A domain-independent interaction integral for fracture analysis of nonhomogeneous piezoelectric materials

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Abstract

Piezoelectric materials and structures contain more or less electromechanical interfaces in engineering applications. It is difficult to obtain the fracture parameters efficiently of the piezoelectric materials with complex interfaces. This paper presents a domain-independent interaction integral for material nonhomogeneity and discontinuity which can be used for solving the stress intensity factors (SIFs) and the electric displacement intensity factor (EDIF) of piezoelectric materials with complex interfaces efficiently. The interaction integral is based on the J-integral by superimposition of two admissible states and the present formulation does not involve any derivatives of mechanical and electric properties. Moreover, it is proved that the interface in the integral domain does not affect the value of the interaction integral and thus, the present method does not require electromechanical parameters of piezoelectric materials to be continuous. The interaction integral method combined with the extended finite element method (XFEM) is used to investigate the influences of material continuity on the SIF and the EDIF and the results show that the material parameters and their first-order derivatives affect both the SIF and the EDIF greatly, while the higher-order derivatives affect both of them slightly.

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1. Introduction

Due to the intrinsic coupling effect between mechanical and electrical fields, piezoelectric materials have been widely used in modern technical areas serving as sensors, actuators or transducers (Duan et al., 2010; Wang and Wu, 2012). Commonly, piezoelectric materials possess many advantages such as good actuating strain, fast response and high stiffness. However, the inherent brittleness and low fracture toughness are the key problem of these materials (Kuna, 2010). Therefore, the fracture analysis is of great importance for piezoelectric materials. The first theoretical work on piezoelectric fracture mechanics was reported by Parton (1976). He derived a fundamental result of a permeable crack in piezoelectric media. Pak (1990) gave a detailed argument for an impermeable crack by neglecting the electric displacement within the crack. Subsequently, Sosa and Pak (1990), Pak (1992), Suo et al. (1992), Park and Sun (1995), Chen and Shioya (1999) conducted a lot investigations on representative fracture problems of piezoelectric materials. These pioneering works provided many exact solutions and meanwhile, established the theoretical fundamentals of linear elastic fracture mechanics (LEFM) of piezoelectric materials. In LEFM, the stresses and electric displacements at crack tips show a $1/\sqrt{r}$ singularity (Suo et al., 1992), where $r$ is the distance from the crack tip. The local asymptotic distributions of these fields can be expressed in terms of stress intensity factors (SIFs) $K_I$, $K_{II}$ and $K_{III}$ corresponding to the three mechanical crack opening modes, and an electric displacement intensity factor (EDIF) $K_{ED}$ (Park and Sun, 1995; Ricoeur and Kuna, 2003). The details can be found in the review articles written by Zhang et al. (2001), Chen and Lu (2003) and Zhang and Gao (2004). However, two-dimensional (2D) crack problems are mostly restricted to be in infinite piezoelectric media and only a few simple three-dimensional (3D) configurations can be solved by theoretical approaches. Therefore, numerical techniques are usually used in actual fracture analyses of piezoelectric materials and structures.

In general, numerical methods such as finite element method (FEM) (Kumar and Singh, 1996; Kuna, 2006), boundary element method (BEM) (Pan, 1999), and extended finite element method (XFEM) (Bechet et al., 2009; Bhargava and Sharma, 2011) are employed to solve the crack-tip fields. Then, displacement extrapolation method (DEM) (Rao and Kuna, 2008) and conservation integrals such as the electromechanical J-integral (Pak, 1990, 1992; Abendroth et al., 2002) and the interaction integral (Kuna, 2006) are applied to extract the fracture parameters. Compared
with the J-integral, the interaction integral has generated a great interest for its convenience in decoupling SIFs $K_1$, $K_2$, $K_\infty$ and EDIF $K_N$. The interaction integral method was proposed by Stern et al. (1976) to solve mode-I and mode-II SIFs separately for 2D static mechanical problems. The interaction integral is based on the J-integral by a superposition of two admissible states (an actual state and an auxiliary state). Recently, the interaction integral method was extended to solve intensity factors of a crack in homogeneous piezoelectric media (Enderlein et al., 2005; Kuna, 2006, 2010; Bank-Sills et al., 2008; Bechet et al., 2009; Janski et al., 2010; Bhargava and Sharma, 2011). By suitable definition of the auxiliary fields, Rao and Kuna (2008) proved rigorously the interaction integral method to be valid for functionally graded piezoelectric materials (FGPMs) which are usually designed to be nonhomogeneous materials with electromechanical properties varying continuously (Wu et al., 1996). They discussed the three optional definitions of FGPMs which are to be nonhomogeneous materials with electromechanical properties varying continuously. (Yu et al., 2009, 2010a, 2010b). In addition, FGPMs are also two- or multi-phase heterogeneous composites in which the microstructures vary spatially or the volume fraction of constituent particles varies in one or several directions. Namely, in a certain scale, piezoelectric materials and structures contain more or less interfaces with discontinuous electromechanical properties. Therefore, the interfaces cannot be ignored when fracture behaviors of piezoelectric materials and structures are concerned. However, to the best knowledge of the authors, all the present fracture mechanics methods can hardly extract the fracture parameters of such piezoelectric materials efficiently without keeping away from material interfaces. For pure elastic media, the authors’ previous work (Yu et al., 2009, 2010a, 2010b) has provided a domain-independent interaction integral for mechanical interfaces. As an in-depth study, this paper aims to develop a domain-independent interaction integral for piezoelectric materials with continuously varying or discontinuous electromechanical properties.

We stress our contributions as follows. (1) This work derives a domain-independent interaction integral for material nonhomogeneity and discontinuity. Therefore, the present interaction integral method may become a promising technique in the fracture analysis of linear piezoelectric materials with complex electromechanical interfaces efficiently. (2) The expression of the present interaction integral does not contain any derivatives of electromechanical properties, which reduces the requirement of material properties and facilitates the practical implementation of the interaction integral method since the derivatives of the properties of actual materials are usually difficult to obtain. The remainder of this paper is organized as follows. Section 2 reviews the basic equations for piezoelectric media. Section 3 describes the interaction integral and the approach to extract the intensity factors. Section 4 derives the domain expression of the interaction integral without any derivatives of material parameters. Section 5 presents the mathematically proof that material interfaces do not affect the validity of the interaction integral. Section 6 introduces the extended finite element method (XFEM) briefly and the numerical discretization of the interaction integral. Section 7 presents several numerical examples to verify the accuracy and check the domain-independence of the interaction integral. Next, the influences of material continuity on the mechanical SIF and the EDIF are investigated. Finally, Section 8 summarizes this work.

2. Basic piezoelectric relations

This work considers a linear piezoelectric medium and at the beginning, the governing equations and the boundary conditions which form the foundation of piezoelectric media are given below.

2.1. Governing equations

- Constitutive equations (Parson, 1976):
  \[ \sigma_{ij} = C_{ijkl}(x)\varepsilon_{kl} - e_{ijkl}(x)\varepsilon_{ki} \]
  \[ D_{ij} = e_{ijkl}(x)\varepsilon_{kl} + k_{ijkl}(x)\varepsilon_{ki} \]

The constitutive relations can also be represented by the following equations (Hwu, 2008)

- Kinematic equations (Pak, 1990):
  \[ e_{ij} = \frac{1}{2}(u_{ij} + u_{ji}) \]
  \[ E_{i} = -\phi_{j} \]

- Equilibrium equations (Pak, 1992):
  \[ \sigma_{ij} + f_{i} = 0 \]
  \[ D_{ij} - \omega_{i} = 0 \]

In this work, the body forces $f_{i}$ and volume charge $\omega_{i}$ are assumed to be zero and then, the equilibrium equations are simplified as

\[ \sigma_{ij} = 0, \quad D_{ij} = 0 \]

where the variables marked by the subscript $i, j, k$ and $l$ ($i, j, k, l = 1, 2, 3$) are the components of the vectors or tensors in a coordinate system; $u_{ij}$, $\sigma_{ij}$, $e_{ijk}$, $D_{ij}$, and $E_{i}$ are the elastic displacement, stress, strain, electric potential, electric displacement and electric field, respectively; $C_{ijkl}$ is the elastic stiffness tensor measured with the electric field held constant; $S_{ijkl}$ is the elastic compliance tensor measured with the electric displacement held constant; $k_{ijkl}$ is the dielectric permittivity measured with the strain held constant; $\omega_{i}$ is the dielectric non-permittivity measured with the stress held constant; $e_{ijkl}$ and $g_{ijkl}$ are the piezoelectric stress/charge and strain/voltage tensors, respectively. A comma denotes partial differentiation and the repetition of an index implies summation with respect to that index over its range. It should be pointed out that the summation convention is only valid for repeated spatial indices ($i,j,k,l$), but not valid for the indices denoting fracture modes ($M,N$) or denoting eigenvalue number ($\lambda$).

2.2. Boundary conditions

Consider a piezoelectric medium occupying the space $\Omega$ enclosed by surface $\partial \Omega$. The boundary surface $\partial \Omega = \partial A_{\sigma} + \partial A_{\omega} = \partial A_{\rho} + \partial A_{\theta}$. On the boundaries $\partial A_{\sigma}$ and $\partial A_{\omega}$, the stresses and electric displacements are (Suo et al., 1992)

\[ \sigma_{ij}n_{i} = t_{ij}^{\rho}, \quad \sigma_{ij}n_{i} = -\omega_{i}^{\theta}, \quad \sigma_{ij}n_{i} = 0, \quad D_{ij}n_{i} = 0 \]

On the boundaries $\partial A_{\rho}$ and $\partial A_{\theta}$, the stresses and electric displacements are (Suo et al., 1992)

\[ \sigma_{ij}n_{i} = t_{ij}^{\rho}, \quad D_{ij}n_{i} = 0 \]

(6)
where \( t_i^p \) and \( \phi^p \) are prescribed boundary traction on \( A_p \) and surface charges on \( A_p \), respectively; \( n_i \) is the outward unit normal vector to \( A \). On the boundaries \( A_u \) and \( A_d \), the displacements and the electric potential are (Suo et al., 1992)
\[
\begin{align*}
  u_i &= u_i^0, \quad \text{on } A_u \\
  \phi &= \phi^0, \quad \text{on } A_d 
\end{align*}
\]
where \( u_i^0 \) and \( \phi^0 \) are prescribed values on \( A_u \) and \( A_d \), respectively.

### 3. Definition of the interaction integral

The present paper focuses on a piezoelectric solid with a mechanical traction-free and electrically impermeable crack. In this section, the definition of the interaction integral and its application to extract the intensity factors are given.

#### 3.1. Auxiliary fields

We first introduce the auxiliary fields that will be used in the interaction integral method. As shown in Fig. 1, the definitions of the auxiliary fields are given in the polar coordinates \( (r, \theta) \) with the origin at the crack tip. The auxiliary stresses \( \sigma_{\text{aux}} \) and electrical displacements \( D_{\text{aux}} \) are defined as (Rao and Kuna, 2008)
\[
\begin{align*}
  \sigma_{\text{aux}}(r, \theta) &= \frac{1}{r} \sum_N K_{ji}^{\text{aux}} \sigma_i^N(r) \\
  D_{\text{aux}}(r, \theta) &= \frac{1}{r} \sum_N K_{ji}^{\text{aux}} E_i^N(r)
\end{align*}
\]
and the auxiliary mechanical displacements \( u_i^{\text{aux}} \) and electric potential \( \phi^{\text{aux}} \) are defined as
\[
\begin{align*}
  u_i^{\text{aux}}(r, \theta) &= \sqrt{2} \sum_N K_{ji}^{\text{aux}} u_i^N(\theta) \\
  \phi^{\text{aux}}(r, \theta) &= \sqrt{2} \sum_N K_{ji}^{\text{aux}} \phi_i^N(\theta)
\end{align*}
\]
where the summation over \( N = \{I, II, III, IV\} \) comprises different fracture opening modes; \( K_{ji}^{\text{aux}}, K_{ji}^{\text{aux}} \) and \( K_{ji}^{\text{aux}} \) denote the auxiliary mode-I, mode-II, mode-III mechanical SIFs and the auxiliary EDIF, respectively. The angular functions \( f_i(\theta), g_i(\theta), d_i^N(\theta) \) and \( v_i^N(\theta) \) are the standard angular functions for a crack in a homogeneous piezoelectric elastic medium, which depend only on the material properties at crack-tip location. The detailed definitions of the angular functions can be found in Appendix A.

![Fig. 1. Schematic illustration of the contour integrals and related equivalent domain integrals.](image)

The auxiliary strains and electric fields are defined by
\[
\begin{align*}
  e_{ij}^{\text{aux}} &= S_{ji}(x) \sigma_{\text{aux}}^{\text{max}} + g_{ji}(x) D_{\text{aux}}^{\text{max}} \\
  E_i^{\text{aux}} &= -g_{ji}(x) \sigma_{\text{aux}}^{\text{max}} + \beta_i(x) D_{\text{aux}}^{\text{max}}
\end{align*}
\]
where \( \sigma_{\text{aux}}^{\text{max}}, \sigma_{\text{aux}}^{\text{max}} \) and \( D_{\text{aux}}^{\text{max}} \) are prescribed values on \( A_u \) and \( A_d \), respectively.

Since the fields defined in Eqs. (8) and (9) are the analytical solutions of a homogeneous cracked piezoelectric body under the assumption that both the body forces and volume charges are zeros, the auxiliary stresses and electrical displacements satisfy the following equilibrium equations
\[
\begin{align*}
  \sigma_{\text{aux}}^{\text{max}} &= 0, \quad D_{\text{aux}}^{\text{max}} = 0
\end{align*}
\]

The constitutive equations of the auxiliary fields use the same elastic, piezoelectric and dielectric material parameters as those of the actual fields.

As shown in Fig. 1, for a 2D nonhomogeneous piezoelectric cracked body, the electromechanical J-integral (Pak and Herrmann, 1986; Kuna, 2010) is
\[
J = \lim_{1 \to 0} \int_{r_s} \left( H \delta_{ij} - \sigma_{ij} u_i - D_{\text{aux}} \phi_i \right) n_j d\Gamma
\]
where \( H = \sigma_{ij} \delta_{ij} / 2 \) is the electric enthalpy density for linear piezoelectric media; \( \delta_{ij} \) is Kronecker delta.

Next, two independent equilibrium states are considered which are actual fields \( (u_i, \sigma_{ij}, \phi, D_i, E_i) \) and auxiliary fields \( (u_i^{\text{aux}}, \sigma_{ij}^{\text{aux}}, \phi^{\text{aux}}, D_{\text{aux}}^{\text{aux}}, E_{\text{aux}}^{\text{aux}}) \). Making superposition of these two states into another equilibrium state (state S), the electromechanical J-integral is
\[
J^S = \lim_{1 \to 0} \int_{r_s} \left\{ \frac{1}{2} \left[ (\sigma_{ij} + \sigma_{ij}^{\text{aux}}) (\epsilon_{ij} + \epsilon_{ij}^{\text{aux}}) - (D_i + D_i^{\text{aux}}) (E_i + E_i^{\text{aux}}) \right] \delta_{ij} n_j d\Gamma \right\}
\]
Expanding Eq. (16), we have
\[
J^S = J + J^{\text{aux}} + I
\]
where \( J \) is the J-integral aroused by actual fields alone with expression in Eq. (15),
\[
J^{\text{aux}} = \lim_{1 \to 0} \int_{r_s} \left\{ \frac{1}{2} \left[ (\sigma_{ij}^{\text{aux}} - D_i^{\text{aux}} E_i^{\text{aux}}) \delta_{ij} - (\sigma_{ij}^{\text{aux}} u_i + D_i^{\text{aux}} \phi_i^{\text{aux}}) \right] n_j d\Gamma \right\}
\]
is the J-integral aroused by auxiliary fields alone, and
\[
I = \lim_{\Gamma \to 0} \int_{\Gamma} \left\{ \frac{1}{\sigma_{ik}^{aux}} \left( \sigma_{ik}^{aux} \phi_k - D_i E^i_{aux} + D_{aux} E^i \right) \right\} \eta_j d\Gamma
\]
(19)
is the interaction integral. According to Eqs. (2) and (10), we have
\[
\sigma_{ik}^{aux} - D_i E^i_{aux} = \sigma_{ik}^{aux} - D_i \phi^{aux}_k - D_{aux} E^i
\]
(20)
and hence, Eq. (19) can be simplified as
\[
I = \lim_{\Gamma \to 0} \int_{\Gamma} \left\{ \frac{1}{\sigma_{ik}^{aux}} \left( \sigma_{ik}^{aux} \phi_k - D_i \phi^{aux}_k - D_i \phi^{aux}_k - D_{aux} E^i \right) \right\} \eta_j d\Gamma
\]
(21)

3.3. Extraction of the intensity factors from the interaction integral

For piezoelectrics, the electromechanical J-integral is equal to the energy release rate and thus, \( J \) can be expressed as (Ricouer and Kuna, 2003)
\[
J = \frac{1}{2} K^Y \mathbf{K}^\text{aux}
\]
(22)
where \( \mathbf{K} = [K_{ij} K_{ij} K_{ij} K_{ij}] \) is the vector of the four field intensity factors; \( \mathbf{Y} \) is the (4 x 4) generalized Irwin matrix which depends on the material constants at the crack-tip location. The definition of Irwin matrix \( \mathbf{Y} \) is given in Appendix A. And the interaction integral (Rao and Kuna, 2008)
\[
I = K^Y \mathbf{K}^\text{aux}
\]
(23)
where \( \mathbf{K}^\text{aux} = [K_{aux} K_{aux} K_{aux} K_{aux}] \).

For 2D case, \( K_{ij} = 0 \). If the auxiliary state is chosen to be mode-II fracture state, namely, \( K_{aux} = 1, K_{aux} = K_{aux} = K_{aux} = 0 \), Eq. (23) reduces to
\[
I^{(0)} = K_2 Y_{11} + K_2 Y_{12} + K_2 Y_{22}
\]
(24)
Similarly, by selecting \( K_{aux} = 1, K_{aux} = K_{aux} = K_{aux} = 0 \) and \( K_{aux} = K_{aux} = K_{aux} = 0, K_{aux} = 1 \), Eq. (23) reduces to, respectively,
\[
I^{(0)} = K_2 Y_{11} + K_2 Y_{12} + K_2 Y_{22} + K_2 Y_{42}
\]
(25)
and
\[
I^{(0)} = K_2 Y_{11} + K_2 Y_{12} + K_2 Y_{22} + K_2 Y_{42}
\]
(26)
By solving the simultaneous Eqs. (24) - (26), the intensity factors \( K_2, K_2, \) and \( K_2 \) can be obtained. In the following, how to obtain the value of the interaction integral will be discussed.

4. Interaction integral for piezoelectrics with continuous properties

The infinitesimal contour integral in Eq. (21) cannot be obtained directly in numerical calculations and thus, it is usually converted into an equivalent domain form which can avoid the potential source of inaccuracy in the computation process of a line integral (Moran and Shih, 1987). For a 2D piezoelectric solid with continuous electromechanical properties, although the domain formulations of the interaction integral have been investigated by Rao and Kuna (2008), this section will derive a new domain expression.

4.1. An equivalent contour form of the interaction integral

To begin, as shown in Fig. 1, consider two domain \( A \) and \( A_0 \) enclosed by the contours \( \Gamma_A \) and \( \Gamma_{A_0} = \Gamma_N + \Gamma_i + \Gamma_{i+1} \), respectively where \( \Gamma_{N-i} \) is the opposite path of the contour \( \Gamma_i \). Therefore, taking the limit \( \Gamma_i \to 0 \) leads to \( A_0 \to A \). Next, we define an integral on a closed contour \( \Gamma_{0} \) as
\[
\tilde{I} = \lim_{\Gamma_{0} \to 0} \int_{\Gamma_{0}} P_{ij} m_{ij} dq d\Gamma
\]
(27)
where the expression of \( P_{ij} \) is identical with that in the bracket in Eq. (21), i.e.,
\[
P_{ij} = \sigma_{ij}^{aux} \delta_{ij} - \sigma_{ij}^{aux} \phi_k - D_i E^i f_{ij} - D_{ij} \phi^{aux}_{ik} - D_{ij} \phi^{aux}_{jk}
\]
(28)
Here, \( P_{ij} \) can be regarded as the mutual piezoelectric energy momentum tensor in the spirit of Eshelby’s concept; \( m_i \) is the unit outward normal vector to the contour \( \Gamma_0 \), \( q \) is an arbitrary weight function with value varying smoothly from 1 on \( \Gamma_c \) to 0 on \( \Gamma_y \).

In this paper, the crack faces are assumed to be mechanical traction-free and electrically impermeable, and this assumption is also valid for the auxiliary fields, i.e.,
\[
\begin{align*}
\sigma_{ij}^{aux} &= 0, \quad D_i n_{ij} = 0, \quad \text{on } \Gamma_c \quad \text{and} \quad \Gamma_c \\
\sigma_{ij}^{aux} &= 0, \quad D_i n_{ij} = 0, \quad \text{on } \Gamma_y \quad \text{and} \quad \Gamma_c
\end{align*}
\]
(29)
According to Eqs. (29) and (30), \( q = 0 \) on the contour \( \Gamma_b \), \( m_i = 0 \) on crack faces, and \( m_i = -n_{ij} \) on \( \Gamma_i \), it can be proved easily that
\[
I = -\tilde{I} = -\lim_{\Gamma_{0} \to 0} \int_{\Gamma_{0}} P_{ij} m_{ij} dq d\Gamma
\]
(31)
The detailed derivations of Eq. (31) are given in Appendix B.

4.2. Domain formulation of the interaction integral

By applying divergence theorem to Eq. (31) one obtains
\[
I = -\int_A P_{ij} q_{ij} dA - \int_A P_{ij} g_{ij} dA
\]
(32)
According to Eq. (3) and the symmetry of the auxiliary stress tensor, it is obtained that
\[
\begin{align*}
\sigma_{ij}^{aux} c_{ij} - \sigma_{ij}^{aux} n_{ij} &= 0, \quad D_i E^i f_{ij} + D_{ij} \phi^{aux}_{k} = 0
\end{align*}
\]
(33)
Substituting Eqs. (5), (11), and (13) into \( P_{ij} \), one obtains
\[
P_{ij} = \sigma_{ij}^{aux} c_{ij} - \sigma_{ij}^{aux} n_{ij} - D_i E^i f_{ij} - D_{ij} \phi^{aux}_{k}
\]
(34)
Substituting Eqs. (2), (13), and (14) into Eq. (34), we have
\[
P_{ij} = \sigma_{ij}^{aux} \left[ S_{ij}(x) - S_{ij}(b_{ij}) b_{ij} + \sigma_{ij}^{aux} \left[ g_{ij}(x) - g_{ij}(b_{ij}) \right] b_{ij} + \sigma_{ij}^{aux} \left[ D_{ij}(x) - D_{ij}(b_{ij}) \right] b_{ij} \right]
\]
(35)
The detailed derivations of Eq. (35) are given in Appendix B. By substituting Eqs. (28) and (35) into Eq. (32), the domain formulation of the interaction integral is finally obtained as
\[
I = \int_A \left( \sigma_{ij}^{aux} c_{ij} - \sigma_{ij}^{aux} n_{ij} - D_i E^i f_{ij} + D_{ij} \phi^{aux}_{k} - \sigma_{ij}^{aux} \left[ S_{ij}(x) - S_{ij}(b_{ij}) \right] b_{ij} + \sigma_{ij}^{aux} \left[ g_{ij}(x) - g_{ij}(b_{ij}) \right] b_{ij} + \sigma_{ij}^{aux} \left[ D_{ij}(x) - D_{ij}(b_{ij}) \right] b_{ij} \right) dq dA
\]
(36)
Since only the electromechanical properties at the crack-tip location are adopted in the auxiliary mechanical displacements, stresses, electric potential and electric displacements, there are no derivatives of material properties in both \( \sigma_{ij}^{aux} \) and \( D_{ij} \). Compared with the domain expression given by Rao and Kuna (2008) for FGPMs which contains the terms \( \partial S_{ij} / \partial x_i \), \( \partial g_{ij} / \partial x_i \) and \( \partial \phi / \partial x_i \), Eq. (36) does not include any terms related to the derivatives of material properties with respect to the coordinates.
5. Interaction integral for piezoelectrics with discontinuous properties

From the above section, it is shown that the interaction integral method does not need the mechanical or electric material properties to be differentiable. However, the material properties are still required to be continuous. In this section, we will discuss whether this continuity condition is necessary in the interaction integral method.

5.1. Interaction integral for discontinuous piezoelectrics

As shown in Fig. 2, there is a perfectly bonded interface $I_{\text{interface}}$ in the integral domain $A$. Thus, the domain $A$ enclosed by a closed contour $\Gamma_0$ is divided by $I_{\text{interface}}$ into two parts, $A_1$ and $A_2$ which are enclosed by $\Gamma_{A1}$ and $\Gamma_{A2}$, respectively. Here, $\Gamma_{G1} = \Gamma_{B1} + I_{\text{interface}} + \Gamma_{B2} + \Gamma_{e} + \Gamma_{e'}$ and $\Gamma_{G2} = \Gamma_{B2} + I_{\text{interface}}$. In order to convert the interaction integral into a domain form, Eq. (31) should first be written as

$$I = \lim_{\Gamma_0 \to \infty} \int_{\Gamma_0} P_i m_j q d\Gamma - \int_{\Gamma_{A1}} P_i m_j q d\Gamma + I_{\text{interface}}$$

where $I_{\text{interface}}$ is a line integral along the expression shown below

$$I_{\text{interface}} = \int_{I_{\text{interface}}} P_i^0 m_j q d\Gamma + \int_{I_{\text{interface}}} P_i^m m_j q d\Gamma$$

$$= \int_{I_{\text{interface}}} (P_i^0 - P_i^m) m_j q d\Gamma$$

Here, the variables or expressions on the interface marked by the superscripts 1 and 2 means that they belong to the domains $A_1$ and $A_2$, respectively. By applying divergence theorem to the first and second integrals in Eq. (37), respectively, we have

$$I = -\int_A P_i q dA - \int_A P_i q dA + I_{\text{interface}}$$

In comparison of Eq. (39) and Eq. (32), it can be found that only a term $I_{\text{interface}}$ is added when the integral domain contains a material interface. The value of $I_{\text{interface}}$ will be discussed in the following part.

5.2. Interface integral $I_{\text{interface}}$

According to the definitions of the auxiliary fields, the auxiliary mechanical displacements, stresses, electric potential and electric displacements and their derivatives are continuous on the interface. Therefore, $(\sigma_{ij}^m)^{\Omega} = (\sigma_{ij}^m)^{\bar{\Omega}}$, $(\partial u_i^m / \partial x_j)^{\Omega} = (\partial u_i^m / \partial x_j)^{\bar{\Omega}}$, $(\partial u_i^m / \partial x_j)^{\Omega} = (\partial u_i^m / \partial x_j)^{\bar{\Omega}}$ and $(\partial u_i^m / \partial x_j)^{\Omega} = (\partial u_i^m / \partial x_j)^{\bar{\Omega}}$. The integral $I_{\text{interface}}$ in Eq. (38) can be written as

$$I_{\text{interface}} = \int_{\Gamma_{\text{interface}}} \left\{ \sigma_{ij}^m (\partial_{ij}^\Omega - \partial_{ij}^{\bar{\Omega}}) m_1 - D_{ij}^m (\partial_{ij}^\Omega - \partial_{ij}^{\bar{\Omega}}) m_1 - m_1 (\sigma_{ij}^m - \sigma_{ij}^{\bar{\Omega}}) n_i m_i - m_1 D_{ij}^m (\sigma_{ij}^m - \sigma_{ij}^{\bar{\Omega}}) n_i m_i \right\} q d\Gamma$$

In order to facilitate describing the continuity conditions on the interface $I_{\text{interface}}$, as shown in Fig. 3, we first define the curvilinear coordinates of a point $p$ as

$$\eta_1 = \sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2}$$
$$\eta_2 = \int_{p_0}^{p} d\Gamma$$

where $p_0(x_{10},x_{20})$ is the point on $I_{\text{interface}}$ closest to the point $p(x_1,x_2)$.

Since the material interface is in equilibrium, the tractions and surface charges on both sides of the interface should be equal. That is

$$m_1 \sigma_{ij}^m = m_1 \sigma_{ij}^{\bar{\Omega}}$$
$$m_1 D_{ij}^m = m_1 D_{ij}^{\bar{\Omega}}$$

The interface is assumed to be perfectly bonded and therefore, the derivatives of both mechanical displacements and electric potential with respect to the coordinate $\eta_2$ are equal on both sides of the interface, i.e.

$$\left( \frac{\partial u_i}{\partial \eta_2} \right)^{\Omega} = \left( \frac{\partial u_i}{\partial \eta_2} \right)^{\bar{\Omega}}$$
$$\left( \frac{\partial \phi}{\partial \eta_2} \right)^{\Omega} = \left( \frac{\partial \phi}{\partial \eta_2} \right)^{\bar{\Omega}}$$

Fig. 2. An integral domain $A$ cut by a material interface $I_{\text{interface}}$. Domains $A_1$, $A_2$, and $A$ are enclosed by $\Gamma_0$, $\Gamma_{A1}$, and $\Gamma_{A2}$ for $\Gamma \to 0$. Here $A = A_1 + A_2$, $A_1 = A_1 + \Gamma_{B1} + \Gamma_{e} + \Gamma_{e'}$, $A_2 = A_2 + \Gamma_{B2} + \Gamma_{e} + \Gamma_{e'}$, and $A = A_1 + A_2 + I_{\text{interface}}$.

Fig. 3. A curvilinear coordinate system based on the interface.
In order to simplify the first term of the integrand in Eq. (40), applying the strain-displacement relations of actual fields in Eq. (3), one obtains

\[ \sigma_y^{\text{aux}}(\epsilon_y^0 - \epsilon_y^g)m_1 = \sigma_y^{\text{aux}} \left( \frac{\partial u_y}{\partial x_1} \right)^\ominus \left( \frac{\partial u_y}{\partial x_1} \right)^\oplus m_1 \]  

(44)

By the chain rule we can write Eq. (44) as

\[ \sigma_y^{\text{aux}}(\epsilon_y^0 - \epsilon_y^g)m_1 = \sigma_y^{\text{aux}} \left[ \left( \frac{\partial u_y}{\partial x_1} \right)^\ominus - \left( \frac{\partial u_y}{\partial x_1} \right)^\oplus \right] \partial u_y / \partial x_1 m_1 \]

(45)

It can be noted from Eq. (41) that \( \partial u_1 / \partial x_1 = m_1 \). Substituting Eq. (43), and \( \partial u_1 / \partial x_1 = m_1 \) into Eq. (45), we have

\[ \sigma_y^{\text{aux}}(\epsilon_y^0 - \epsilon_y^g)m_1 = \sigma_y^{\text{aux}} \left[ \left( \frac{\partial u_y}{\partial x_1} \right)^\ominus - \left( \frac{\partial u_y}{\partial x_1} \right)^\oplus \right] m_1 \]

(46)

Similarly, substituting Eq. (3), and \( \partial u_1 / \partial x_1 = m_1 \) into the second term of the integrand in Eq. (40), one obtains

\[ D_{IJ}^{\text{aux}}(E^J - E^g)m_1 = -m_1 D_{IJ}^{\text{aux}} \left[ \left( \frac{\partial \phi}{\partial x_1} \right)^\ominus - \left( \frac{\partial \phi}{\partial x_1} \right)^\oplus \right] m_1 \]

Using the chain rule and substituting Eq. (43), and \( \partial u_1 / \partial x_1 = m_1 \) into the fifth term of the integrand in Eq. (40), we have

\[ m_1 \sigma_y^{\text{aux}} \left[ \left( \frac{\partial u_y}{\partial x_1} \right)^\ominus - \left( \frac{\partial u_y}{\partial x_1} \right)^\oplus \right] = m_1 \sigma_y^{\text{aux}} \left[ \left( \frac{\partial u_y}{\partial x_1} \right)^\ominus - \left( \frac{\partial u_y}{\partial x_1} \right)^\oplus \right] m_1 \]

(48)

Similarly, substituting Eq. (43) and \( \partial u_1 / \partial x_1 = m_1 \) into the sixth term of the integrand in Eq. (40), one obtains

\[ m_1 D_{IJ}^{\text{aux}} \left[ \left( \frac{\partial \phi}{\partial x_1} \right)^\ominus - \left( \frac{\partial \phi}{\partial x_1} \right)^\oplus \right] = m_1 D_{IJ}^{\text{aux}} \left[ \left( \frac{\partial \phi}{\partial x_1} \right)^\ominus - \left( \frac{\partial \phi}{\partial x_1} \right)^\oplus \right] m_1 \]

(49)

Substituting Eq. (82) and Eqs. (46)–(49) into Eq. (40) yields

\[ I_{\text{interface}} = 0 \]

(50)

Similarly to the above derivations, the same result in Eq (50) will be obtained for the interface penetrating the crack faces.

5.3. Discussion on the interaction integral

Substituting Eq. (50) into Eq. (39), the same expression as Eq. (36) is obtained. It implies that Eq. (36) is still valid for nonhomogeneous piezoelectric materials with interfaces on which all electromechanical parameters may be discontinuous. Namely, the interaction integral method does not require material properties to be continuous and hence, its applicable range is greatly enlarged. Moreover, the expression in Eq. (36) can facilitate the numerical implementation for the piezoelectric materials with complex interfaces around the crack tip since the integral domain can be chosen arbitrarily.

If the crack faces in the integral domain \( A \) are curved as shown in Fig. 4, the interaction integral can be written as

\[ I = -\int_A P_y q dA - \int_A P_y q dA + I_{\text{crackface}} \]

(51)

where \( I_{\text{crackface}} \) is a line integral on the crack faces and its expression is

\[ I_{\text{crackface}} = \int_{\Gamma_c} P_y m_y q d\Gamma \]

(52)

where, \( \Gamma_c \) is a fictitious crack face tangent to the crack tip and \( \Gamma_c \) is its opposite path. Considering the boundary conditions, traction- and charge-free for actual fields on \( \Gamma_c^+ \) and \( \Gamma_c^- (m_y = 0); m_D = 0 \), traction- and charge-free for auxiliary fields on \( \Gamma_c^+ \) and \( \Gamma_c^- (m_y = 0); m_D = 0 \), and \( m_1 \neq 0 \) on \( \Gamma_c^+ \) and \( \Gamma_c^- \), Eq. (52) can be simplified as

\[ I_{\text{crackface}} = \int_{\Gamma_c^+} \left( \sigma_y^{\text{aux}} e_y - D_{IJ}^{\text{aux}} E^I \right) m_1 - m_1 \sigma_y^{\text{aux}} u_1 - m_1 D_{IJ}^{\text{aux}} \phi_1 \right) q dA \]

(53)

It should be pointed out that the present interaction integral employs the stress and electric displacement fields with inverse square root singularity and thus, its effective implementation requires that the crack tip cannot be too close to the interface. According to the previous investigation on the pure elastic media (Hwu, 2008), the interaction integral should be effective and accurate for piezoelectric media when the distance from the crack tip to an interface is no less than 0.03 times of the crack length. In addition, if the crack lies along an interface in piezoelectric materials with complex interfaces, the interaction integral derived in this paper should be also effective by selecting a suitable auxiliary field.

5.4. Expanded form of the interaction integral

In order to simplify the expression of Eq. (36) in this paper, let

\[ u_4 = \phi \]

(54)

\[ \sigma_4 = \sigma_4 = I_1, \quad \sigma_4 = 0 \]

\[ 2e_4 = 2e_4 = u_4 = -E_1, \quad e_4 = 0 \]

\[ C_{44} = C_{44}, \quad C_{44} = -K_{44}, \quad C_{44} \text{ arbitrary} \]

\[ 2S_{44} = S_{44}, \quad 4S_{44} = -E_{44}, \quad S_{44} \text{ arbitrary} \]

Here, the expanded elastic stiffness \( C_{ijkl} \) and compliance \( S_{ijkl} \) have the symmetry relations \( C_{ijkl} = C_{ijlk} = C_{klij} \) and \( S_{ijkl} = S_{ljik} = S_{klij} = S_{iklj} \), respectively, and they meet the relation \( C_{ijkl} \delta_{kij} = \delta_{ij} \phi_4 \) (Zhang et al., 2001), where the subscripts \( I, J, K, L, P, Q = 1, 2, 3, 4 \). And define

\[ (\cdot)_4 = 0 \]

(55)

where \( (\cdot)_4 \) is an arbitrary variable or expression. According to these definitions, Eq. (36) can be rewritten in an expanded tensor notation as

\[ I = \int_A \left( \sigma_4^{\text{aux}} + \sigma_4^{\text{aux}} u_1 - \sigma_4^{\text{aux}} \phi_4 \right) q dA \]

\[ + \int_A \sigma_4^{\text{aux}} \phi_4^{\text{aux}} q dA \]

(57)
It can be observed that the present interaction integral formulation for piezoelectric media is of the same form as that for pure elastic media given by Yu et al. (2009) only by extending the range of indices from 1–3 to 1–4.

6. Numerical implementation of the interaction integral

The interaction integral method is implemented in conjunction with the extended finite element method (XFEM) since the XFEM can greatly simplify the analysis of fracture problems, especially,

Table 1

<table>
<thead>
<tr>
<th>Properties</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic constants (10^9 Pa)</td>
<td></td>
</tr>
<tr>
<td>$C_{110}$</td>
<td>126</td>
</tr>
<tr>
<td>$C_{120}$</td>
<td>55</td>
</tr>
<tr>
<td>$C_{130}$</td>
<td>53</td>
</tr>
<tr>
<td>$C_{220}$</td>
<td>117</td>
</tr>
<tr>
<td>$C_{440}$</td>
<td>35.3</td>
</tr>
<tr>
<td>Piezoelectric constants (C/m²)</td>
<td></td>
</tr>
<tr>
<td>$e_{210}$</td>
<td>-6.5</td>
</tr>
<tr>
<td>$e_{220}$</td>
<td>23.3</td>
</tr>
<tr>
<td>$e_{160}$</td>
<td></td>
</tr>
<tr>
<td>Permittivity (10⁻⁹ C/Vm)</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon_{110}$</td>
<td>15.1</td>
</tr>
<tr>
<td>$\varepsilon_{220}$</td>
<td>17.0</td>
</tr>
</tbody>
</table>
crack propagation problems. Therefore the XFEM is introduced briefly.

### 6.1. XFEM for piezoelectrics

For elastic media, the extended finite element method (XFEM) was developed by Belytschko and Black (1999) and Moës et al. (1999) who introduced the local enrichment functions into standard displacement-based approximation to characterize the local features. Therefore, the XFEM allows the discontinuous boundaries, such as cracks or material interfaces, to be independent of the mesh. Recently, the XFEM is extended to the piezoelectric materials (Bechet et al., 2009; Bhargava and Sharma, 2011). The approximations of the displacements and electric potential without near-tip enrichment functions are adopted as

\[
\begin{align*}
\mathbf{u}^h(x) &= \sum_{e=1}^{N_e} \left( N_e(x) \mathbf{u}_e^h + \sum_{p \in D_e} N_e(x) \varphi_p(x) \mathbf{u}_e^p + \sum_{k \in D_e} N_e(x) \psi_k(x) \mathbf{b}_e^k \right) , \\
\phi^h(x) &= \sum_{e=1}^{N_e} \left( N_e(x) \varphi_e^h + \sum_{p \in D_e} N_e(x) \varphi_p(x) \mathbf{b}_e^p + \sum_{k \in D_e} N_e(x) \psi_k(x) \mathbf{b}_e^k \right) \\
\end{align*}
\]

Here, \(x\) is an arbitrary point in the mesh; \(x\) is a point on the discontinuous surface (crack or interface) which is closest to point \(x\); \(x_s\) is the point located at node \(S\); \(N_e(x)\) is the standard finite element shape function; \(\varphi_p(x)\) and \(\psi_k(x)\) are the shifted enrichment functions for material interfaces and cracks, respectively; \(D_e\) is the set of all nodes in mesh; \(D_p\) and \(D_c\) are the sets of the nodes enriched with \(\varphi_p(x)\) and \(\psi_k(x)\), respectively; \(\mathbf{u}_e^h\) and \(\phi^h\) are the standard nodal displacements and nodal electric potential, respectively; \(a^p\) and \(b^k\) are additional degrees of freedom for the nodes in \(D_p\); \(c^p\) and \(d^k\) are additional degrees of freedom for the nodes in \(D_c\). \(H(x)\) is a Heaviside step function. In order to improve the numerical precision, the mesh around the crack tip is refined as shown in Fig. 5.

### 6.2. Numerical discretization of the interaction integral

In order to compute the interaction integral, Eq. (36) should be discretized as

\[
I = \sum_{e_1=1}^{e_1} \sum_{e_2=1}^{e_2} \left\{ \begin{array}{l}
\sigma_{e_1} u_{e_1}^i + \sigma_{e_1} u_{e_1}^i u_{e_1}^j + \\
-\sigma_{e_2} u_{e_2}^i + \sigma_{e_2} u_{e_2}^i u_{e_2}^j + \\
+D_{e_1} \phi_{e_1} + D_{e_1} \phi_{e_1}^i + \\
+D_{e_2} \phi_{e_2} + D_{e_2} \phi_{e_2}^i \\
\end{array} \right\} q_{ij} \left\{ \begin{array}{l}
\mathbf{J}_p \mathbf{w}_p \end{array} \right\} w_p
\]

Here, \(e_1\) is the number of elements in the integral domain \(A\); \(p\) is the number of integration points in one element; \(\mathbf{J}_p\) represents the determinant of Jacobian matrix; \(w_p\) is the corresponding weight factor at the integration point \(p\).

In this paper, the nonhomogeneous element method and the quadratures used by Yu et al. (2009) are adopted for piezoelectric materials.

### 7. Numerical examples and discussions

At first, two benchmark fracture problems are given to verify the accuracy of the interaction integral. Then, the domain-independence of the interaction integral is checked for material nonhomogeneity and discontinuity. Finally, our attention will be focused on the influences of the material continuity on the SIFs and the EDIF.
7.1. Example 1: A center crack in a homogeneous piezoelectric plate

As shown in Fig. 6(a), first example is a center-cracked homogeneous piezoelectric plate of length 2L and width 2W subjected to far-field tensile stress $\sigma_{\infty}$ and electric displacement $D_{\infty}$ on the remote boundary. Dimension L remains fixed at two times the larger of W to simulate an infinite-length plate here. The plate contains a horizontal crack of length 2a with the center coinciding with the origin. The analysis solutions of the mode-I SIF and the EDIF at the right crack tip for this problem (Tada, 1971) are, respectively,

$$K_I = k_I \sigma_{\infty} \sqrt{\pi a} \quad \text{and} \quad K_W = k_W D_{\infty} \sqrt{\pi a}$$

where $k_I$ is determined by

$$k_I = \sqrt{\frac{\pi}{2}} \left[ 1 - 0.025 \left( \frac{a}{W} \right)^2 + 0.06 \left( \frac{a}{W} \right)^4 \right]$$

For all examples in this paper, the polarization direction is along $x_2$-axis. Using the relation between the indices $1 \to 2, 22 \to 2, 33 \to 3, 23 \to 4, 31 \to 5$ and $12 \to 6$, the constitutive Eq. (1) can be written in Voigt notation as:

$$\sigma_{ij} = C_{ij \ell} \varepsilon_{\ell}, \quad D_{ij} = C_{ij \ell}^{D} \varepsilon_{\ell}$$

The material considered in this example is PZT-5H and for which material parameters are defined by

$$(C_{\ell \ell}, e_{\ell p}, K_{\ell}) = (C_{220}, e_{220}, K_{220})$$

where $C_{220}, e_{220}$ and $K_{220}$ are given in Table 1. In this example, $\sigma_{\infty}/D_{\infty} = \sqrt{C_{220}/K_{220}}$ is adopted to hold that the mechanical energy is equal to the electric energy. The data used in the analysis is as follows: $W = 1, L = 2, a = 0.1 \cdots 0.4; \sigma_{\infty} = 3 \times 10^6 \text{ Pa}; D_{\infty} = 10^{-3} \text{ C/m}^2$; generalized plane strain.

The mesh configuration is shown in Fig. 6(b). Eight-node quadrilateral (Q8) elements are used over most of the mesh. Since the stress and electric displacement fields exhibit an inverse square root singularity, six-node quarter-point (T6qp) singular elements are employed to improve the accuracy. The mesh consists of 1917 Q8 and 24 T6qp elements, with a total of 1941 elements.

The geometry and boundary conditions are shown in Figs. 8(a) and (b) for two types of loading combinations: Type (1): the far-field normal load $\sigma_{\infty}$ and the far-field electrical displacement load $D_{\infty}$. Type (2): the far-field shear load $\tau_{\infty}$ and the far-field electrical displacement load $D_{\infty}$. And the load combination parameters $\lambda_{\sigma} = D_{\infty} C_{220}/\sigma_{\infty}$ and $\lambda_{\tau} = D_{\infty} C_{16}/\tau_{\infty}$ are used for Type (1) and Type (2), respectively, to reflect the combination between the mechanical load and the electric load. The data used in the analysis are: $W = 10, a = 1; \lambda_{\sigma} = (5, 0.5); \zeta = 1/2 \cdots 1/2$; generalized plane strain. The same fracture problem has been investigated by Rao and Kuna (2008).

The mesh consists of 960 elements and 2926 nodes. The normalized intensity factors $K_I^0 = K_I/\sigma_{\infty} \sqrt{\pi a}$, $K_W^0 = K_W/D_{\infty} \sqrt{\pi a}$ and $K_{IV}^0 = K_{IV}/D_{\infty} \sqrt{\pi a}$ at the right crack tip are given in this example. Before giving the results, we first select seven integral domains of different size $(R/h_o = 1 \cdots 2^5)$ to compute the intensity factors for verifying the convergence of the interaction integral method. Fig. 9 shows the varying curves of the normalized intensity factors versus the integral domain size $R/h_o$. It can be found that as the

![Fig. 9. Normalized intensity factors vs the integral domain size $R/h_o$ for a functionally graded piezoelectric plate with a horizontal crack: (a) Type (1) loading; (b) Type (2) loading.](image-url)
integral domain increases, the intensity factors \( K_0^I, K_0^{II} \) and \( K_0^{IV} \) converge to a stable value and moreover, their relative changes are no more than 0.2% when \( R_i/h_i > 2 \). It indicates that the interaction integral is domain-independent for homogeneous and nonhomogeneous piezoelectric materials when the integral domain size reaches a threshold value (in this example, \( R_i/h_i > 2 \)).

### Table 3
Normalized intensity factors at the right crack tip for a functionally graded piezoelectric plate under far-field normal load \( \sigma_\infty \) and far-field electrical displacement load \( D_\infty \) (Example 2: Type (1) loading, \( K_0^I = K_I/\sigma_\infty \sqrt{\pi a} \) and \( K_0^{IV} = K_{IV}/D_\infty \sqrt{\pi a} \)).

<table>
<thead>
<tr>
<th>( \zeta )</th>
<th>Present</th>
<th>Rao and Kuna (2008)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_\infty = -5 )</td>
<td>( \lambda_\infty = 0 )</td>
<td>( \lambda_\infty = 5 )</td>
</tr>
<tr>
<td>( K_0^I )</td>
<td>( K_0^{II} )</td>
<td>( K_0^I )</td>
</tr>
<tr>
<td>( -1/2 )</td>
<td>1.0181</td>
<td>0.0913</td>
</tr>
<tr>
<td>( -1/4 )</td>
<td>1.2764</td>
<td>0.3913</td>
</tr>
<tr>
<td>( -1/8 )</td>
<td>1.1278</td>
<td>0.7490</td>
</tr>
<tr>
<td>0 \</td>
<td>1.0122</td>
<td>1.0020</td>
</tr>
<tr>
<td>( 1/8 )</td>
<td>1.2435</td>
<td>0.8686</td>
</tr>
<tr>
<td>( 1/4 )</td>
<td>1.4074</td>
<td>0.5166</td>
</tr>
<tr>
<td>( 1/2 )</td>
<td>1.6328</td>
<td>0.1326</td>
</tr>
</tbody>
</table>

### Table 4
Normalized intensity factors at the right crack tip for a functionally graded piezoelectric plate under far-field shear load \( \tau_\infty \) and far-field electrical displacement load \( D_\infty \) (Example 2: Type (2) loading, \( K_0^{II} = K_{II}/\tau_\infty \sqrt{\pi a} \) and \( K_0^{IV} = K_{IV}/D_\infty \sqrt{\pi a} \)).

<table>
<thead>
<tr>
<th>( \zeta )</th>
<th>Present</th>
<th>Rao and Kuna (2008)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_\infty = -5 )</td>
<td>( \lambda_\infty = 0 )</td>
<td>( \lambda_\infty = 5 )</td>
</tr>
<tr>
<td>( K_0^{II} )</td>
<td>( K_0^{IV} )</td>
<td>( K_0^{II} )</td>
</tr>
<tr>
<td>( -1/2 )</td>
<td>0.8721</td>
<td>0.2387</td>
</tr>
<tr>
<td>( -1/4 )</td>
<td>0.9396</td>
<td>0.5555</td>
</tr>
<tr>
<td>( -1/8 )</td>
<td>0.9743</td>
<td>0.8224</td>
</tr>
<tr>
<td>0 \</td>
<td>1.0075</td>
<td>1.0043</td>
</tr>
<tr>
<td>( 1/8 )</td>
<td>1.0376</td>
<td>0.9301</td>
</tr>
<tr>
<td>( 1/4 )</td>
<td>1.0648</td>
<td>0.0958</td>
</tr>
<tr>
<td>( 1/2 )</td>
<td>1.1148</td>
<td>0.3388</td>
</tr>
</tbody>
</table>

\[ \sigma_\infty, D_\infty \]

Polarization

\[ x_2 \]

\[ \theta \]

\[ x_1 \]

\[ 2W \]

\[ 2L \]

**Fig. 10.** A piezoelectric plate with an inclined crack AB: (a) geometry and boundary conditions; (b) finite element mesh.
Then, the intensity factor results obtained by letting \( R_l/h_e = 4 \) are listed in Table 3 and Table 4, respectively, for Type (1) loading and Type (2) loading. In comparison of the present results and those given by Rao and Kuna (2008), it can be found that the relative errors are all within 0.1% and 0.2%, respectively, for Type (1) loading; the relative errors of \( K_{pI} \) and \( K_{pII} \) are all within 0.4% and 0.1%, respectively, for Type (2) loading. Excellent agreements in Examples 1 and 2 demonstrate that the present interaction integral is valid for the fracture analysis of piezoelectric materials with continuous properties.

7.3. Example 3: Domain-independence of the interaction integral

In order to check the domain-independence of the interaction integral for material nonhomogeneity and discontinuity, as shown in Fig. 10(a), we select a piezoelectric plate on which a vertical interface exists at \( x_1 = 0 \) and the material parameters vary according to the following relations

\[
(C_{ij}, e_{ij}, K_{ij}) = \begin{cases} 
(C_{i0j}, e_{i0j}, K_{i0j}) & (x_1 \leq 0) \\
2(C_{i0j}, e_{i0j}, K_{i0j}) & (x_1 > 0) 
\end{cases}
\] (65)

In Eq.(65), \( C_{i0j}, e_{i0j} \) and \( K_{i0j} \) given in Table 1 are the values of the material parameters at \( x_1 = 0 \). The above definitions imply that each parameter varies continuously on the left-half plate, jumps at the middle line of the plate and then, holds a constant on the right-half plate. The plate of length 2L and width 2W is subjected to far-field tensile stress \( \sigma_\infty \) and electric displacement \( D_\infty \) on the remote boundary. The load combination parameter \( \lambda_\alpha = D_\infty C_{220}/\sigma_\infty C_{120} \) is still adopted. In this example, an inclined crack AB of length 2a occupies the segment from \( A (-4.6, -1) \) to \( B (0.6, -1) \). The following data are used for numerical analysis: \( L = 30; W = 10; \zeta = \ln(10)/2W; \lambda_\alpha = (-5.5); \) generalized plane strain.

Fig. 10(b) shows the corresponding mesh configuration which consists of 1993 elements and 6082 nodes. As shown in Figs. 11(a) and (b), eight integral domains (\( R_l/h_e = 3 \sim 3 \times 2^3 \)) are selected to check the variations of the intensity factors. According to Examples 1 and 2, it can be concluded that good accuracy can be obtained for \( R_l/h_e = 3 \). Since the domains \( R_l/h_e = 3 \sim 3 \times 2^3 \) do not contain the vertical interface, they are mainly used to check the domain-independence of the interaction integral for material nonhomogeneity. The domains \( R_l/h_e = 3 \sim 2^5 \sim 3 \times 2^7 \) are employed to check the domain-independence for material discontinuity.

In order to estimate the deviation of the results obtained by different integral domains, the relative error can be defined as

\[
\text{Err} = \frac{K_{\text{max}} - K_{\text{min}}}{K_{\text{mean}}} \times 100\%
\] (66)

where \( K_{\text{max}}, K_{\text{min}} \) and \( K_{\text{mean}} \) denote the maximum, minimum and mean of the intensity factors, respectively, obtained by different integral domains. Table 5 lists the normalized intensity factors \( K_{pI}(B) = K_{pI}(B)/\sigma_\infty \sqrt{\rho_0 a}, \ K_{pII}(B) = K_{pII}(B)/\sigma_\infty \sqrt{\rho_0 a}, \ K_{pIV}(B) = K_{pIV}(B)/D_\infty \sqrt{\rho_0 a} \) and the relative errors \( \text{Err} \). It can be observed that the relative errors are all within 0.22%, which implies the interaction integral should be domain-independent for nonhomogeneous and discontinuous piezoelectric materials. Therefore, it can be deduced that the interaction integral method is reliable for the piezoelectric materials with electromechanical interfaces.

7.4. Example 4: Influences of material continuity on the intensity factors

In this part, the influences of the material continuity on the intensity factors will be investigated. The model shown in Fig. 10(a) is still adopted. According to the continuity of the material parameters and their derivatives, we select four types of material parameters as shown in Fig. 12.

(1) Case 1: The material parameters \( C_{i0j}, e_{i0j} \) and \( K_{i0j} \) are discontinuous at \( x_1 = 0 \). Their definitions are given in Eq. (65).

<table>
<thead>
<tr>
<th>( h_l )</th>
<th>( \lambda_\alpha = -5 )</th>
<th>( \lambda_\alpha = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_{pI}(B) )</td>
<td>( K_{pII}(B) )</td>
<td>( K_{pIV}(B) )</td>
</tr>
<tr>
<td>3</td>
<td>0.89632</td>
<td>0.34375</td>
</tr>
<tr>
<td>3</td>
<td>0.89601</td>
<td>0.33360</td>
</tr>
<tr>
<td>3</td>
<td>0.89536</td>
<td>0.34346</td>
</tr>
<tr>
<td>3</td>
<td>0.89598</td>
<td>0.34349</td>
</tr>
<tr>
<td>3</td>
<td>0.89626</td>
<td>0.34302</td>
</tr>
<tr>
<td>3</td>
<td>0.89674</td>
<td>0.34341</td>
</tr>
<tr>
<td>3</td>
<td>0.89644</td>
<td>0.34350</td>
</tr>
<tr>
<td>3</td>
<td>0.89642</td>
<td>0.34347</td>
</tr>
<tr>
<td>( E_r (%) )</td>
<td>0.15</td>
<td>0.21</td>
</tr>
</tbody>
</table>
Case 2: The material parameters are continuous at $x_1 = 0$, but their derivatives are discontinuous. They are defined as

$$
(C_{ij}, e_{ij}, K_{ij}) = \begin{cases} 
(C_{ij0}, e_{ij0}, K_{ij0}) & (x_1 \leq 0) \\
(C_{ij}, e_{ij0}, K_{ij0}) & (x_1 > 0)
\end{cases}
$$

(67)

Here, it should be noted that every material parameter at right half-plate is the constant term truncated from Taylor series of that at left half-plate.

Case 3: The material parameters and their first-order derivatives are continuous at $x_1 = 0$, but their high-order derivatives are discontinuous. They are defined as

$$
(C_{ij}, e_{ij}, K_{ij}) = \begin{cases} 
(C_{ij0}, e_{ij0}, K_{ij0}) e^{x_1} & (x_1 \leq 0) \\
(C_{ij}, e_{ij0}, K_{ij0}) (1 + \zeta x_1) & (x_1 > 0)
\end{cases}
$$

(68)

Here, every material parameter at right half-plate is the first-order polynomial truncated from Taylor series of that at left half-plate.

Case 4: The material parameters and their derivatives are continuous at $x_1 = 0$. Their definitions are given in Eq. (64).

---

**Fig. 12.** Four types of material parameters with different continuity.

**Fig. 13.** Normalized mode-I SIFs $K_{ij}$ vs crack center $c/W$ for four types of materials with different continuity: (a) at crack tip A; (b) at crack tip B.

**Fig. 14.** Normalized EDIFs $K_{ij}$ vs $c/W$ for four types of materials with different continuity: (a) at crack tip A; (b) at crack tip B.
In this example, a horizontal crack with center denoted by \( C(0, 0) \) is considered. The crack length is fixed and its location moves from the center of the left-half plate to the center of the right-half plate, which leads to \( c/W = -0.5 \) to 0.5. As a result, the two crack tips will cross the vertical interface at \( x_1 = 0 \) one after the other. The data used in numerical analysis are: \( L = 30; \ W = 10; \ a/W = 0.1; \ \theta = 0; \ \zeta = \ln(10)/2W; \ \lambda_\sigma = 5; \ K_{I}^0 = K_{I}/\sqrt{\pi a}; \ K_{IV}^0 = K_{IV}/\sqrt{\pi a}; \ \) generalized plane strain.

Figs. 13 and 14 show the normalized mode-I SIF \( K_I^0 \) and the normalized EDIF \( K_{IV}^0 \) varying with crack center \( c/W \), respectively. It can be observed from Fig. 10(a) that for the crack length \( a/W = 0.1 \), the crack tip \( A(B) \) just reaches the vertical interface for the crack center \( c/W = 0.1 \). Therefore, in Figs. 13 and 14, a vertical dash line at \( c/W = 0.1 \) is used to indicate where the crack tip \( A(B) \) is just on the vertical interface for the crack center line at \( x_1 = 0 \) one after the other. The data used in numerical analysis are: \( L = 30; \ W = 10; \ a/W = 0.1; \ \theta = 0; \ \zeta = \ln(10)/2W; \ \lambda_\sigma = 5; \ K_{I}^0 = K_{I}/\sqrt{\pi a}; \ K_{IV}^0 = K_{IV}/\sqrt{\pi a}; \ \) generalized plane strain.

Figs. 13 and 14 show the normalized mode-I SIF \( K_I^0 \) and the normalized EDIF \( K_{IV}^0 \) varying with crack center \( c/W \), respectively. It can be observed from Fig. 10(a) that for the crack length \( a/W = 0.1 \), the crack tip \( A(B) \) just reaches the vertical interface for the crack center \( c/W = 0.1 \). Therefore, in Figs. 13 and 14, a vertical dash line at \( c/W = 0.1 \) is used to indicate where the crack tip \( A(B) \) is just on the vertical interface for the crack center line at \( x_1 = 0 \) one after the other. The data used in numerical analysis are: \( L = 30; \ W = 10; \ a/W = 0.1; \ \theta = 0; \ \zeta = \ln(10)/2W; \ \lambda_\sigma = 5; \ K_{I}^0 = K_{I}/\sqrt{\pi a}; \ K_{IV}^0 = K_{IV}/\sqrt{\pi a}; \ \) generalized plane strain.

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Figs. 13 and 14 show the normalized mode-I SIF \( K_I^0 \) and the normalized EDIF \( K_{IV}^0 \) varying with crack center \( c/W \), respectively. It can be observed from Fig. 10(a) that for the crack length \( a/W = 0.1 \), the crack tip \( A(B) \) just reaches the vertical interface for the crack center \( c/W = 0.1 \). Therefore, in Figs. 13 and 14, a vertical dash line at \( c/W = 0.1 \) is used to indicate where the crack tip \( A(B) \) is just on the vertical interface for the crack center line at \( x_1 = 0 \) one after the other. The data used in numerical analysis are: \( L = 30; \ W = 10; \ a/W = 0.1; \ \theta = 0; \ \zeta = \ln(10)/2W; \ \lambda_\sigma = 5; \ K_{I}^0 = K_{I}/\sqrt{\pi a}; \ K_{IV}^0 = K_{IV}/\sqrt{\pi a}; \ \) generalized plane strain.
Appendix A

In the local polar coordinate system shown in Fig. 1, the angular functions \( f_{ij}^\theta (\varphi) \), \( g_{ij}^\theta (\varphi) \), \( d^\theta (\varphi) \) and \( p^\theta (\varphi) \) can be obtained by means of the extended stroph formalism and semi-analytical calculations. Only 2D problems are considered in this paper. Therefore, \( K_{yy} = 0 \) and the subscripts \( i, j = 1, 2 \) in these angular functions. The functions can be expressed in terms of complex material eigenvalues \( P_e \) eigenvectors \( A_{me} \), and matrices \( M_{ee} \) and \( N_{ee} \) (Park and Sun, 1995; Rao and Kuna, 2008).

\[
\begin{align*}
    f_{ij}^\theta &= \frac{4}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{M_{e}N_{e}P_{e}}{\cos^2 \theta + p_{e} \sin^2 \theta} \right\}, \\
    g_{ij}^\theta &= \frac{4}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{M_{e}N_{e}Q_{e}}{\cos \theta + p_{e} \sin \theta} \right\}, \\
    d^\theta &= \frac{4}{\pi} \sum_{n=1}^{\infty} \left\{ A_{me}N_{e} \sqrt{\cos \theta + p_{e} \sin \theta} \right\}, \\
    p^\theta &= \frac{4}{\pi} \sum_{n=1}^{\infty} \left\{ A_{me}N_{e} \sqrt{\cos \theta + p_{e} \sin \theta} \right\}.
\end{align*}
\] (A1)

Here, \( \text{Re}(\bullet) \) and \( \text{Im}(\bullet) \) denote the real part and the imaginary part respectively of the quantity in brackets. The four conjugate pairs of eigenvalues \( P_e \) and the \((4 \times 4)\) matrix of eigenvectors \( A_{me} \) can be obtained by solving the following quadratic eigenvalue problem:

\[
\begin{bmatrix}
    Q + (R + R^T)p + Tp^T
\end{bmatrix}
\begin{bmatrix}
    A_i
\end{bmatrix}
= 0
\] (A3)

where

\[
Q = \begin{bmatrix}
    C_{ij1}^{\text{tip}} & e_{ij1}^{\text{tip}} \\
    e_{ij1}^{\text{tip}} & -k_{ij1}^{\text{tip}}
\end{bmatrix},
R = \begin{bmatrix}
    C_{1k1}^{\text{tip}} & e_{1k1}^{\text{tip}} \\
    e_{1k1}^{\text{tip}} & -k_{1k1}^{\text{tip}}
\end{bmatrix},
T = \begin{bmatrix}
    C_{2k2}^{\text{tip}} & e_{2k2}^{\text{tip}} \\
    e_{2k2}^{\text{tip}} & -k_{2k2}^{\text{tip}}
\end{bmatrix}
\] (A4)

where \( C_{ij1}^{\text{tip}}, e_{ij1}^{\text{tip}} \) and \( k_{ij1}^{\text{tip}} \) are the elastic stiffness, piezoelectric coefficient and dielectric permittivity tensors respectively, evaluated at the crack tip location. Eq. (A3) can be converted into the following eigenrelations (Hwu, 2008):

\[
\begin{bmatrix}
    -T^{-1} R & T^{-1}
\end{bmatrix}
\begin{bmatrix}
    \text{RT} & -\text{Q}
\end{bmatrix}
\begin{bmatrix}
    \xi
\end{bmatrix}
= \begin{bmatrix}
    p
\end{bmatrix}
\] (A5)

where the eigenvector \( \xi = \begin{bmatrix} A_1^T & B_1^T \end{bmatrix}^T \), \( A_1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \end{bmatrix}^T \) and \( B_1 = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \end{bmatrix}^T \). \( A_2 \) and \( B_2 \) satisfy the following relation

\[
\begin{bmatrix}
    A_2
\end{bmatrix}
= \begin{bmatrix}
    R & p_z T
\end{bmatrix}
\begin{bmatrix}
    A_1
\end{bmatrix}
= \begin{bmatrix}
    -1
\end{bmatrix}
\begin{bmatrix}
    Q & p_z R
\end{bmatrix}
\begin{bmatrix}
    A_1
\end{bmatrix}
\] (A6)

Only the four eigenvalues \( p_z \) having positive imaginary part and the corresponding eigenvectors are used in Eqs. (A1) and (A2). The \((4 \times 4)\) matrices \( M_{ee} \) and \( N_{ee} \) are calculated by (Park and Sun, 1995).

\[
N^{-1} = M = \begin{bmatrix}
    (C_{121} + C_{122} P_2) A_{kw} + (e_{121} + e_{122} P_2) A_{kl} \\
    (C_{121} + C_{122} P_2) A_{kw} + (-k_{21} - k_{22} P_2) A_{kl}
\end{bmatrix}
\] (A7)

In Eq. (A6), the summation convention is valid only on \( k \), but not on \( x \).

Appendix B

The Irwin matrix \( Y \) is defined as (Ricoeur and Kuna, 2003)

\[
Y = Y_{mn} = -\sum_{n=1}^{4} \text{Im}(A_{mn} N_{mn})
\] (A8)

It is necessary to pointed out that in the symbols \( A_{me}, M_{ee}, N_{ee} \) and \( Y_{mn} \), the indices \( M, N = \{ I, II, III, IV \} \) denote the crack opening modes with the values corresponding to a general index \( i = \{ 1, 2, 3, 4 \} \), respectively.

References


