Some expressions are given for the determinant of an $m \times m$ block-Toeplitz band matrix $L = [L_{i-j}]$, with bandwidth $(p + q + 1)n < mn$, in terms of the $n \times n$ generating matrix polynomial $L(\lambda) = \sum_{j=0}^{p} \lambda^{j} L_{p-j}$, $\det L_{p-q} \neq 0$. In the scalar case this yields formulas for the determinant expressed via the zeros of the generating (scalar) polynomial. The approach adopted in this work leans heavily on the recently developed spectral theory of matrix polynomials.

0. PRELIMINARIES

The spectrum of an $n \times n$ matrix polynomial $L(\lambda) = \sum_{j=0}^{l} \lambda^{j} L_{j}$, $L_{j} \in \mathbb{C}^{n \times n}$, $j = 0, 1, \ldots, l$ of degree $l$ is defined by the set $\sigma(L) = \{ \lambda \in \mathbb{C} : \det L(\lambda) = 0 \}$. It will be assumed throughout the work that the leading coefficient $L_{1}$ of $L(\lambda)$ is an invertible $n \times n$ matrix.

Recall (see [3]) that any matrix polynomial $L(\lambda)$ with an invertible leading coefficient can be associated with a triple of matrices $(X, T, Y)$ having the following properties: The matrix $X$ has size $n \times nl$; the $nl \times nl$ matrix $T$ is such that the matrix

$$
\Omega = \text{col}(XT^{j})_{j=0}^{l} := \begin{bmatrix}
X \\
XT \\
\vdots \\
XT^{l-1}
\end{bmatrix}
$$

The matrix $T$ is such that the matrix

$$
\Omega = \text{col}(XT^{j})_{j=0}^{l} := \begin{bmatrix}
X \\
XT \\
\vdots \\
XT^{l-1}
\end{bmatrix}
$$

The same holds for the case when $L(\lambda)$ has complex coefficients.
is nonsingular and

$$\sum_{j=0}^{l} L_{j}X^{T}Y = 0. \quad (0.1)$$

Further, the matrix $Y$ is $nl \times n$ and is the unique solution of the equation $\Omega Y = [0 \cdots 0 (L_{l}^{-1})^{T}]^{T}$. Any triple $(X, T, Y)$ satisfying the above conditions is called a standard triple for $L(\lambda)$. The pair $(X, I')$ is referred to as a standard pair for $L(\lambda)$.

It is clear from the definition that

$$XT^{j}Y = \begin{cases} 0, & j = 0, 1, \ldots, l - 2, \\ L_{j}^{T}, & j = l - 1. \end{cases} \quad (0.2)$$

Also,

$$\sum_{j=0}^{l} T^{j}YL_{j} = 0. \quad (0.3)$$

To ease the use of our main reference [3] it is relevant to point out the relation between the standard triples of $L(\lambda)$ and the associated monic matrix polynomial $\hat{L}(\lambda) = L_{l}^{-1}L(\lambda)$. Namely, $(X, T, Y)$ is a standard triple for $L(\lambda)$ if and only if $(X, T, YT_{l})$ is a standard triple for $\hat{L}(\lambda)$.

Note two important examples of a standard triple for $L(\lambda)$. The first is the Jordan triple $(Q, J, R)$, where $Q = [Q_{1}Q_{2} \cdots Q_{s}]$, $J = [\delta_{jk}J_{j}]_{j,k=1}^{s}$, and $Q_{j}, 1 \leq j \leq s$, consists of a canonical system of Jordan chains for $L(\lambda)$ corresponding to an eigenvalue $\lambda_{j}$ of $L(\lambda)$. The matrix $J_{j}$ is here a direct sum of the Jordan blocks associated with $\lambda_{j}$ (see [3] for definitions). The matrix $R$ is uniquely defined by the matrices $Q$ and $J$. Note that in the linear case $l = 1$ and $L_{1} = -I$ the matrix $J$ just coincides with the Jordan canonical form for $L_{0}$, while $Q = R^{-1}$ performs the corresponding similarity transformation.

Another example of a standard triple for $L(\lambda)$ is provided by the companion triple $(X_{0}, C_{L}, Y_{0})$, where

$$X_{0} = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}, \quad Y_{0} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} (L_{l}^{-1})^{T}.$$ 

$I$ stands for the $n \times n$ identity matrix, and $C_{L}$ denotes a companion matrix.
DETERMINANT OF BAND MATRICES

associated with the matrix polynomial $L(\lambda)$:

$$C_L = \begin{bmatrix}
0 & I & & \\
& 0 & I & \\
& & \ddots & I \\
-\tilde{L}_0 & -\tilde{L}_1 & \cdots & -\tilde{L}_l \\
\end{bmatrix}$$

where $\tilde{L}_j = L_{j+1}^{-1}L_j$, $j = 0, 1, \ldots, l$.

Note that the standard triples $(X_1, T_1, Y_1)$ and $(X_0, T_0, Y_0)$ of a given $n \times n$ matrix polynomial $L(\lambda)$ of degree $l$ are similar. That is,

$$X_1 = X_2S, \quad T_1 = S^{-1}T_2S, \quad Y_1 = S^{-1}Y_2$$

for some invertible $n \times n$ matrix $S$.

It turns out [3] that a triple of matrices $(X, T, Y)$ of sizes $n \times nl$, $nl \times nl$, $nl \times n$, respectively, is a standard triple for $L(\lambda)$ if and only if the following representation of the resolvent holds:

$$L^{-1}(\lambda) = X(\lambda I - T)^{-1}Y$$

(0.4)

where $\lambda \notin \sigma(T)$, the spectrum of $T$.

Let $\Gamma$ denote a rectifiable simple closed contour in $\mathbb{C}$ (for brevity, contour, in the sequel) containing $\sigma(T)$ [or, what is equivalent, $\sigma(L)$] in its interior. Since

$$\frac{1}{2\pi i} \int_{\Gamma} \lambda ^j (\lambda I - T)^{-1} d\lambda = T^j, \quad j = 0, 1, \ldots$$

(0.5)

it follows from (0.4) that

$$\frac{1}{2\pi i} \int_{\Gamma} \lambda ^j L^{-1}(\lambda) d\lambda = XT^jY$$

(0.6)

for $j = 0, 1, \ldots$.

We adopt the following notational conventions. Given an $s \times s$ matrix $A$, the submatrix of $A$ containing rows numbered $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_r$ is denoted by $A[i_1, \ldots, i_k|j_1, \ldots, j_r]$. We also use the notation
\[ \text{diag}[Z]_{1} := [\delta_{jk}Z]_{j,k-1}, \delta_{jk} \text{ being the Kronecker symbol, and} \]

\[ \begin{bmatrix} Z_{1} \\ Z_{2} \\ \vdots \\ Z_{k} \end{bmatrix}, \quad \text{row}(Z)_{j=1} := [Z_{1} \ Z_{2} \ \cdots \ Z_{j}] \]  

Throughout the paper the integer \( n \) stands for the size of the matrix polynomial and is fixed. Further, \( I_{q} \) denotes the \( qn \times qn \) identity matrix, \( I_{r} = I \), while \( P_{s} \) stands for the \( sn \times sn \) reverse unit matrix \([\delta_{jk}, j=1]_{j-1}\).

Also,

\[ \tilde{L}_{s,p} = \begin{bmatrix} L_{s} \\ L_{s-1} \ L_{s} \\ \vdots \ & \ddots \ & \ddots \\ L_{p} \ & \cdots \ & L_{s-1} \ L_{s} \end{bmatrix} \]  

1. BLOCK-TOEPLITZ-HESSENBERG MATRIX

Consider an \( mn \times mn \) block-Toeplitz band matrix of Hessenberg's type

\[ \mathcal{L}_{m} = \begin{bmatrix} L_{k} & 0 & \cdots \\ L_{k-2} & L_{k} & \cdots \\ \vdots & \vdots & \ddots \\ L_{0} & \cdots & \cdots & L_{k} \\ 0 & \cdots & \cdots & \cdots & L_{k} \end{bmatrix} \]

\[ \mathcal{L}_{m} \in \mathbb{C}^{mn \times mn}, \]  

where \( L_{j} \in \mathbb{C}^{n \times n}, j = 0,1,\ldots, k, \) and \( 1 \leq k < m \). Introduce the \( n \times n \) matrix
polynomial

\[ L(\lambda) = L_0 + \lambda L_1 + \cdots + \lambda^k L_k \]

associated with the matrix \( \mathcal{L}_m \).

**Theorem 1.** If \( \det L_k \neq 0 \), then the determinant of \( \mathcal{L}_m \) in (1.1) is given by the formula

\[
\det \mathcal{L}_m = (-1)^{(m-1)n^2 + 1} (\det L_k)^{m+1} \left( \frac{1}{2\pi i} \int_{\Gamma} \lambda^{m+k-l} L^{-1}(\lambda) d\lambda \right),
\]

(1.2)

where \( \Gamma \) stands for a contour containing the spectrum of the generating matrix polynomial \( L(\lambda) \) in its interior.

**Proof.** Denote by \( F_m \) the matrix obtained from the \( mn \times mn \) identity matrix on replacing its first \( n \) columns by the matrix \( [S_0^T S_1^T \cdots S_{m-1}^T]^T \), where \( S_0 = I \) and \( S_i \in \mathbb{C}^{n \times n} \), \( i = 1, 2, \ldots, m - 1 \). Consider the product \( G_m = \mathcal{L}_m F_m \). Clearly, the elements of \( G_m \), except those of the first block column, coincide with the corresponding elements of \( \mathcal{L}_m \). Note also that

\[
\det \mathcal{L}_m = \det G_m
\]

(1.3)

for any choice of the matrices \( S_i \) \( (i = 1, 2, \ldots, m - 1) \).

Aiming at an easy computation of the determinant of \( G_m \), we choose the blocks \( S_i \) \( (i = 1, 3, \ldots, m - 1) \) so that the \( n \times n \) blocks \( G_{ij} \) \( (i = 1, 2, \ldots, m - 1) \) equal the zero matrix. This yields the relations

\[
\sum_{i=0}^{j} L_{k-i} S_{j-i} = 0 \quad (j = 1, 2, \ldots, k),
\]

(1.4)

\[
\sum_{i=0}^{k} L_i S_{i+j} = 0 \quad (j = 1, 2, \ldots, m - k - 1).
\]

(1.5)

Now view the equations (1.5) as a part of the matrix difference equation

\[
L_0 S_j + L_1 S_{j+1} + \cdots + L_k S_{j+k} = 0 \quad (j = 1, 2, \ldots)
\]

(1.6)
with initial conditions (1.4) rewritten in the form [see (0.8)]

\[
\text{col}(S_i)_{i-1}^k = -I_{k-1}^{-1}\text{col}(L_i)_{i-1}^k.
\]

(1.7)

Since

\[
G_{m1} = L_0S_{m-k} + L_1S_{m-k+1} + \cdots + L_kS_{m-1},
\]

it follows from (1.6) that \(G_{m1} = -I_kS_m\) and hence, in view of (1.3),

\[
\det \mathcal{L}_m = (-1)^{m-1} (\det I_k)^{m-1} (\det S_m),
\]

(1.8)

where \((S_1, S_2, \ldots, S_u, \ldots)\) is the solution of the matrix difference equation (1.6) with initial conditions (1.7).

The desired solution is given by the Lancaster formula [9]

\[
S_m = X T^{m-1} c,
\]

(1.9)

where

\[
c = \text{row}((T^k)'Y)_{i=1}^k I_{k-1}^{-1}\text{col}(S_i)_{i=1}^k
\]

(1.10)

and \((X, T, Y)\) stands for a standard triple for the matrix polynomial \(L(\lambda)\).

Substituting from (1.7) in (1.10), we obtain from (0.3)

\[
c = -\sum_{j=0}^{k-1} T^jYL_j = T^kYL_k.
\]

Appealing to (1.9) and (1.8), we thus have

\[
\det \mathcal{L}_m = (-1)^{m-1} (\det I_k)^{m-1} \det(X T^{m-k} Y).
\]

(1.11)

In view of (0.6), the proof is complete. 

It is easily seen that for \(|\lambda|\) sufficiently large, the \(n \times n\) matrix

\[
Z_j = \frac{1}{2\pi i} \int_{\Gamma} \lambda^j L^{-1}(\lambda) d\lambda
\]
is the coefficient of $\lambda^{-j-1}$ in the Laurent expansion

$$L^{-1}(\lambda) = \lambda^{-k}Z_{k-1} + \lambda^{-k+1}Z_k + \cdots \quad (|\lambda| > |\lambda_0| > 0) \quad (1.12)$$

of the resolvent $L^{-1}(\lambda)$. Proceeding to the reverse matrix polynomial

$$L_\infty^{-1}(\lambda) := \lambda^k L(\lambda^{-1}) = \lambda^k L_0 + \lambda^{k-1}L_1 + \cdots + \lambda L_{k-1} + L_k,$$

we obtain from (1.12) the following expansion in power series:

$$L_\infty^{-1}(\lambda) = \sum_{j=k}^{\infty} \lambda^{j-k-1}Z_j = \sum_{j=0}^{\infty} \lambda^j Z_{j+k-1} \left( |\lambda| < \frac{1}{|\lambda_0|} \right).$$

Setting $L_k = I$ in (1.2), we arrive at a generalization of the well-known Wronsky formula [5] for monic matrix polynomials.

**Corollary 1.** Let $M(\lambda) = I + \lambda M_1 + \cdots + \lambda^k M_k$, and for $|\lambda|$ sufficiently small

$$M^{-1}(\lambda) = \sum_{j=0}^{\infty} (-1)^{i} \lambda^j W_j.$$  

Then for $j = 0, 1,\ldots$

$$W_j = \frac{(-1)^j}{2\pi i} \int_{\Gamma} \lambda^{i+k-1} M_\infty^{-1}(\lambda) \, d\lambda,$$

where the contour $\Gamma$ is defined in Theorem 1, and for $j > k$

$$\det W_j = \det \begin{bmatrix} M_1 & I \\ M_2 & M_1 & I \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & I \\ & & & & M_k & \cdots & \cdots & M_1 \end{bmatrix} \quad (\in \mathbb{C}^{in \times jn}).$$
Concerning the scalar case \( n = 1 \), we deduce the following result stated in a different form in [7].

**COROLLARY 2.** If \( n = 1 \) in (1.1) and the (scalar) polynomial \( L(\lambda) \) \((L_k \neq 0)\) has distinct zeros \( \lambda_1, \lambda_2, \ldots, \lambda_s \) with multiplicities \( r_1, r_2, \ldots, r_s \), \((r_1 + r_2 + \cdots + r_s = k)\) respectively, then

\[
\det \mathcal{L}_m = (-1)^m L_k^{m+1} \sum_{i=1}^{s} \frac{1}{(r_i - 1)!} \lim_{\lambda \to \lambda_i} \frac{d^{r_i-1}}{d\lambda^{r_i-1}} \left[ (\lambda - \lambda_i)^{r_i} f(\lambda) \right] \quad (1.13)
\]

where \( f(\lambda) = \lambda^{m+k-1} / L(\lambda) \) and \( k < m \). In particular, if the zeros of \( L(\lambda) \) are all simple, then

\[
\det \mathcal{L}_m = (-1)^m L_k^{m+1} \sum_{i=1}^{k} \frac{\lambda_i^{m+k-1}}{L(\lambda_i)} \quad (1.14)
\]

**Proof.** By the residue theorem,

\[
\frac{1}{2\pi i} \int \frac{f(\lambda)}{L(\lambda)} d\lambda = \sum_{i=1}^{s} \text{Res} f(\lambda_i),
\]

and by the relation

\[
\text{Res} f(\lambda_i) = \frac{1}{(r_i - 1)!} \lim_{\lambda \to \lambda_i} \frac{d^{r_i-1}}{d\lambda^{r_i-1}} \left[ (\lambda - \lambda_i)^{r_i} f(\lambda) \right]
\]

for \( i = 1, 2, \ldots, s \), the formula (1.13) follows.

**Example 1.** Let \( \mathcal{L}_m \) be a tridiagonal Toeplitz matrix generated by the (scalar) polynomial \( L(\lambda) = 3 - \lambda - 2\lambda^2 \) with zeros \( \lambda_1 = 1, \lambda_2 = -\frac{3}{2} \). Then by (1.14)

\[
\det \mathcal{L}_m = (-1)^m (-2)^{m+1} \left[ \frac{1}{-5} + \frac{\left(-\frac{3}{2}\right)^{m+1}}{5} \right] = \frac{2^{m+1}}{5} \left[ 1 - \left(-\frac{3}{2}\right)^{m+1} \right]
\]

for \( m > 2 \).
As noted by the referee, Theorem 1, as stated, can be proved without using the spectral theory of matrix polynomials. Indeed,

\[ L(\lambda)(Z_0\lambda^{-1} + Z_1\lambda^{-2} + \cdots) = I, \]

and hence the Laurent coefficients \( Z_j \) of \( L(\lambda)^{-1} \) at infinity solve (1.6). It turns out, however, that in this case we lose the possibility to compute \( \det \mathcal{L}_m \) by (1.11) in which the companion triple is taken, for instance.

2. GENERAL BLOCK-TOEPLITZ BAND MATRIX

Let now

\[ L = \begin{bmatrix}
L_{k-1} & L_k & & L_l \\
L_{k-2} & L_{k-1} & & \vdots \\
& \ddots & \ddots & \vdots \\
& & \ddots & L_l \\
L_0 & & & \\
& & \ddots & L_k \\
& & & L_0 & \cdots & L_{k-1}
\end{bmatrix} \]

\[ \left( L_j \in \mathbb{C}^{n \times n}, \quad j = 0, 1, \ldots, l \right) \] \hspace{1cm} (2.1)

denote an \( mn \times mn \) block-Toeplitz band matrix such that \( 1 \leq k \leq l < m \), and let

\[ L(\lambda) = L_0 + \lambda L_1 + \cdots + \lambda^l L_l, \quad \det L_l \neq 0. \]

**Theorem 2.** If \( \det L_l \neq 0 \), then the determinant of the matrix \( \mathcal{L}_m \) in (2.1) is given by the formula

\[ \det \mathcal{L}_m = (-1)^{m-q}(\det L_l)^{m-q}(\det M_q), \] \hspace{1cm} (2.2)
where \( k \leq l < m, \quad q = l - k + 1, \) and

\[
M_q = \frac{1}{2\pi i} \int \lambda^{m-l-q} \begin{bmatrix}
\lambda^{q-1} L^{-1}(\lambda) & \cdots & \lambda L^{-1}(\lambda) & L^{-1}(\lambda) \\
\lambda^q L^{-1}(\lambda) & \cdots & \lambda^2 L^{-1}(\lambda) & \lambda L^{-1}(\lambda) \\
\vdots & & \vdots & \vdots \\
\lambda^{2q} L^{-1}(\lambda) & \cdots & \lambda^q L^{-1}(\lambda) & \lambda^q L^{-1}(\lambda)
\end{bmatrix} d\lambda,
\]

in which \( \Gamma \) denotes a contour in the complex plane containing the spectrum of the generating matrix polynomial \( L(\lambda) \) in its interior.

**Proof.** To compute the determinant of \( \mathcal{L}_m \) in (2.1), we first note that the matrices \( S_l \) used in the proof of Theorem 1 have been chosen to be of the form \( XT^i Y L^{-i} \), where \((X, T, Y)\) is a standard triple for the generating matrix polynomial with the leading coefficient \( L^{-i} \). This observation prompts how to generalize the procedure exploited in Theorem 1 for calculating \( \det \mathcal{L}_m \).

Define \( mn \times m \) matrices

\[
R_j = \text{col}(XT^i Y L^{-i})_{i=0}^{m-1} \quad (j = l - 1, l - 2, \ldots, k - 1),
\]

where \((X, T, Y)\) is a standard triple for \( L(\lambda) \). Denote by \( F_m \) the matrix obtained from \( L_m \) on replacing its first \( qn \) columns by the matrix \([R_{t-1} \ R_{t-2} \ \cdots \ R_{k-1}]\). The relations (0.2) show that \( F_m \) is a lower triangular matrix with ones on the main diagonal. Let \( \mathcal{L}_m^{(S)} = \mathcal{L}_m[1, 2, \ldots, sn, 1, 2, \ldots, mn] \). The relation (0.1) yields for \( j > k - 1 \)

\[
\mathcal{L}_m^{(k)} R_j = - P_{k-1} \mathcal{L}_m^{(k)} R_{k-1} \text{col}(XT^j Y L^{-j})_{i=0}^{m-1},
\]

and it thus follows from (0.2) that

\[
\mathcal{L}_m^{(k-1)} R_j = 0
\]

for \( j \leq l - 1 \). Further, the relation (0.1) implies

\[
\mathcal{L}_m^{(m-q)} R_j = 0,
\]

(2.4)
and for $k - 1 \leq j \leq l - 1$

$$\mathcal{L}_m^{|1|} R_j = -I_{l, l - q - 1} \text{col}(XT^{m - j + 1})^T_{i = 0}^q, \quad (2.6)$$

where $\mathcal{L}_m^{|1|} = \mathcal{L}_m[(m \quad q)n \mid 1, \ldots, mn][1, 2, \ldots, mn]$. Combining (2.4)-(2.6), we obtain

$$\mathcal{L}_m F_m = \begin{bmatrix} I_{m - q} & 0 \\ 0 & -L_{l, l - q - 1} \end{bmatrix} G_m, \quad (2.7)$$

where $G_m$ differs from $\mathcal{L}_m$ by the first $qn$ columns:

$$G_m[1, 2, \ldots, mn][1, 2, \ldots, qn] = \begin{bmatrix} 0 \\ \tilde{X} \end{bmatrix} \text{diag}[L_{l}]_{1}^{q}.$$

with the $qn \times qn$ block-Toeplitz matrix

$$\tilde{X} = \begin{bmatrix} XT^{m - l + 1}Y & \cdots & XT^{m - k + 1}Y \\ XT^{m - l}Y & \cdots & XT^{m - k}Y \\ \vdots & \vdots & \vdots \\ XT^{m - l + q}2Y & \cdots & XT^{m - l + 1}Y \end{bmatrix}, \quad (2.8)$$

It is now easily checked that in view of (2.7)

$$\det \mathcal{L}_m = \det \mathcal{L}_m F_m = (-1)^{(m - q)n^2 + q}(\det L_{l})^{m - q}(\det \tilde{X}), \quad (2.9)$$

which along with (0.6) gives the required result.

The following immediate consequence of Theorem 2 extends the assertion of Corollary 2 to an arbitrary Toeplitz band matrix. This formula, even for the case of arbitrary rational functions, was given by Day [2]. For other proofs see [1] and [6]. It is stated in [1] that Day's formula was generalized in [4]. The latter article is difficult to obtain, and the author has not seen it.

**Corollary 3 [2].** Let $n = 1$, and the (scalar) polynomial $L(\lambda)$ ($L_{l} \neq 0$) have distinct zeros $\lambda_1, \lambda_2, \ldots, \lambda_s$ with multiplicities $r_1, r_2, \ldots, r_s$ ($r_1 + r_2$
+ \cdots + r_s = 1), respectively. Then for \( m > l \)

\[
\det \mathcal{L}_m = (-1)^m L_m^{m+q} \det \left( \sum_{i=1}^{s} A_i \right),
\]

where \( i = 1, 2, \ldots, s \)

\[
A_i = \frac{1}{(r_i - 1)!} \lim_{\lambda \to \lambda_i} \frac{d^{r_i-1}}{d\lambda^{r_i-1}} \left[ (\lambda - \lambda_i)^q f(\lambda) \right]
\]

\[
\times \begin{bmatrix}
\lambda_i^{q-1} & \cdots & \cdots & \lambda_i & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
\lambda_i^{2q-2} & \cdots & \cdots & \lambda_i^{q-1} & \\
\end{bmatrix}
\]

with \( f(\lambda) = \lambda^{m-t-q} / L(\lambda) \).

In particular, if the zeros of \( I(\lambda) \) are all simple, then for \( m > l \)

\[
\det \mathcal{L}_m = (-1)^m L_m^{m+q} \det \left( \sum_{j=1}^{l} A_j \right),
\]

where

\[
A_j = \frac{\lambda_j^{m-t-q}}{L'(\lambda_j)} \begin{bmatrix}
\lambda_j^{q-1} & \cdots & \cdots & \lambda_j & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
\lambda_j^{2q-2} & \cdots & \cdots & \lambda_j^{q-1} & \\
\end{bmatrix}
\]

3. EXPRESSION FOR \( \det \mathcal{L}_m \) IN TERMS OF SOLVENTS

Following [8], we say that an \( n \times n \) matrix \( Z \) is called a (right) solvent for \( I(\lambda) \) if

\[
L_0 + L_1 Z + \cdots + L_l Z^l = 0.
\]
An \( n \times n \) matrix \( Z \) is a (right) solvent for \( L(\lambda) \) if and only if

\[
L(\lambda) = L_1(\lambda)(\lambda I - Z)
\]

for some matrix polynomial \( L_1(\lambda) \) of degree \( l - 1 \). If, in addition, \( \sigma(L_1) \cap \sigma(Z) = \emptyset \), then \( Z \) is referred to as a (right) spectral solvent for \( L(\lambda) \). Let \( \gamma \) denote a rectifiable simple closed contour such that \( \sigma(Z) \) and \( \sigma(L_1) \) are inside and outside \( \gamma \), respectively. As stated in [10],

\[
\int_{\gamma} \lambda L^{-1}(\lambda) \, d\lambda = Z^s \int_{\gamma} L^{-1}(\lambda) \, d\lambda \quad (s = 1, 2, \ldots) \tag{3.1}
\]

for any (right) spectral solvent of \( L(\lambda) \). Further, if \( Z_1, Z_2, \ldots, Z_I \) are (right) solvents for \( L(\lambda) \), \( \sigma(Z_i) \cap \sigma(Z_j) = \emptyset \) \( (i, j = 1, 2, \ldots, I, i \neq j) \), and the \( I \times I \) "Vandermonde" matrix

\[
V = V(Z_1, Z_2, \ldots, Z_I) := \begin{bmatrix}
1 & 1 & \cdots & 1 \\
Z_1 & Z_2 & \cdots & Z_I \\
\vdots & \vdots & \ddots & \vdots \\
Z_1^{-1} & Z_2^{-1} & \cdots & Z_I^{-1}
\end{bmatrix}
\]

is invertible, then \( Z_1, Z_2, \ldots, Z_I \) generate a complete set of (right) solvents for \( L(\lambda) \). It is shown in [10] that the matrices in the complete set of solvents for \( L(\lambda) \) are spectral solvents for the polynomial and that \( \sigma(L) = \bigcup_{j=1}^{I} \sigma(Z_j) \). Clearly, in the scalar case \( n = 1 \), the complete set of solvents for \( L(\lambda) \) (if it exists) coincides with the set of all its distinct roots.

The theorem below generalizes formula (2.10) for block-Toeplitz band matrices.

**Theorem 3.** Let \( Z_1, Z_2, \ldots, Z_I \) constitute a complete set of (right) solvents for \( L(\lambda) \). If \( \det L_1 \neq 0 \), then, preserving the above notation,

\[
\det \mathcal{L}_m = (-1)^{m - q} \, q^{m - q} \, (\det L_1)^{m + q} \, \det \left( \frac{1}{2\pi i} \sum_{j=1}^{I} B_j \right), \tag{3.2}
\]
where for \( j = 1,2,\ldots l \)

\[
B_j = \text{diag} \left[ Z_i^{n+l-q} \right]_l^{q} \begin{bmatrix}
Z_j^{-1} & \cdots & Z_j & I \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
Z_j^{q-2} & \cdots & Z_j^{q-1} 
\end{bmatrix} \times \text{diag} \int_{\gamma_j} L^{-1}(\lambda) \, d\lambda
\]  

(3.3)

and \( \gamma_j \) contains \( G(Z_j) \) inside and \( G(L) \backslash G(Z_j) \) outside.

**Proof.** By the definition, the spectra of \( Z_i \) \((i = 1,2,\ldots,l)\) do not intersect, and therefore there exist contours \( \gamma_i \) \((i = 1,2,\ldots,l)\) containing \( \sigma(Z_i) \) inside and \( \sigma(L) \backslash \sigma(Z_i) \) outside. Since \( \sigma(L) = \bigcup_{j=1}^{l} (Z_j) \), it follows, using the residue theorem and the relation (3.1), that

\[
\int_{\Gamma} \lambda^{e} L^{-1}(\lambda) \, d\lambda = \sum_{j=1}^{l} Z_j^{e} \int_{\gamma_j} L^{-1}(\lambda) \, d\lambda.
\]

As before, \( \Gamma \) stands for a contour containing \( \sigma(L) \) in its interior. The proof follows on applying (2.2) and (2.3). \( \square \)

**Corollary 4.** Let \( L_m \) denote an \( mn \times mn \) block-Toeplitz tridiagonal matrix generated by the \( n \times n \) matrix polynomial \( L(\lambda) = L_0 + \lambda L_1 + \lambda^2 L_2 \) with \( \det L_2 \neq 0 \). If \( Z_1 \) and \( Z_2 \) are (right) solvents for \( L(\lambda) \) and \( \sigma(Z_1) \cap \sigma(Z_2) = \emptyset \), then for \( m > 2 \)

\[
\det L_m = (-1)^{m-1} m^{n+1} \left( \frac{\det(Z_2^{m+1} - Z_1^{m+1})}{\det(Z_2 - Z_1)} \right)
\]

(3.4)

**Proof.** The matrices \( B_j \) \((j = 1,2)\) in (3.3) become in this case

\[
B_j = Z_j^{m+1} \int_{\gamma_j} L^{-1}(\lambda) \, d\lambda.
\]

(3.5)
According to [8] (see also [1, Section 2.5]) the matrix $Z_1 - Z_2$ is invertible [along with $V(Z_1, Z_2)$] and

$$L^{-1}(\lambda) = \left( (\lambda I - Z_2)^{-1} - (\lambda I - Z_1)^{-1} \right) (Z_2 - Z_1)^{-1} L^{-1}_I.$$  

Thus, for $j = 1, 2$

$$\int_{\gamma_j} L^{-1}(\lambda) \, d\lambda = (-1)^j \int_{\gamma_j} (\lambda I - Z_1)^{-1} d\lambda (Z_2 - Z_1)^{-1} L^{-1}_I,$$

and it follows from (0.5) that

$$\int_{\gamma_j} L^{-1}(\lambda) \, d\lambda = (-1)^j 2\pi i (Z_2 - Z_1)^{-1} L^{-1}_I.$$  

The substitution in (3.5) and subsequently in (3.2) gives (3.4).  

\[ \text{Example 2.} \] Let $\mathcal{L}^{m}_{m}$ denote an $m \times m$ tridiagonal Toeplitz matrix generated by $L(\lambda) = a + b\lambda + c\lambda^2$, $c \neq 0$. If $d^2 = b^2 - 4ac \neq 0$, then $L(\lambda)$ has two distinct zeros $z_{1,2} = (-b \pm d)/2c$. Then by (3.4)

$$\det \mathcal{L}^{m}_{m} = (-1)^m c^m \left[ \frac{-c}{d} \right] \left[ (b - d)^{m+1} - (b + d)^{m+1} \right] \left[ 2c \right]^{-m-1}$$

$$= 2^{-m} \left[ (b + d)^{m+1} - (b - d)^{m+1} \right] \frac{1}{d}.$$  

\[ \text{Example 3.} \] Let $\mathcal{L}^{m}_{m}$ stand for a $2m \times 2m$ block-Toeplitz tridiagonal matrix generated by the matrix polynomial $L(\lambda) = L_0 + \lambda L_1 + \lambda^2 L_2$, where

$$L_0 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  

The matrices

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$
form a Jordan pair for $L(\lambda)$. Decompose $Q$ and $J$ into $2 \times 2$ matrices

$$Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}, \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix},$$

and observe that (0.1) implies

$$L_0Q_j + L_1Q_jJ_j + L_2Q_jJ_j^2 = 0 \quad (j = 1, 2).$$

Hence the matrices

$$Z_1 = Q_1J_1Q_1^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix},$$

$$Z_2 = Q_2J_2Q_2^{-1} = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

are (right) solvents for $L(\lambda)$. Since $\sigma(Z_1) \cap \sigma(Z_2) = \emptyset$ and $Z_j^t = Z_j$ for all positive integers $j$, it follows by (3.4) that

$$\det \mathcal{L}_m = \det \begin{bmatrix} 1 - (-1)^{m+1} & 0 \\ 2 & -2^{m+1} \end{bmatrix} / \det \begin{bmatrix} -2 & 0 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{cases} 2^m & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}. \end{cases} \quad (3.6)$$

Note that the knowledge of a whole standard triple $(Q, J, R)$ for $L(\lambda)$ allows one to compute $\det \mathcal{L}_m$ by a straightforward use of (2.8) [or (1.11) in this case]. Indeed,

$$J^{m+1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (-1)^{m+1} & 0 \\ 0 & 0 & 0 & 2^{m+1} \end{bmatrix}, \quad R = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ -1 & 0 \\ -1 & 1 \end{bmatrix},$$

and hence $\det(QJ^{m+1}R)$ coincides with the right-hand expression in (3.6).

Another possibility is, of course, to compute the determinant of the matrix $X_0C_{i_j}^{-1}Y_0$ (see Preliminaries for definition), but this is usually a difficult task.
4. A DIFFERENT APPROACH

In the preceding sections the computation of the determinant of an \( mn \times mn \) block-Toeplitz band matrix \( \mathcal{L}_m \) is reduced to that of a \( qn \times qn \) determinant. We now derive a formula for \( \det \mathcal{L}_m \) in terms of the determinant of an \( ln \times ln \) generalized Vandermonde matrix. In spite of \( l = q + k - 1 \geq q \), the latter determinant has some computational advantages.

**Theorem 4.** Let \( \mathcal{L}_m \) be defined by (2.1), and let \( L_1 \neq 0 \). Assume, in addition, that \( \det L_0 \neq 0 \). If \( (X, T) \) is a standard pair for the generating matrix polynomial \( L(\lambda) \), then for \( m > l \geq k \geq 2 \)

\[
\det \mathcal{L}_m = (-1)^r \frac{\left(\det L_1\right)^m \left(\det V\right)}{\det \tilde{X}}, \tag{4.1}
\]

where \( r = (m - k)(l - k + 1)n^2 \) and

\[
V = \begin{bmatrix}
X \\
\vdots \\
XT^{k - 2} \\
XT^{m + k - 1} \\
\vdots \\
XT^{m + l - 1}
\end{bmatrix}, \quad \tilde{X} = \text{col}(XT^i)_{i=0}^l
\tag{4.2}
\]

**Proof.** Define an \( mn \times mn \) matrix

\[
\mathcal{F}_m = \begin{bmatrix}
\tilde{X} \\
\vdots \\
0 \\
I_{m-1}
\end{bmatrix},
\]

where \( \tilde{X} = \text{col}(XT^{k+i-1})_{i=0}^{m-1} \). Note that since \( \det L_0 \neq 0 \), it follows that \( \det T \neq 0 \) (take, for instance, \( T = C_L \), the companion matrix) and therefore the matrix \( \mathcal{F}_m \) is invertible. Now use (0.1) to obtain

\[
\mathcal{L}_m \mathcal{F}_m = \begin{bmatrix}
G_1 \\
0 \\
G_2
\end{bmatrix}, \tag{4.3}
\]
where \( \hat{\mathcal{L}}_m = \mathcal{L}_m[1,2,\ldots, mn|ln+1,\ldots, mn] \) and
\[
G_1 = -P_{k-1} \hat{L}_{0,k-2} P_{k-1} \cot(X^T)^k, \\
G_2 = -\hat{L}_{l,k} \cot(X^T)^{m+k+1-k}.
\]

Permuting rows in (4.3), we obtain, exploiting the notation in (4.2), that
\[
(\det \mathcal{L}_m)(\det \hat{X})(\det T)^{k-1} = (-1)^v \det \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} (\det L_l)^{m-1}, \tag{4.4}
\]
where \( v = [(m-k+1)n+1]q_n, \ q = 1-k+1.\) Represent
\[
\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = - \begin{bmatrix} P_{k-1} \hat{L}_{0,k-2} P_{k-1} & 0 \\ 0 & \hat{L}_{l-k} \end{bmatrix} V,
\]
where \( V \) is defined in (4.2), and observe that
\[
\det \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = (-1)^n (\det L_0)^{k-1} (\det L_l)^q (\det V).
\]

It remains to observe that
\[
\det T = \det G_l = (-1)^n \frac{\det L_0}{\det L_l}. \tag*{□}
\]

Theorem 4 applied to a scalar polynomial gives the expression for \( \det \mathcal{L}_m \) obtained in [11] and used there in studying the eigenvalue problem for Toeplitz band matrices.

Let \( n = 1 \) in (2.1) and \( L_0 \neq 0, \ L_l \neq 0.\) Suppose that \( \lambda_1, \lambda_2,\ldots, \lambda_s \) are the distinct zeros of \( L(\lambda) \) with multiplicities \( r_1, r_2,\ldots, r_s (r_1 + r_2 + \cdots + r_s = l), \) respectively. Given integers \( m \) and \( k, \) define the \( l \)th-order vector function
\[
A(x) = \begin{bmatrix} 1 & x & \cdots & x^{k-2} & x^{m+k-1} & \cdots & x^{m+l-1} \end{bmatrix}^T,
\]
and denote by \( A^{(j)}(x) \) its \( j \)th derivative. Now construct the generalized Vandermonde matrix \( V_m \) associated with the given polynomial \( L(\lambda) \) as follows: the first \( r_1 \) columns of \( V_m \) are \( A(\lambda_1), A'(\lambda_1),\ldots, A^{r_1-1}(\lambda_1), \) respectively; the next \( r_2 \) columns are \( A(\lambda_2), A'(\lambda_2),\ldots, A^{(r_2-1)}(\lambda_2); \) and so on.
Corollary 5 [11]. With the notation of the preceding paragraph,

$$\det \mathcal{L}_m = (-1)^r L_i^m \frac{\det V_m}{\det V_0},$$

where $r$ is defined in Theorem 4.

In particular, if all the zeros of $L(\lambda)$ are distinct, then

$$\det \mathcal{L}_m = (-1)^r L_i^m \det \begin{bmatrix}
1 & \cdots & 1 \\
\lambda_1 & \cdots & \lambda_i \\
\vdots & \ddots & \vdots \\
\lambda_{i-1}^{k-1} & \cdots & \lambda_{i-1}^{k-2} \\
\lambda_i & \cdots & \lambda_i^{k-1}
\end{bmatrix} (\det V^{(i)})^{-1},$$

where $V^{(i)}$ stands for the ordinary Vandermonde matrix constructed from $\lambda_1, \lambda_2, \ldots, \lambda_i$.

For the proof it suffices to observe that the Jordan pair $\{Q_i, J_i\}$ associated with the zero $\lambda_i$ of $L(\lambda)$ with multiplicity $r_i$ is given by the formula

$$Q_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \quad J_i = \begin{bmatrix} \lambda_i & 1 & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\lambda_i & \cdots & \lambda_i & 1 \\
& & \lambda_i 
\end{bmatrix},$$

where $Q_i \in \mathbb{C}^{1 \times r_i}$, $J_i \in \mathbb{C}^{r_i \times r_i}$. Hence

$$Q = \begin{bmatrix} Q_1 & Q_2 & \cdots & Q_s \end{bmatrix}, \quad J = [\delta_{jk} J_j]_{j, k = 1}^r$$

is a Jordan pair for $L(\lambda)$ in the general case. If all the zeros of $L(\lambda)$ are simple, then

$$Q = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}, \quad J = [\delta_{jk} \lambda_j]_{j, k = 1}^r$$

form a Jordan pair for $L(\lambda)$.

Now apply Theorem 4 with $X = Q$, $T = J$. $\blacksquare$
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REFERENCES

4. M.B. Gorodeckii, Toeplitz determinants generated by rational functions (in Russian), Report 5451-81, VINITI.

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