $k = i + 1$, follows from the definition.

Now let $k - i > 1$. By induction hypothesis $a_k a_{i+1} = a_{i+1}$. Then

$$a_k a_i = a_k(a_{i+1}a_i) = (a_k a_{i+1})a_i = a_{i+1}a_i = a_i.$$

(2) readily follows from $a_{i+1}a_i = a_i$.

(3) (cf. [6, Lemma 3.1]) Let $a_k = a_i g$ for $k > i$ and some $g \in R$. Multiplying by a_i on the right we obtain $a_k a_i = a_i g a_i$. But $a_k a_i = a_i$ by (1), hence $a_i = a_i g a_i$. Thus $a_i g = a_k$ is an idempotent.

(4) Since e is central, $a_{i+1}e \cdot a_i e = a_{i+1} a_i e = a_i e$. □

We say that a right a-sequence *converges* (to the right ideal $a_k R$) if the corresponding ascending chain of right ideals of R stabilizes at $a_k R$.

The following is obvious and well known.

Remark 2.4. Let e, f be idempotents of a ring R. Then $eR \subseteq fR$ if and only if $fe = e$. Therefore $eR = fR$ if and only if $fe = e$ and $ef = f$.

Next we show that every convergent right a-sequence eventually consists of idempotents.

Lemma 2.5. *A right a-sequence $\langle a\rangle$ converges if and only if there is an index k such that $a_i = e_i$ is idempotent for every $i > k$ and $e_i \cdot e_j = e_j$ for all $j > i > k$.*

Proof. Both directions follow from Remark 2.4. For the less obvious one, let $a_k R = a_{k+1} R = \cdots$. By Lemma 2.3, every $a_i = e_i$ is an idempotent. Now $e_i e_j = e_j$ $(j > i > k)$ by Remark 2.4. □

More can be said in the commutative case: every convergent a-sequence is eventually constant.

Lemma 2.6. *Let $\langle a\rangle$ be a convergent right a-sequence over a ring R all of whose idempotents are central. Then $\langle a\rangle \sim \langle \bar{e}\rangle$ for some idempotent $e \in R$.*

Proof. By Lemma 2.5, there is a k such that every $a_i = e_i$, $i > k$, is an idempotent, and $e_i e_j = e_j$ for all $j > i > k$. Further, by the definition of a-sequence, $e_j e_i = e_i$. Then $e_i = e_j e_i = e_i e_j = e_j$ for all $i, j > k$. □

Now we dualize the notion of right a-sequence. A sequence $\langle b\rangle = \langle b_1, b_2, \ldots\rangle$ of ring elements is said to be a *left d-sequence* if $b_{i+1} b_i = b_{i+1}$, $i = 1, 2, \ldots$.

We collect the properties corresponding to those of Lemmas 2.1 and 2.3 in a lemma, whose proof we omit, since it is dual to the ones above.

Lemma 2.7. *Let $\langle b\rangle$ be a left d-sequence over a ring R.*

(1) *If b_i is left invertible, then $b_k = 1$ for every $k < i$.*
(2) *If $1 - b_i$ is right invertible, then $b_k = 0$ for every $k > i$; in particular $\langle b\rangle \sim \langle \bar{0}\rangle$. This is the case, for instance, when b_i is nilpotent or $b_i \in \mathrm{Jac}(R)$.*
(3) *$b_k b_i = b_k$ for every $k > i$.*
(4) *$Rb_1 \supseteq Rb_2 \supseteq \cdots$ is a **d**escending chain of left ideals of R.*
(5) *If $b_i \in Rb_k$ for $k > i$, i.e., $Rb_i = Rb_k$, then b_i is an idempotent.*
(6) *If $e \in R$ is a central idempotent, then $\langle b\rangle e = \langle b_1 e, b_2 e, \ldots\rangle$ is a left d-sequence.*

In particular, any left d-sequence leads to a **d**escending chain of *left* ideals, and we can dually define *convergence* of such a sequences by demanding that this chain stabilize. The following lemma then corresponds to Lemmas 2.5 and 2.6, and we again omit the proof.

Lemma 2.8. *Let $\langle b\rangle$ be a left d-sequence over a ring R.*

(1) *$\langle b\rangle$ converges if and only if there exists an index k such that $b_i = f_i$ is an idempotent for every $k > i$, and $f_i f_j = f_i$ for all $j > i > k$.*
(2) *If $\langle b\rangle$ converges and all idempotents of R are central, then $\langle b\rangle \sim \langle \bar{e}\rangle$ for some idempotent $e \in R$.*

The following exhibits a useful connection between a-sequences and d-sequences.

Lemma 2.9. *$\langle a\rangle$ is a right a-sequence if and only if $\langle 1 - a_i\rangle$ is a left d-sequence.*

Proof. Let $b_i = 1 - a_i$. If $\langle a\rangle$ is a right a-sequence, then

$$b_{i+1}b_i = (1 - a_{i+1})(1 - a_i) = 1 - a_{i+1} - a_i + a_{i+1}a_i = 1 - a_{i+1} = b_{i+1}.$$

Thus $\langle b\rangle$ is a left d-sequence. The proof of the converse is similar. □

Note that any idempotent $e \in R$ gives rise to a right a-sequence $\langle \bar{e}\rangle$ and a left d-sequence $\langle \overline{1 - e}\rangle$.

Lemma 2.10. *A right a-sequence $\langle a\rangle$ converges if and only if the left d-sequence $\langle 1 - a_i\rangle$ does.*

Proof. Suppose that $\langle a\rangle$ converges. By Lemma 2.5 we may assume that each $a_i = e_i$ is an idempotent such that $e_i e_j = e_j$ and $e_j e_i = e_i$ for all $i < j$. Then $1 - e_i = f_i$ is an idempotent. If $i < j$ then

$$f_i f_j = (1 - e_i)(1 - e_j) = 1 - e_i - e_j + e_i e_j = 1 - e_i = f_i,$$

and also

$$f_j f_i = (1 - e_j)(1 - e_i) = 1 - e_j - e_i + e_j e_i = 1 - e_j = f_j.$$

$R(1 - a_i) = R(1 - a_j)$ follows.

The converse is dual and left to the reader. □

To conclude this section we state some results that connect the behavior of these sequences with projectivity—the original topic of interest.

Fact 2.11 [6, Lemma 3.1]. *Let $\langle a\rangle$ be a right a-sequence over a ring R. Then the right ideal $P_{\langle a\rangle} = \sum_{i=1}^{\infty} a_i R$ is a projective right R-module. Further, $\langle a\rangle$ converges if and only if $P_{\langle a\rangle}$ is finitely generated (hence generated by an idempotent a_k).*

The following result can be easily derived from [6, proof of Proposition 3.5].

Fact 2.12. *Let R be a ring. Then the following are equivalent.*

(1) *Every cyclic flat right R-module is projective.*
(2) *Every right a-sequence over R converges.*
(3) *Every left d-sequence over R converges.*

We call a ring R a *right S-ring*, if every finitely generated flat right R-module is projective. The corresponding matrix version of the previous result characterizes such rings.

Fact 2.13 [6, Proposition 3.5]. *Let R be a ring. Then the following are equivalent.*

(1) *R is an S-ring.*
(2) *For each n, every right a-sequence over the ring R_n (of $n \times n$ matrices over R) converges.*
(3) *For each n, every left d-sequence over R_n converges.*

3. Examples

First we prove that the class of right S-rings is closed under taking subrings, which yields a rich supply of examples.

Lemma 3.1. *Let R be a subring of a ring T (where the units of R and T need not be the same). If T is a right S-ring, then R is a right S-ring.*

Proof. If the units of R and T are the same, we may use the following: if M is a finitely generated flat R-module such that $M \otimes_R T$ is a projective T-module, then M_R is projective. But, even in this case, it is instructive to see a proof using the above criterion.

By Fact 2.13, it suffices to prove that every right $\langle a\rangle$-sequence over R_n converges. Since T is a right S-ring, $\langle a\rangle$ converges over T_n. By Lemma 2.10, we may assume that every $a_i = e_i$ is an idempotent such that $e_i e_j = e_j$ and $e_j e_i = e_i$ holds for all $j > i$. But then $e_i R_n = e_j R_n$ for all i, j, hence $\langle a\rangle$ converges over R_n. □

Example 3.2. Since the free algebra $A = \Bbbk\langle X\rangle$ over a field $\Bbbk$, where X is a set of non-commuting variables, is embeddable in a skew field, A is a left and right S-ring.

From Fact 2.12 and Lemma 2.2 it follows that every cyclic flat module over a domain is projective. We can do better if the domain is also (one-sided) Ore, since such domains are embedded in a skew field (which obviously is an S-ring).

Example 3.3. Every right Ore domain R is a right and left S-ring.

In fact, we can extend this to a wider class of rings.

Example 3.4. Let R be a right nonsingular ring of finite right Goldie dimension. Then R is a right and left S-ring.

Proof. Since R is right nonsingular, it is embedded in its right maximal quotient ring Q. Since R is of finite right Goldie dimension, Q is a semisimple artinian ring by [11, 13.4]. Thus R is a right and left S-ring by Lemma 3.1. □

Next we see that the Ore condition cannot be entirely dropped in the above.

Example 3.5. Let $\Bbbk$ be a field, and let R be the (noncommutative) $\Bbbk$-algebra with generators $x, y, u, v, x', y', u', v'$ and the relation

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix} \cdot \begin{pmatrix} x' & y' \\ u' & v' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then R is neither a right nor a left S-ring.

Proof. Shepherdson [16] proved that R is a domain which is not stably finite (see also [11, §1.1, Exercise 18]). By Corollary 4.8 below, R is neither a right nor a left S-ring. □

The next example was suggested to us by D. Herbera.

Example 3.6. There is a domain R that is, though a left and right S-ring, not embeddable in a skew field (and hence not Ore).

Proof. By [5, Example 5.7] there is a hereditary semilocal domain R which is embeddable in a simple artinian ring R' (of length 2), but not in a skew field. Since R' is a left and right S-ring, R is a left and right S-ring by Lemma 3.1. □

For the following, note that semiperfectness is a left-right symmetric property of rings generalizing that of (one-sided) perfectness.

Example 3.7 (*Bass, see also* [11, §4, Exercise 21]). Every semiperfect ring R is a right and left S-ring.

Proof (*with Ivo Herzog*). Let M be a finitely generated flat module over R. By semiperfectness, M has a projective cover $c : P \to M$ (cf. [10, Proposition 24.12]). Then the kernel K is a pure small submodule of the projective module P. The assertion will follow once we show $K = 0$. For this we may as well assume (by adding on an appropriate direct summand) that P is free, which allows us to use [11, Theorem 4.23] as follows. Given any $k \in K$, there is an endomorphism f of P fixing k whose image is in K. Then $k \in \ker(1 - f)$. Since $c(1 - f) = c$, as is easily verified, properties of the projective cover (cf. [10, Proposition 24.10]) force the endomorphism $(1 - f)$ to be an automorphism, hence $\ker(1 - f) = 0$. But then $k = 0$ and therefore $K = 0$, as desired. □

We conclude this section with two more preservation properties and an open question.

Lemma 3.8.

(1) *The property of being a right S-ring is preserved under Morita equivalence.*
(2) *A finite direct product of rings, $R = \prod_{i=1}^{n} R_i$, is a right S-ring if and only if each R_i is a right S-ring.*

Proof. (2) is obvious and so is (1), for being flat, being finitely generated, and being projective are Morita invariant properties. □

Lemma 4.5 below shows that the class of right S-rings is not closed under infinite direct products (as any such ring would contain an infinite set of orthogonal idempotents).

Question 3.9. Is every right S-ring a left S-ring? (We do not even know the answer for domains.)

We will answer this question by verifying symmetry in various particular cases, see 4.9, 4.10, 6.4, below.

4. S-rings via idempotents

Lemma 4.1. *Let R be a ring with the a.c.c. on right annihilators of elements or the d.c.c. on left annihilators of elements. Then every a-sequence eventually consists of idempotents.*

Proof. Suppose that R has the d.c.c. on left annihilators of elements. Then the ascending chain $a_1R \subseteq a_2R \subseteq \cdots$ gives rise to a descending chain of left annihilators, $\operatorname{ann}_R(a_1) \supseteq \operatorname{ann}_R(a_2) \supseteq \cdots$. By hypothesis, this chain stabilizes, i.e., there is an i such that $\operatorname{ann}_R(a_i) = \operatorname{ann}_R(a_k)$ for every $k > i$.

Now $a_k a_i = a_i$ implies $1 - a_k \in \operatorname{ann}_R(a_i) = \operatorname{ann}_R(a_k)$. Then $(1 - a_k)a_k = 0$ shows that a_k is an idempotent.

Analogously if R has the a.c.c. on right annihilators of elements, just consider a left d-sequence $\langle b\rangle$ instead. □

The following proposition shows that over many classical rings at least cyclic flat modules are projective.

Proposition 4.2. *Let R be a ring with the a.c.c. on right annihilators of elements or the d.c.c. on left annihilators of elements. Then every cyclic flat right R-module is projective.*

Proof. Otherwise there exists a divergent right a-sequence $\langle a\rangle$ over R. By Lemma 4.1, we may assume that every $a_i = e_i$ is an idempotent. Since $\langle a\rangle$ diverges, we may suppose that every inclusions $e_iR \subset e_{i+1}R$ is proper. Note that e_iR is a right annihilator of $1 - e_i$. Hence, if R has the a.c.c. on right annihilators of elements, we obtain a contradiction. If R has the d.c.c. on left annihilators of elements, we obtain a contradiction considering the descending chain $R(1 - e_1) \supset R(1 - e_2) \supset \cdots$. □

Corollary 4.3. *Let R be a ring such that every ring R_n has the a.c.c. on right annihilators of elements or the d.c.c. on left annihilators of elements. Then R is a right S-ring.*

Remark 4.4. The d.c.c. part of this is contained in [6, Corollary 3.6], and the a.c.c. part in Zhus's [18, Proposition 9]. However, the proofs of the three previous results show that they hold true for rings with apparently weaker chain conditions and thus strengthen both of the cited results (with a uniform proof). Namely, all we used was the d.c.c. on left annihilators of right a-sequences or the a.c.c. on right annihilators of left d-sequences.

Zhu, in fact, works with another a.c.c., the a.c.c. on right annihilators of sequences of ring elements of the form $b_1, b_2b_1, b_3b_2b_1, \ldots$. However, d-sequences are clearly of this form, and so his a.c.c. may be slightly stronger than ours (on right annihilators of left d-sequences).

Note that Zhu's a.c.c. is equivalent to the a.c.c. on right annihilators of sequences of ring elements $c_1, c_2, c_3, \ldots$ such that $Rc_1 \supseteq Rc_2 \supseteq Rc_3 \supseteq \cdots$. The corresponding d.c.c. is that on left annihilators of sequences of ring elements $a_1, a_2, a_3, \ldots$ such that $a_1R \subseteq a_2R \subseteq a_3R \subseteq \cdots$, a d.c.c. that seems slightly stronger than ours (on left annihilators of right a-sequences).

We are going to answer Question 3.9 for the case of exchange rings and for the case of semihereditary rings and show symmetry for these.

To this end we first establish the fact that S-rings are *I-finite* in the sense that they contain no infinite set of orthogonal (nonzero) idempotents.

Lemma 4.5. *If every cyclic flat right R-module is projective, then R is I-finite.*

Proof. Suppose that $e_1, e_2, \ldots$ is an infinite set of orthogonal idempotents of R. Set $a_i = e_1 + \cdots + e_i$. Then

$$a_{i+1}a_i = (e_1 + \cdots + e_i + e_{i+1})(e_1 + \cdots + e_i) = e_1 + \cdots + e_i = a_i,$$

hence $\langle a \rangle = \langle a_1, a_2, \ldots \rangle$ is a right a-sequence. But $a_ia_{i+1} = a_i \neq a_{i+1}$ hence, by Lemma 2.5, $\langle a \rangle$ diverges. □

Corollary 4.6. *If R is a right S-ring, then for every n, the ring R_n is I-finite.*

Proof. Since R is an S-ring, R_n is an S-ring for every n. Now the result follows from Lemma 4.5. □

Corollary 4.7. *A von Neumann regular ring is a right S-ring if and only if it is semisimple artinian* (*if and only if it is a left S-ring*).

This also follows from the fact that every module over a von Neumann regular ring is flat.

Recall that a ring R is called *Dedekind finite* if $rs = 1$ for $r, s \in R$ implies $sr = 1$. If the same property holds for every pair of $n \times n$ matrices over R, the ring R is called *stably finite*. Corollary 4.6 together with [11, Proposition 6.60(2)] yields at once

Corollary 4.8. *Every right S-ring is stably finite.*

Now we are in a position to prove that for exchange rings (see, e.g., [13]) the S-property is indeed left-right symmetric (cf. Question 3.9 above). Note that the concept of exchange ring is itself left-right symmetric. A proper subclass of that of exchange rings is the class of semiregular rings, i.e., rings R such that $R/\operatorname{Jac}(R)$ is von Neumann regular and whose idempotents may be lifted modulo $\operatorname{Jac}(R)$. For example, endomorphism rings of pure-injective modules are semiregular. More generally, Guil Asensio and Herzog [7] proved that endomorphism rings of cotorsion modules are semiregular as well.

Proposition 4.9. *An exchange ring is a right S-ring if and only if it is semiperfect* (*if and only if it is a left S-ring*).

Proof. Right S-rings are I-finite by Lemma 4.5. But Camillo and Yu Hua-Ping [1] proved that I-finite exchange rings are semiperfect. It remains to apply Example 3.7. □

We conclude this section by showing that symmetry also holds for (one-sided) semihereditary rings.

Proposition 4.10. *A right semihereditary ring R is a right S-ring if and only if R_n is I-finite for every n, if and only if it is a left S-ring. In this case R is also left semihereditary.*

Proof. If R is a right S-ring, Corollary 4.6 shows that every ring R_n is I-finite.

Since R is right semihereditary, by [11, 7.63], the right annihilator of any matrix in R_n is generated by an idempotent. So if, conversely, R_n is I-finite, it has the a.c.c. on right annihilators of elements. Hence R is a right S-ring by Corollary 4.3, this proving the first equivalence.

On the other hand, by [12, Proposition 5.4.3], for I-finite rings semi-heriditarity is a left-right symmetric property. So R is two-sided semihereditary. But then, since I-finiteness of R_n is left-right symmetric, the first equivalence (on the other side) shows that it also is equivalent to the fact that R is a left S-ring. □

5. Lifting the S-property

The following fact helps to lift the S-property modulo various (two-sided) ideals. (Although the statement differs from that of the original lemma, it is precisely what is proved there.)

Fact 5.1 [8, Proposition 2.1]. *Let P be a projective right module over an arbitrary ring R. If I is a nil ideal such that P/PI is cyclic, then P is cyclic.*

Lemma 5.2. *Let I be a nil ideal of a ring R such that every cyclic flat right R/I-module is projective. Then every cyclic flat right R-module is projective.*

Proof. Let $\langle a\rangle$ be a right a-sequence over R. By Fact 2.11 it suffices to prove that the projective right R-module $P = P_{\langle a\rangle}$ is finitely generated.

Since every cyclic flat right R/I-module is projective, the right a-sequence $\langle\bar{a}\rangle = \langle\bar{a}_1, \bar{a}_2, \ldots\rangle$ over R/I converges. Hence the projective right R/I-module $\overline{P} = P_{\langle\bar{a}\rangle}$ is cyclic. But $\overline{P} = P/PI$, and so the previous fact implies that P is also cyclic. □

It is not known if being nil passes over to matrices (in fact, this is equivalent to Köthe's conjecture), but being included in the prime radical does, and so we may infer that the S-property can be lifted modulo (nil) ideals contained in the prime radical.

Corollary 5.3. *Let I be an ideal contained in the prime radical of a ring R* (*e.g., any nilpotent ideal*). *If R/I is a right S-ring, then R is a right S-ring.*

Now that we know one can lift the S-property modulo the prime radical we turn to the problem of lifting it modulo the Jacobson radical. Here we have only partial results, based on the following

Fact 5.4 [8, Lemma 2.4]. *Let P be a projective right module over an arbitrary ring R. If $P/P\,\mathrm{Jac}(R)$ is finitely generated and so is P/PI for every prime ideal I, then P is finitely generated.*

If, in the above proof, Fact 5.1 is replaced by this fact (from the same paper), we at once obtain the next result. (Note that here passing to matrix rings is no problem.)

Proposition 5.5. *Let every prime factor of the ring R be a right S-ring. If $R/\mathrm{Jac}(R)$ is a right S-ring, then R is a right S-ring.*

Since, being embeddable in a semisimple artinian ring, a prime right Goldie ring is an S-ring, this readily yields

Corollary 5.6. *Let every prime factor of the ring R be a right Goldie-ring* (*this is the case, e.g., when R has a polynomial identity, in particular, when R is commutative*). *If $R/\mathrm{Jac}(R)$ is an S-ring, then R is an S-ring.*

We are ready to give some more example of S-rings.

Example 5.7. Endomorphism rings of a right artinian modules are left and right S-rings.

Proof. If R is the endomorphism ring of an artinian module M, then R_n is the endomorphism of the artinian module M^n. So it suffices to prove that every cyclic flat right or left R-module is projective.

By [4, Proposition 10.6] R contains a two-sided nilpotent ideal H such that every chain of left annihilators of the ring R/H is uniformly bounded. Then every chain of right annihilators of R/H is uniformly bounded. Hence, by Proposition 4.2, every cyclic flat left or right R/H-module is projective, and it remains to apply Lemma 5.2. □

It is easy to show (see the remark in the introduction) that every right noetherian ring is a right S-ring. It turns out that it must be also a left S-ring. In fact, more can be said.

Proposition 5.8. *Any ring with right Krull dimension is a left and right S-ring.*

Proof. Let N be the prime radical of R. By Corollary 5.3 it suffices to prove that R/N is an S-ring. But by [4, Corollary 7.19], R/N is a semiprime Goldie ring, hence R/N is embeddable into a semisimple artinian ring. It remains to apply Lemma 3.1. □

Next we investigate when triangular matrix rings are S-rings.

Proposition 5.9. *Let ${}_RM_T$ be an R-T-bimodule, and let $U = \begin{pmatrix} R & M \\ 0 & T \end{pmatrix}$ be a triangular matrix ring. Then U is a right S-ring if and only if R and T are right S-rings.*

Proof. If U is a right S-ring, then R and T are right S-rings by Lemma 3.1. Now assume that R and S are right S-rings. Note that $N = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ is a nilpotent (of index 2) ideal of U such that $U/N \cong R \oplus S$. Hence we may apply Corollary 5.3 (and Lemma 3.8). □

If M is an R-R-bimodule, then the ring $\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \mid r \in R, m \in M\}$ is called *trivial extension* (of M).

Proposition 5.10. *Let M be an R-R-bimodule. Then the trivial extension of M is a right S-ring if and only if R is a right S-ring.*

Proof. Similar to Proposition 5.9. □

6. L-rings

Following Zöschinger [19], a ring R is an *L-ring*, if it has the following property. If P is a projective right R-module such that $P/\operatorname{Jac}(P)$ is finitely generated, then P is finitely generated.

Zöschinger [19] proved that the property is two-sided. He also gave the following characterization.

Fact 6.1 [19, Satz 2.3]. *The following are equivalent for any ring R.*

(1) *R is an L-ring.*
(2) *If F is a finitely generated flat right R-module such that $F/F\operatorname{Jac}(R)$ is a projective right $R/\operatorname{Jac}(R)$-module, then F is projective.*

Corollary 6.2. *Every right S-ring is an L-ring.*

Lemma 6.3. *Let $R/\operatorname{Jac}(R)$ be a right S-ring. Then R is a right S-ring if and only if R is an L-ring.*

Proof. By Corollary 6.2 we need to prove that if R is an L-ring, then R is a right S-ring.

By Fact 6.1 it suffices to check that $F' = F/F\operatorname{Jac}(R)$ is a projective $R' = R/\operatorname{Jac}(R)$-module for every finitely generated flat right R-module F.

Clearly F' is a finitely generated R'-module. Since $F' = F \otimes_R R'$, this R'-module is also flat. But R' is a right S-ring, so F' is indeed projective. □

The symmetry of the property of being an L-ring allows us to prove symmetry as addressed in Question 3.9 for the case of *semilocal* rings, i.e., rings R such that $R/\operatorname{Jac}(R)$ is semisimple artinian. This is implicit also in [6, Remark 3.7].

Proposition 6.4. *A semilocal ring is a right S-ring if and only if it is a left S-ring.*

Proof. Since R is a right S-ring, R is an L-ring by Corollary 6.2. Since $R/\operatorname{Jac}(R)$ is a left S-ring, R is a left S-ring by Lemma 6.3. □

Not every semilocal ring is an S-ring. Indeed, the first author has an example of a semilocal ring of Goldie dimension one (on both sides) which is not an L-ring, [14]. Such a ring can be neither a left nor a right S-ring (cf. Lemma 6.3).

However, if we add an extra condition, we do get the S-property. To this end, call R *homogeneous semilocal* if $R/\operatorname{Jac}(R)$ is a simple artinian ring. For examples of such rings see Corisello and Facchini [2].

Example 6.5. Every homogeneous semilocal ring is a right and left S-ring.

Proof. Lemma 6.3 tells us that we need only prove that R is an L-ring.

By [2, Theorem 2.3] every projective right R-module P is a direct sum of copies of a unique cyclic indecomposable projective R-module. Thus if P is not finitely generated, then $P/\operatorname{Jac}(R)$ is not finitely generated either. □

7. Commutative S-rings

In the commutative case things considerably simplify due to a result of Vasconcelos.

Fact 7.1 [17, Corollary 1.7]. *Let R be a commutative ring such that every cyclic flat R-module is projective. Then R is an S-ring.*

Thus, in the commutative case every a-sequence of square matrices converges whenever every a-sequence of ring elements does.

Further, one easily reduces the general commutative case as follows to that of rings without nontrivial idempotents.

Lemma 7.2. *Let R be an I-finite ring such that all idempotents of R are central. Then R is a finite direct sum $\bigoplus_{i=1}^{n} R_i$ of rings without nontrivial idempotents. Moreover, R is an S-ring if and only if each R_i is an S-ring.*

Proof. We say that a nonzero idempotent $e \in R$ is an *atom*, if the ring $eRe = eR$ contains no nontrivial idempotents (other then 0 and e).

It is easy to prove that two distinct commuting atoms are orthogonal. Hence there are only finitely many atoms (for R is I-finite), say $e_1, \dots, e_n$. If $e = e_1 + \cdots + e_n$, then $R = e_1R \oplus \cdots \oplus e_nR \oplus (1-e)R$ is the desired decomposition. It remains to invoke Lemma 3.8. □

Not all commutative rings without idempotents are S-rings, as we exemplify next.

Example 7.3. Let R be a commutative algebra over a field $\Bbbk$ with generators $x_1, x_2, \dots$ and relations $x_{i+1}x_i = x_i$. Then

(1) R is reduced.
(2) R has no nontrivial idempotents.
(3) R is not an S-ring.

Proof. Every element $r \in R$ has a canonical form $f_0 + \sum_{i=1}^{k} f_i$, where $f_0 \in \Bbbk$, and f_i is a polynomial in x_i whose free term is equal to zero, for all $i \geqslant 1$.

(1) and (2). If $n > 0$ is the degree of f_k in the above representation of $r \in R$, then $r^m = g_0 + \sum_{i=1}^{k} g_k$, where the degree of g_k is equal to mn. Hence neither $r = r^2$ nor is r nilpotent.

(3) Clearly $x_1, x_2, \dots$ is an a-sequence in R. If it stabilized, it would follow that eventually either $x_i = 0$, or $x_i = 1$, a contradiction. □

Next we prove that a-sequences over commutative rings of Goldie dimension one behave like those over local rings, that is, we prove Lemma 2.2(2) for the commutative Goldie dimension one case.

Proposition 7.4. *Every commutative ring of Goldie dimension one is an S-ring. Moreover, every nonzero right a-sequence over such a ring is of the form $(0, 0, \dots, 0, r, s, 1, 1, 1, \dots)$, where $0 \neq r \in R$ and $sr = r$.*

Proof. Assuming the contrary, we would have ring elements $a_1 \neq 0$, $a_2 \neq 0, 1$, and $a_3 \neq 1$ such that $(0, 0, \dots, 0, a_1, a_2, a_3, 1, 1, 1, \dots)$ is a right a-sequence. This would lead to a contradiction as follows.

From $a_2a_1 = a_1$ it follows that $(1 - a_2)a_1 = 0$. Hence the annihilator of (the nonzero element) $1 - a_2$ in R is nonzero.

Similarly $a_3a_2 = a_2$ implies $(1 - a_3)a_2 = 0$, whence the annihilator of (the nonzero element) a_2 is nonzero as well.

Since R has Goldie dimension one, there is a nonzero $s \in R$ such that $(1 - a_2)s = 0$ and $a_2 s = 0$ (this is where commutativity is used). But then $s = (1 - a_2)s + a_2 s = 0$, a contradiction. □

The following fact is known, but will be improved on below.

Fact 7.5 [17]. *Every semilocal commutative ring is an S-ring.*

Proof. By Corollary 5.3 it suffices to prove that R/N is an S-ring, where N is the prime radical of R. Thus we may assume that R is semilocal and reduced. Then R is embedded into a finite product of local rings (localizations of R with respect to maximal ideals).

Now every local ring is an S-ring, and every subring of an S-ring is an S-ring. □

After Proposition 6.4 above we mentioned an example of a (noncommutative) semilocal ring R of Goldie dimension one which is not an S-ring. Thus neither Proposition 7.4 nor Fact 7.5 hold in general.

In order to generalize the previous result, let $\mathrm{Max}(R)$ denote the set of maximal ideals of (the commutative ring) R endowed with the topology induced by the Zariski topology on the prime spectrum of R. Then for every $a \in R$, the set $V(a) = \{I \in \mathrm{Max}(R) \mid a \in I\}$ is closed, and every closed set of $\mathrm{Max}(R)$ is an intersection of such sets.

Proposition 7.6. *Let R be a commutative ring such that* $\mathrm{Max}(R)$ *has the a.c.c. or the d.c.c. on subsets of the form $V(a)$, where $a \in R$. Then R is an S-ring.*

Proof. By Corollary 5.6 it suffices to prove that $R/\mathrm{Jac}(R)$ is an S-ring. Since $R/\mathrm{Jac}(R)$ has the a.c.c. (the d.c.c.) on subsets of the form $V(a)$ iff R does, we may assume that $\mathrm{Jac}(R) = 0$ from the very beginning.

Let $a_1, a_2, \ldots$ be an a-sequence over R. Put $V_i = V(a_i)$ and $W_i = V(1 - a_i)$. Then $V_i \cap W_i = \emptyset$, for if $I \in V_i \cap W_i$, then $a_i \in I$, and $1 - a_i \in I$, hence $1 \in I$, a contradiction.

Further, $V_i \cup W_{i+1} = \mathrm{Max}(R)$ for every i. Indeed, from $(1 - a_{i+1})a_i = 0$ it follows that for every $I \in \mathrm{Max}(R)$, either $1 - a_{i+1} \in I$, i.e., $I \in W_{i+1}$, or $a_i \in I$, i.e., $I \in V_i$.

We see that $W_1 \subseteq W_2 \subseteq \cdots$ is an ascending chain:

$$W_i = W_i \cap \mathrm{Max}(R) = W_i \cap (V_i \cup W_{i+1}) = (W_i \cap V_i) \cup (W_i \cap W_{i+1}) = W_i \cap W_{i+1}.$$

If $\mathrm{Max}(R)$ has the a.c.c. on subsets of the form $V(a)$, then $W_i = W_{i+1} = \cdots$ for some i. We may assume that $W_1 = W_2 = \cdots$. Then for every i, $V_i \cup W_{i+1} = \mathrm{Max}(R)$ implies $V_i \cup W_i = \mathrm{Max}(R)$. It follows that $a_i(1 - a_i) \in I$ for every maximal ideal I, therefore $a_i(1 - a_i) \in \mathrm{Jac}(R) = 0$.

Thus every $a_i = e_i$ is an idempotent. Aiming for a contradiction, we may assume that all inclusions $e_i R \subset e_{i+1} R$ are proper. But then R contains an infinite set of orthogonal idempotents, which clearly violates the a.c.c.

If $\mathrm{Max}(R)$ has the d.c.c. on subsets of the form $V(a)$, the argument is analogous, using the descending chain $V_1 \supseteq V_2 \supseteq \cdots$ instead. □

Remark 7.7. The hypotheses on the topological space $\mathrm{Max}(R)$ in this proposition are met once this space is artinian or noetherian. The latter is true in case the commutative ring R is noetherian; but we know already that even one-sided noetherian rings are two-sided S-rings, see Proposition 5.8 and the remarks preceding it.

We conclude with two more examples. For the first one, recall that the (total) quotient ring $Q(R)$ of a commutative ring R is the localization of R with respect to the set of all nonzero divisors. Endo [3] proved that if $Q(R)$ is semilocal, then R is an S-ring and asked if the converse were also true [3, p. 289]. The answer is no, as the next example shows.

Example 7.8. Consider the $\mathbb{Z}$-$\mathbb{Z}$-bimodule $M = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ and its trivial extension $R = \{\left(\begin{smallmatrix} z & m \\ 0 & z \end{smallmatrix}\right) \mid z \in \mathbb{Z}, m \in M\}$. Then R is an S-ring, whose total quotient ring $Q(R) = R$ is not semilocal.

Proof. Since $\mathbb{Z}$ is an S-ring, so is R, by Proposition 5.10. Further, since $\mathbb{Z}$ is not semilocal, neither is R. But $Q(R) = R$. □

Example 7.9. There is a commutative S-ring which is a Goldie ring and whose ring R_2 of 2×2 matrices does not have the a.c.c. on right annihilators.

Proof. Kerr [9] constructed an example of a commutative Goldie ring R (of Goldie dimension two) such that R_2 does not have an a.c.c. on right annihilators. Since R has the a.c.c. on annihilators, by Proposition 4.2, every cyclic flat R-module is projective. Thus R is an S-ring by Fact 7.1. □

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