Determinant of Block-Toeplitz Band Matrices

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ABSTRACT

Some expressions are given for the determinant of an $mn \times mn$ block-Toeplitz band matrix $\mathscr{L} = [L_{i-j}]$, with bandwidth $(p+q+1)n < mn$, in terms of the $n \times n$ generating matrix polynomial $L(\lambda) = \sum_{j=0}^{p+q} \lambda^j L_{p-j}$, $\det L_{-q} \neq 0$. In the scalar case this yields formulas for the determinant expressed via the zeros of the generating (scalar) polynomial. The approach adopted in this work leans heavily on the recently developed spectral theory of matrix polynomials.

0. PRELIMINARIES

The *spectrum* of an $n \times n$ matrix polynomial $L(\lambda) = \sum_{j=0}^{l} \lambda^j L_j$, $L_j \in \mathbb{C}^{n \times n}$, $j = 0, 1, \ldots, l$ of degree l is defined by the set $\sigma(L) = \{\lambda \in \mathbb{C}^1 : \det L(\lambda) = 0\}$. It will be assumed throughout the work that the leading coefficient L_l of $L(\lambda)$ is an invertible $n \times n$ matrix.

Recall (see [3]) that any matrix polynomial $L(\lambda)$ with an invertible leading coefficient can be associated with a triple of matrices (X, T, Y) having the following properties: The matrix X has size $n \times nl$; the $nl \times nl$ matrix T is such that the matrix

$$\Omega = \operatorname{col}(XT^j)_{j=0}^{l-1} := \begin{bmatrix} X \\ XT \\ \vdots \\ XT^{l-1} \end{bmatrix}$$

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is nonsingular and

$$\sum_{j=0}^{l} L_j X T^j = 0. \tag{0.1}$$

Further, the matrix Y is $nl \times n$ and is the unique solution of the equation $\Omega Y = [0 \ \cdots \ 0 \ (L_l^T)^{-1}]^T$. Any triple (X, T, Y) satisfying the above conditions is called a *standard triple* for $L(\lambda)$. The pair (X, T) is referred to as a *standard pair* for $L(\lambda)$.

It is clear from the definition that

$$XT^jY = \begin{cases} 0, & j = 0, 1, \dots, l-2, \\ L_l^{-1}, & j = l-1. \end{cases} \tag{0.2}$$

Also,

$$\sum_{j=0}^{l} T^j Y L_l = 0. \tag{0.3}$$

To ease the use of our main reference [3] it is relevant to point out the relation between the standard triples of $L(\lambda)$ and the associated *monic* matrix polynomial $\tilde{L}(\lambda) = L_l^{-1}L(\lambda)$. Namely, (X, T, Y) is a standard triple for $L(\lambda)$ if and only if (X, T, YL_l) is a standard triple for $\tilde{L}(\lambda)$.

Note two important examples of a standard triple for $L(\lambda)$. The first is the *Jordan triple* (Q, J, R), where $Q = [Q_1 \ Q_2 \cdots Q_s]$, $J = [\delta_{jk} J_j]_{j,k=1}^{s}$, and Q_j, $1 \leqslant j \leqslant s$, consists of a canonical system of Jordan chains for $L(\lambda)$ corresponding to an eigenvalue λ_j of $L(\lambda)$. The matrix J_j is here a direct sum of the Jordan blocks associated with λ_j (see [3] for definitions). The matrix R is uniquely defined by the matrices Q and J. Note that in the linear case $l = 1$ and $L_1 = -I$ the matrix J just coincides with the Jordan canonical form for L_0, while $Q = R^{-1}$ performs the corresponding similarity transformation.

Another example of a standard triple for $L(\lambda)$ is provided by the *companion triple* (X_0, C_L, Y_0), where

$$X_0 = [I \quad 0 \quad \cdots \quad 0], \qquad Y_0 = \begin{bmatrix} 0 & \cdots & 0 & (L_l^{-1})^T \end{bmatrix}^T,$$

I stands for the $n \times n$ identity matrix, and C_L denotes a companion matrix associated with the matrix polynomial $L(\lambda)$:

$$C_L = \begin{bmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \ddots & \ddots & \\ & & & 0 & I \\ -\tilde{L}_0 & -\tilde{L}_1 & \cdots & -\tilde{L}_{l-2} & -\tilde{L}_{l-1} \end{bmatrix},$$

where $\tilde{L}_j = L_l^{-1} L_j$, $j = 0, 1, \dots, l$.

Note that the standard triples (X_1, T_1, Y_1) and (X_2, T_2, Y_2) of a given $n \times n$ matrix polynomial $L(\lambda)$ of degree l are similar. That is,

$$X_1 = X_2 S, \qquad T_1 = S^{-1} T_2 S, \qquad Y_1 = S^{-1} Y_2$$

for some invertible $ln \times ln$ matrix S.

It turns out [3] that a triple of matrices (X, T, Y) of sizes $n \times nl$, $nl \times nl$, $nl \times n$, respectively, is a standard triple for $L(\lambda)$ if and only if the following representation of the resolvent holds:

$$L^{-1}(\lambda) = X(\lambda I - T)^{-1} Y \tag{0.4}$$

where $\lambda \notin \sigma(T)$, the spectrum of T.

Let Γ denote a rectifiable simple closed contour in $\mathbb{C}^1$ (for brevity, contour, in the sequel) containing $\sigma(T)$ [or, what is equivalent, $\sigma(L)$] in its interior. Since

$$\frac{1}{2\pi i} \int_\Gamma \lambda^j (\lambda I - T)^{-1} d\lambda = T^j, \qquad j = 0, 1, \dots, \tag{0.5}$$

it follows from (0.4) that

$$\frac{1}{2\pi i} \int_\Gamma \lambda^j L^{-1}(\lambda)\, d\lambda = X T^j Y \tag{0.6}$$

for $j = 0, 1, \dots$.

We adopt the following notational conventions. Given an $s \times s$ matrix A, the submatrix of A containing rows numbered $i_1, i_2, \dots, i_k$ and columns $j_1, j_2, \dots, j_r$ is denoted by $A[i_1, \dots, i_k | j_1, \dots, j_r]$. We also use the notation

$\operatorname{diag}[Z]_1^s := [\delta_{jk} Z]_{j,k=1}^s$, δ_{jk} being the Kronecker symbol, and

$$\operatorname{col}(Z_j)_{j=1}^l := \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_l \end{bmatrix}, \qquad \operatorname{row}(Z_j)_{j=1}^l := \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_l \end{bmatrix}. \quad (0.7)$$

Throughout the paper the integer n stands for the size of the matrix polynomial and is fixed. Further, I_q denotes the $qn \times qn$ identity matrix, $I_1 \equiv I$, while P_s stands for the $sn \times sn$ reverse unit matrix $[\delta_{j,s-j+1} I]_{j=1}^s$. Also,

$$\tilde{L}_{s,p} = \begin{bmatrix} L_s & & & \\ L_{s-1} & L_s & & \\ \vdots & \ddots & \ddots & \\ L_p & \cdots & L_{s-1} & L_s \end{bmatrix}. \quad (0.8)$$

1. BLOCK-TOEPLITZ-HESSENBERG MATRIX

Consider an $mn \times mn$ block-Toeplitz band matrix of Hessenberg's type

$$\mathscr{L}_m = \begin{bmatrix} L_{k-1} & L_k & & & & & 0 \\ L_{k-2} & L_{k-1} & L_k & & & & \\ \vdots & & \ddots & \ddots & & & \\ L_0 & & & \ddots & \ddots & & \\ & \ddots & & & \ddots & \ddots & \\ & 0 & \ddots & & & \ddots & L_k \\ & & & L_0 & \cdots & L_{k-2} & L_{k-1} \end{bmatrix}$$

$$(\mathscr{L}_m \in \mathbb{C}^{mn \times mn}), \quad (1.1)$$

where $L_j \in \mathbb{C}^{n \times n}$, $j = 0, 1, \ldots, k$, and $1 \leqslant k < m$. Introduce the $n \times n$ matrix polynomial

$$L(\lambda) = L_0 + \lambda L_1 + \cdots + \lambda^k L_k$$

associated with the matrix $\mathscr{L}_m$.

THEOREM 1. *If* $\det L_k \neq 0$, *then the determinant of* $\mathscr{L}_m$ *in* (1.1) *is given by the formula*

$$\det \mathscr{L}_m = (-1)^{(m-1)n^2+1} (\det L_k)^{m+1} \det\left(\frac{1}{2\pi i} \int_\Gamma \lambda^{m+k-1} L^{-1}(\lambda)\, d\lambda \right), \tag{1.2}$$

where Γ *stands for a contour containing the spectrum of the generating matrix polynomial* $L(\lambda)$ *in its interior.*

Proof. Denote by F_m the matrix obtained from the $mn \times mn$ identity matrix on replacing its first n columns by the matrix $[S_0^T \; S_1^T \; \cdots \; S_{m-1}^T]^T$, where $S_0 = I$ and $S_i \in \mathbb{C}^{n \times n}$, $i = 1,2,\ldots, m-1$. Consider the product $G_m = \mathscr{L}_m F_m$. Clearly, the elements of G_m, except those of the first block column, coincide with the corresponding elements of $\mathscr{L}_m$. Note also that

$$\det \mathscr{L}_m = \det G_m \tag{1.3}$$

for any choice of the matrices S_i $(i = 1,2,\ldots, m-1)$.

Aiming at an easy computation of the determinant of G_m, we choose the blocks S_i $(i = 1,3,\ldots, m-1)$ so that the $n \times n$ blocks G_{i1} $(i = 1,2,\ldots, m-1)$ equal the zero matrix. This yields the relations

$$\sum_{i=0}^{j} L_{k-i} S_{j-i} = 0 \qquad (j = 1,2,\ldots, k), \tag{1.4}$$

$$\sum_{i=0}^{k} L_i S_{i+j} = 0 \qquad (j = 1,2,\ldots, m-k-1). \tag{1.5}$$

Now view the equations (1.5) as a part of the matrix difference equation

$$L_0 S_j + L_1 S_{j+1} + \cdots + L_k S_{j+k} = 0 \qquad (j = 1,2,\ldots) \tag{1.6}$$

with initial conditions (1.4) rewritten in the form [see (0.8)]

$$\operatorname{col}(S_i)_{i=1}^{k} = -\tilde{L}_{k,1}^{-1}\operatorname{col}(L_{k-i})_{i=1}^{k}. \tag{1.7}$$

Since

$$G_{m1} = L_0 S_{m-k} + L_1 S_{m-k+1} + \cdots + L_{k-1} S_{m-1},$$

it follows from (1.6) that $G_{m1} = -L_k S_m$ and hence, in view of (1.3),

$$\det \mathscr{L}_m = (-1)^{(m-1)n^2+1}(\det L_k)^m(\det S_m), \tag{1.8}$$

where $(S_1, S_2, \ldots, S_m, \ldots)$ is the solution of the matrix difference equation (1.6) with initial conditions (1.7).

The desired solution is given by the Lancaster formula [9]

$$S_m = XT^{m-1}c, \tag{1.9}$$

where

$$c = \operatorname{row}(T^{k-i}Y)_{i=1}^{k}\,\tilde{L}_{k,1}\operatorname{col}(S_i)_{i=1}^{k} \tag{1.10}$$

and (X, T, Y) stands for a standard triple for the matrix polynomial $L(\lambda)$. Substituting from (1.7) in (1.10), we obtain from (0.3)

$$c = -\sum_{j=0}^{k-1} T^j Y L_j = T^k Y L_k.$$

Appealing to (1.9) and (1.8), we thus have

$$\det \mathscr{L}_m = (-1)^{(m-1)n^2+1}(\det L_k)^{m+1}\det(XT^{m+k-1}Y). \tag{1.11}$$

In view of (0.6), the proof is complete. ■

It is easily seen that for $|\lambda|$ sufficiently large, the $n \times n$ matrix

$$Z_j = \frac{1}{2\pi i}\int_{\Gamma} \lambda^j L^{-1}(\lambda)\,d\lambda$$

is the coefficient of λ^{-j-1} in the Laurent expansion

$$L^{-1}(\lambda) = \lambda^{-k} Z_{k-1} + \lambda^{-k-1} Z_k + \cdots \qquad (|\lambda| > |\lambda_0| > 0) \tag{1.12}$$

of the resolvent $L^{-1}(\lambda)$. Proceeding to the reverse matrix polynomial

$$L_\infty(\lambda) := \lambda^k L(\lambda^{-1}) = \lambda^k L_0 + \lambda^{k-1} L_1 + \cdots + \lambda L_{k-1} + L_k,$$

we obtain from (1.12) the following expansion in power series:

$$L_\infty^{-1}(\lambda) = \sum_{j=k-1}^{\infty} \lambda^{j-k+1} Z_j = \sum_{j=0}^{\infty} \lambda^j Z_{j+k-1} \quad \left(|\lambda| < \frac{1}{|\lambda_0|} \right).$$

Setting $L_k = I$ in (1.2), we arrive at a generalization of the well-known Wronsky formula [5] for monic matrix polynomials.

Corollary 1. *Let $M(\lambda) = I + \lambda M_1 + \cdots + \lambda^k M_k$, and for $|\lambda|$ sufficiently small*

$$M^{-1}(\lambda) = \sum_{j=0}^{\infty} (-1)^j \lambda^j W_j.$$

Then for $j = 0, 1, \ldots$

$$W_j = \frac{(-1)^j}{2\pi i} \int_\Gamma \lambda^{j+k-1} M_\infty^{-1}(\lambda)\, d\lambda,$$

where the contour Γ is defined in Theorem 1, and for $j > k$

$$\det W_j = \det \begin{bmatrix} M_1 & I & & & & & \\ M_2 & M_1 & I & & & & \\ \vdots & & \ddots & \ddots & & & \\ \vdots & & & \ddots & \ddots & & \\ M_k & & & & \ddots & \ddots & \\ & \ddots & & & & \ddots & I \\ & & M_k & \cdots & \cdots & \cdots & M_1 \end{bmatrix} \quad (\in \mathbb{C}^{jn \times jn}).$$

Concerning the scalar case $n = 1$, we deduce the following result stated in a different form in [7].

COROLLARY 2. *If $n = 1$ in (1.1) and the (scalar) polynomial $L(\lambda)$ ($L_k \neq 0$) has distinct zeros $\lambda_1, \lambda_2, \ldots, \lambda_s$ with multiplicities $r_1, r_2, \ldots, r_s$ ($r_1 + r_2 + \cdots + r_s = k$) respectively, then*

$$\det \mathscr{L}_m = (-1)^m L_k^{m+1} \sum_{i=1}^{s} \frac{1}{(r_1 - 1)!} \lim_{\lambda \to \lambda_i} \frac{d^{r_i - 1}}{d\lambda^{r_i - 1}} \left[(\lambda - \lambda_i)^{r_i} f(\lambda) \right] \quad (1.13)$$

where $f(\lambda) = \lambda^{m+k-1}/L(\lambda)$ and $k < m$. In particular, if the zeros of $L(\lambda)$ are all simple, then

$$\det \mathscr{L}_m = (-1)^m L_k^{m+1} \sum_{i=1}^{k} \frac{\lambda_i^{m+k-1}}{L'(\lambda_i)}. \quad (1.14)$$

Proof. By the residue theorem,

$$\frac{1}{2\pi i} \int_{\Gamma} f(\lambda)\, d\lambda = \sum_{i=1}^{s} \operatorname{Res} f(\lambda_i),$$

and by the relation

$$\operatorname{Res} f(\lambda_i) = \frac{1}{(r_i - 1)!} \lim_{\lambda \to \lambda_i} \frac{d^{r_i - 1}}{d\lambda^{r_i - 1}} \left[(\lambda - \lambda_i)^{r_i} f(\lambda) \right]$$

for $i = 1, 2, \ldots, s$, the formula (1.13) follows. ■

EXAMPLE 1. Let $\mathscr{L}_m$ be a tridiagonal Toeplitz matrix generated by the (scalar) polynomial $L(\lambda) = 3 - \lambda - 2\lambda^2$ with zeros $\lambda_1 = 1, \lambda_2 = -\frac{3}{2}$. Then by (1.14)

$$\det \mathscr{L}_m = (-1)^m (-2)^{m+1} \left[\frac{1}{-5} + \frac{\left(-\frac{3}{2}\right)^{m+1}}{5} \right] = \frac{2^{m+1}}{5} \left[1 - \left(-\tfrac{3}{2}\right)^{m+1} \right]$$

for $m > 2$.

As noted by the referee, Theorem 1, as stated, can be proved without using the spectral theory of matrix polynomials. Indeed,

$$L(\lambda)\left(Z_0\lambda^{-1} + Z_1\lambda^{-2} + \cdots\right) = I,$$

and hence the Laurent coefficients Z_j of $L(\lambda)^{-1}$ at infinity solve (1.6). It turns out, however, that in this case we lose the possibility to compute $\det \mathscr{L}_m$ by (1.11) in which the companion triple is taken, for instance.

2. GENERAL BLOCK-TOEPLITZ BAND MATRIX

Let now

$$\mathscr{L}_m = \begin{bmatrix} L_{k-1} & L_k & \cdot & \cdot & \cdot & L_l & & & \\ L_{k-2} & L_{k-1} & \cdot & & & & \cdot & & \\ \cdot & & \cdot & \cdot & & & & \cdot & \\ \cdot & & & \cdot & \cdot & & & & \cdot \\ \cdot & & & & \cdot & \cdot & & & L_l \\ L_0 & & & & & \cdot & \cdot & & \cdot \\ & \cdot & & & & & \cdot & \cdot & \cdot \\ & & \cdot & & & & & \cdot & \cdot \\ & & & \cdot & & & & \cdot & L_k \\ & & & & L_0 & \cdot & \cdot & \cdot & L_{k-1} \end{bmatrix}$$

$$\left(L_j \in \mathbb{C}^{n\times n}, \quad j = 0,1,\ldots,l\right) \quad (2.1)$$

denote an $mn \times mn$ block-Toeplitz band matrix such that $1 \leqslant k \leqslant l < m$, and let

$$L(\lambda) = L_0 + \lambda L_1 + \cdots + \lambda^l L_l, \qquad \det L_l \neq 0.$$

THEOREM 2. *If* $\det L_l \neq 0$, *then the determinant of the matrix* $\mathscr{L}_m$ *in* (2.1) *is given by the formula*

$$\det \mathscr{L}_m = (-1)^{(m-q)n^2+q}(\det L_l)^{m+q}(\det M_q), \qquad (2.2)$$

where $k \leqslant l < m$, $q = l - k + 1$, *and*

$$M_q = \frac{1}{2\pi i}\int_\Gamma \lambda^{m+l-q}\begin{bmatrix} \lambda^{q-1}L^{-1}(\lambda) & \cdots & \lambda L^{-1}(\lambda) & L^{-1}(\lambda) \\ \lambda^{q}L^{-1}(\lambda) & \cdots & \lambda^{2}L^{-1}(\lambda) & \lambda L^{-1}(\lambda) \\ \vdots & & \vdots & \vdots \\ \lambda^{2q-2}L^{-1}(\lambda) & \cdots & \lambda^{q}L^{-1}(\lambda) & \lambda^{q-1}L^{-1}(\lambda) \end{bmatrix} d\lambda, \tag{2.3}$$

in which Γ *denotes a contour in the complex plane containing the spectrum of the generating matrix polynomial* $L(\lambda)$ *in its interior.*

Proof. To compute the determinant of $\mathscr{L}_m$ in (2.1), we first note that the matrices S_i used in the proof of Theorem 1 have been chosen to be of the form XT^jYL_k, where (X,T,Y) is a standard triple for the generating matrix polynomial with the leading coefficient L_k. This observation prompts how to generalize the procedure exploited in Theorem 1 for calculating det $\mathscr{L}_m$.

Define $mn \times m$ matrices

$$R_j = \operatorname{col}(XT^{i+j}YL_l)_{i=0}^{m-1} \qquad (j = l-1, l-2, \dots, k-1),$$

where (X,T,Y) is a standard triple for $L(\lambda)$. Denote by F_m the matrix obtaining from I_m on replacing its first qn columns by the matrix $[R_{l-1}\ R_{l-2}\ \cdots\ R_{k-1}]$. The relations (0.2) show that F_m is a lower triangular matrix with ones on the main diagonal. Let $\mathscr{L}_m^{(S)} = \mathscr{L}_m[1,2,\dots,sn|1,2,\dots,mn]$. The relation (0.1) yields for $j \geqslant k-1$

$$\mathscr{L}_m^{(k-1)}R_j = -P_{k-1}\tilde{\mathscr{L}}_{0,k-2}P_{k-1}\operatorname{col}(XT^{j-k-i}YL_l)_{i=1}^{k-1},$$

and it thus follows from (0.2) that

$$\mathscr{L}_m^{(k-1)}R_j = 0 \tag{2.4}$$

for $j \leqslant l-1$. Further, the relation (0.1) implies

$$\mathscr{L}_m^{(m-q)}R_j = 0, \tag{2.5}$$

and for $k-1 \leqslant j \leqslant l-1$

$$\mathscr{L}_m^{[q]} R_j = -\tilde{L}_{l,l-q+1} \operatorname{col}(XT^{m-j+i})_{i=0}^{q-1}, \tag{2.6}$$

where $\mathscr{L}_m^{[q]} = \mathscr{L}_m[(m-q)n+1,\ldots,mn|1,2,\ldots,mn]$. Combining (2.4)–(2.6), we obtain

$$\mathscr{L}_m F_m = \begin{bmatrix} I_{m-q} & 0 \\ 0 & -\tilde{L}_{l,l-q+1} \end{bmatrix} G_m, \tag{2.7}$$

where G_m differs from $\mathscr{L}_m$ by the first qn columns:

$$G_m[1,2,\ldots,mn|1,2,\ldots,qn] = \begin{bmatrix} 0 \\ \tilde{X} \end{bmatrix} \operatorname{diag}[L_l]_1^q$$

with the $qn \times qn$ block-Toeplitz matrix

$$\tilde{X} = \begin{bmatrix} XT^{m+l-1}Y & \cdots & XT^{m+k-1}Y \\ XT^{m+l}Y & \cdots & XT^{m+k}Y \\ \vdots & & \vdots \\ XT^{m+l+q-2}Y & \cdots & XT^{m+l-1}Y \end{bmatrix}. \tag{2.8}$$

It is now easily checked that in view of (2.7)

$$\det \mathscr{L}_m = \det \mathscr{L}_m F_m = (-1)^{(m-q)n^2+q} (\det L_l)^{m+q} (\det \tilde{X}), \tag{2.9}$$

which along with (0.6) gives the required result. ■

The following immediate consequence of Theorem 2 extends the assertion of Corollary 2 to an arbitrary Toeplitz band matrix. This formula, even for the case of arbitrary rational functions, was given by Day [2]. For other proofs see [1] and [6]. It is stated in [1] that Day's formula was generalized in [4]. The latter article is difficult to obtain, and the author has not seen it.

COROLLARY 3 [2]. *Let $n=1$, and the (scalar) polynomial $L(\lambda)$ ($L_l \neq 0$) have distinct zeros $\lambda_1, \lambda_2, \ldots, \lambda_s$ with multiplicities $r_1, r_2, \ldots, r_s$ ($r_1 + r_2$*

$+ \cdots + r_s = l$), respectively. Then for $m > l$

$$\det \mathscr{L}_m = (-1)^m L_l^{m+q} \det\left(\sum_{i=1}^{s} A_i \right),$$

where for $i = 1, 2, \ldots, s$

$$A_i = \frac{1}{(r_i - 1)!} \lim_{\lambda \to \lambda_i} \frac{d^{r_i - 1}}{d\lambda^{r_i - 1}} \left[(\lambda - \lambda_i)^{r_i} f(\lambda) \right]$$

$$\times \begin{bmatrix} \lambda_i^{q-1} & \cdot & \cdot & \cdot & \lambda_i & 1 \\ \cdot & \cdot & & & & \lambda_i \\ \vdots & & \cdot & & & \vdots \\ \vdots & & & \cdot & & \vdots \\ \cdot & & & & \cdot & \cdot \\ \lambda_i^{2q-2} & \cdot & \cdot & \cdot & \cdot & \lambda_i^{q-1} \end{bmatrix}$$

with $f(\lambda) = \lambda^{m+l-q}/L(\lambda)$.

In particular, if the zeros of $L(\lambda)$ *are all simple, then for* $m > l$

$$\det \mathscr{L}_m = (-1)^m L_l^{m+q} \det\left(\sum_{j=1}^{l} A_j \right),$$

where

$$A_j = \frac{\lambda_j^{m+l-q}}{L'(\lambda_j)} \begin{bmatrix} \lambda_j^{q-1} & \cdot & \cdot & \cdot & \lambda_j & 1 \\ \cdot & \cdot & & & & \lambda_j \\ \vdots & & \cdot & & & \vdots \\ \vdots & & & \cdot & & \vdots \\ \cdot & & & & \cdot & \cdot \\ \lambda_j^{2q-2} & \cdot & \cdot & \cdot & \cdot & \lambda_j^{q-1} \end{bmatrix}.$$

3. EXPRESSION FOR $\det \mathscr{L}_m$ IN TERMS OF SOLVENTS

Following [8], we say that an $n \times n$ matrix Z is called a (right) *solvent* for $L(\lambda)$ if

$$L_0 + L_1 Z + \cdots + L_l Z^l = 0.$$

An $n \times n$ matrix Z is a (right) solvent for $L(\lambda)$ if and only if

$$L(\lambda) = L_1(\lambda)(\lambda I - Z)$$

for some matrix polynomial $L_1(\lambda)$ of degree $l-1$. If, in addition, $\sigma(L_1) \cap \sigma(Z) = \varnothing$, then Z is referred to as a (right) *spectral solvent* for $L(\lambda)$. Let γ denote a rectifiable simple closed contour such that $\sigma(Z)$ and $\sigma(L_1)$ are inside and outside γ, respectively. As stated in [10],

$$\int_\gamma \lambda^s L^{-1}(\lambda)\, d\lambda = Z^s \int_\gamma L^{-1}(\lambda)\, d\lambda \qquad (s = 1,2,\dots) \tag{3.1}$$

for any (right) spectral solvent of $L(\lambda)$. Further, if $Z_1, Z_2, \dots, Z_l$ are (right) solvents for $L(\lambda)$, $\sigma(Z_i) \cap \sigma(Z_j) = \varnothing \quad (i, j = 1,2,\dots, l,\ i \neq j)$, and the $ln \times ln$ "Vandermonde" matrix

$$V \equiv V(Z_1, Z_2, \dots, Z_l) := \begin{bmatrix} I & I & \cdots & I \\ Z_1 & Z_2 & & Z_l \\ \vdots & \vdots & & \vdots \\ Z_1^{l-1} & Z_2^{l-1} & \cdots & Z_l^{l-1} \end{bmatrix}$$

is invertible, then $Z_1, Z_2, \dots, Z_l$ generate a *complete set* of (right) solvents for $L(\lambda)$. It is shown in [10] that the matrices in the complete set of solvents for $L(\lambda)$ are spectral solvents for the polynomial and that $\sigma(L) = \bigcup_{j=1}^{l} \sigma(Z_j)$. Clearly, in the scalar case $n = 1$, the complete set of solvents for $L(\lambda)$ (if it exists) coincides with the set of all its distinct roots.

The theorem below generalizes formula (2.10) for block-Toeplitz band matrices.

THEOREM 3. *Let $Z_1, Z_2, \dots, Z_l$ constitute a complete set of (right) solvents for $L(\lambda)$. If* $\det L_l \neq 0$, *then, preserving the above notation,*

$$\det \mathscr{L}_m = (-1)^{(m-q)n^2+q} (\det L_l)^{m+q} \det\left(\frac{1}{2\pi i} \sum_{j=1}^{l} B_j \right), \tag{3.2}$$

where for $j = 1,2,\ldots l$

$$B_j = \operatorname{diag}[Z^{m+l-q}]_1^q \begin{bmatrix} Z_j^{q-1} & \cdot & \cdot & \cdot & Z_j & I \\ \cdot & \cdot & & & & Z_j \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ Z_j^{2q-2} & \cdot & \cdot & \cdot & \cdot & Z_j^{q-1} \end{bmatrix}$$

$$\times \operatorname{diag}\left[\int_{\gamma_j} L^{-1}(\lambda)\, d\lambda\right]_1^q \tag{3.3}$$

and γ_j *contains* $G(Z_j)$ *inside and* $G(L)\backslash G(Z_j)$ *outside.*

Proof. By the definition, the spectra of Z_j ($j = 1,2,\ldots,l$) do not intersect, and therefore there exist contours γ_j ($j = 1,2,\ldots,l$) containing $\sigma(Z_j)$ inside and $\sigma(L)\backslash\sigma(Z_j)$ outside. Since $\sigma(L) = \bigcup_{j=1}^{l}(Z_j)$, it follows, using the residue theorem and the relation (3.1), that

$$\int_{\Gamma} \lambda^s L^{-1}(\lambda)\, d\lambda = \sum_{j=1}^{l} Z_j^s \int_{\gamma_j} L^{-1}(\lambda)\, d\lambda.$$

As before, Γ stands for a contour containing $\sigma(L)$ in its interior. The proof follows on applying (2.2) and (2.3). ∎

COROLLARY 4. *Let* $\mathscr{L}_m$ *denote an* $mn \times mn$ *block-Toeplitz tridiagonal matrix generated by the* $n \times n$ *matrix polynomial* $L(\lambda) = L_0 + \lambda L_1 + \lambda^2 L_2$ *with* $\det L_2 \neq 0$. *If* Z_1 *and* Z_2 *are (right) solvents for* $L(\lambda)$ *and* $\sigma(Z_1) \cap \sigma(Z_2) = \varnothing$, *then for* $m > 2$

$$\det \mathscr{L}_m = (-1)^{(m-1)n^2+1} (\det L_2)^m \frac{\det(Z_2^{m+1} - Z_1^{m+1})}{\det(Z_2 - Z_1)} \tag{3.4}$$

Proof. The matrices B_j ($j = 1,2$) in (3.3) become in this case

$$B_j = Z_j^{m+1} \int_{\gamma_j} L^{-1}(\lambda)\, d\lambda. \tag{3.5}$$

According to [8] (see also [1, Section 2.5]) the matrix $Z_1 - Z_2$ is invertible [along with $V(Z_1, Z_2)$] and

$$L^{-1}(\lambda) = \left\{(\lambda I - Z_2)^{-1} - (\lambda I - Z_1)^{-1}\right\}(Z_2 - Z_1)^{-1} L_l^{-1}.$$

Thus, for $j = 1,2$

$$\int_{\gamma_j} L^{-1}(\lambda)\, d\lambda = (-1)^j \int_{\gamma_j} (\lambda I - Z_j)^{-1}\, d\lambda\, (Z_2 - Z_1)^{-1} L_l^{-1},$$

and it follows from (0.5) that

$$\int_{\gamma_j} L^{-1}(\lambda)\, d\lambda = (-1)^j 2\pi i (Z_2 - Z_1)^{-1} L_l^{-1}.$$

The substitution in (3.5) and subsequently in (3.2) gives (3.4). ∎

EXAMPLE 2. Let $\mathscr{L}_m$ denote an $m \times m$ tridiagonal Toeplitz matrix generated by $L(\lambda) = a + b\lambda + c\lambda^2$, $c \neq 0$. If $d^2 = b^2 - 4ac \neq 0$, then $L(\lambda)$ has two distinct zeros $z_{1,2} = (-b \pm d)/2c$. Then by (3.4)

$$\begin{aligned} \det \mathscr{L}_m &= (-1)^m c^m \left(-\frac{c}{d}\right)\left[(-b-d)^{m+1} - (-b+d)^{m+1}\right](2c)^{-m-1} \\ &= 2^{-m-1}\left[(b+d)^{m+1} - (b-d)^{m+1}\right]\frac{1}{d}. \end{aligned}$$

EXAMPLE 3. Let $\mathscr{L}_m$ stand for a $2m \times 2m$ block-Toeplitz tridiagonal matrix generated by the matrix polynomial $L(\lambda) = L_0 + \lambda L_1 + \lambda^2 L_2$, where

$$L_0 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad L_1 = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrices

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

form a Jordan pair for $L(\lambda)$. Decompose Q and J into 2×2 matrices

$$Q=\begin{bmatrix} Q_1 & Q_2 \end{bmatrix}, \qquad J=\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix},$$

and observe that (0.1) implies

$$L_0Q_j + L_1Q_jJ_j + L_2Q_jJ_j^2 = 0 \qquad (j=1,2).$$

Hence the matrices

$$Z_1 = Q_1J_1Q_1^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix},$$

$$Z_2 = Q_2J_2Q_2^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

are (right) solvents for $L(\lambda)$. Since $\sigma(Z_1)\cap\sigma(Z_2)=\varnothing$ and $Z_1^j = Z_1$ for all positive integers j, it follows by (3.4) that

$$\det \mathscr{L}_m = \det\begin{bmatrix} 1-(-1)^{m+1} & 0 \\ 2 & -2^{m+1} \end{bmatrix} \Big/ \det\begin{bmatrix} -2 & 0 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{cases} 2^m & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases} \tag{3.6}$$

Note that the knowledge of a whole standard triple (Q, J, R) for $L(\lambda)$ allows one to compute $\det \mathscr{L}_m$ by a straightforward use of (2.8) [or (1.11) in this case]. Indeed,

$$J^{m+1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (-1)^{m+1} & 0 \\ 0 & 0 & 0 & 2^{m+1} \end{bmatrix}, \qquad R = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -1 & -1 \\ -1 & 0 \\ -1 & 1 \end{bmatrix},$$

and hence $\det(QJ^{m+1}R)$ coincides with the right-hand expression in (3.6).

Another possibility is, of course, to compute the determinant of the matrix $X_0C_L^{m+1}Y_0$ (see Preliminaries for definition), but this is usually a difficult task.

4. A DIFFERENT APPROACH

In the preceding sections the computation of the determinant of an $mn \times mn$ block-Toeplitz band matrix $\mathscr{L}_m$ is reduced to that of a $qn \times qn$ determinant. We now derive a formula for $\det \mathscr{L}_m$ in terms of the determinant of an $ln \times ln$ generalized Vandermonde matrix. In spite of $l = q + k - 1 \geqslant q$, the latter determinant has some computational advantages.

THEOREM 4. *Let $\mathscr{L}_m$ be defined by* (2.1), *and let $L_l \neq 0$. Assume, in addition, that* $\det L_0 \neq 0$. *If (X, T) is a standard pair for the generating matrix polynomial $L(\lambda)$, then for $m > l \geqslant k \geqslant 2$*

$$\det \mathscr{L}_m = (-1)^r \frac{(\det L_l)^m (\det V)}{\det \tilde{X}}, \tag{4.1}$$

where $r = (m-k)(l-k+1)n^2$ and

$$V = \begin{bmatrix} X \\ \vdots \\ XT^{k-2} \\ XT^{m+k-1} \\ \vdots \\ XT^{m+l-1} \end{bmatrix}, \qquad \tilde{X} = \operatorname{col}(XT^i)_{i=0}^{l-1} \tag{4.2}$$

Proof. Define an $mn \times mn$ matrix

$$\mathscr{F}_m = \begin{bmatrix} \hat{X} & \vdots & \begin{matrix} 0 \\ I_{m-l} \end{matrix} \end{bmatrix},$$

where $\hat{X} = \operatorname{col}(XT^{k+i-1})_{i=0}^{m-1}$. Note that since $\det L_0 \neq 0$, it follows that $\det T \neq 0$ (take, for instance, $T = C_L$, the companion matrix) and therefore the matrix $\mathscr{F}_m$ is invertible. Now use (0.1) to obtain

$$\mathscr{L}_m \mathscr{F}_m = \begin{bmatrix} G_1 & \vdots & \\ 0 & \vdots & \tilde{\mathscr{L}}_m \\ G_2 & \vdots & \end{bmatrix}, \tag{4.3}$$

where $\tilde{\mathscr{L}}_m = \mathscr{L}_m[1,2,\dots,mn|ln+1,\dots,mn]$ and

$$G_1 = -P_{k-1}\tilde{L}_{0,k-2}P_{k-1}\operatorname{col}(XT^i)_{i=0}^{k-2},$$

$$G_2 = -\tilde{L}_{l,k}\operatorname{col}(XT^{m+k+i-1})_{i=0}^{l-k}.$$

Permuting rows in (4.3), we obtain, exploiting the notation in (4.2), that

$$(\det \mathscr{L}_m)(\det \tilde{X})(\det T)^{k-1} = (-1)^{\nu}\det\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}(\det L_l)^{m-l}, \qquad (4.4)$$

where $\nu = [(m-k+1)n+1]qn$, $q = l-k+1$. Represent

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = -\begin{bmatrix} P_{k-1}\tilde{L}_{0,k-2}P_{k-1} & 0 \\ 0 & \tilde{L}_{l-k} \end{bmatrix} V,$$

where V is defined in (4.2), and observe that

$$\det\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = (-1)^{ln}(\det L_0)^{k-1}(\det L_l)^q(\det V).$$

It remains to observe that

$$\det T = \det C_L = (-1)^{ln^2}\frac{\det L_0}{\det L_l}. \qquad \blacksquare$$

Theorem 4 applied to a scalar polynomial gives the expression for det $\mathscr{L}_m$ obtained in [11] and used there in studying the eigenvalue problem for Toeplitz band matrices.

Let $n = 1$ in (2.1) and $L_0 \neq 0$, $L_l \neq 0$. Suppose that $\lambda_1, \lambda_2, \dots, \lambda_s$ are the distinct zeros of $L(\lambda)$ with multiplicities $r_1, r_2, \dots, r_s$ ($r_1 + r_2 + \cdots + r_s = l$), respectively. Given integers m and k, define the lth-order vector function

$$A(x) = \begin{bmatrix} 1 & x & \cdots & x^{k-2} & x^{m+k-1} & \cdots & x^{m+l-1} \end{bmatrix}^T,$$

and denote by $A^{(j)}(x)$ its jth derivative. Now construct the generalized Vandermonde matrix V_m associated with the given polynomial $L(\lambda)$ as follows: the first r_1 columns of V_m are $A(\lambda_1), A'(\lambda_1), \dots, A^{(r_1-1)}(\lambda_1)$, respectively; the next r_2 columns are $A(\lambda_2), A'(\lambda_2), \dots, A^{(r_2-1)}(\lambda_2)$; and so on.

COROLLARY 5 [11]. *With the notation of the preceding paragraph,*

$$\det \mathscr{L}_m = (-1)^r L_l^m \frac{\det V_m}{\det V_0},$$

where r is defined in Theorem 4.

In particular, if all the zeros of $L(\lambda)$ *are distinct, then*

$$\det \mathscr{L}_m = (-1)^r L_l^m \det \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_l \\ \vdots & & \vdots \\ \lambda_1^{k-2} & \cdots & \lambda_l^{k-2} \\ \lambda_1^{m+k-1} & \cdots & \lambda_l^{m+k-1} \\ \vdots & & \vdots \\ \lambda_1^{l-1} & \cdots & \lambda_l^{l-1} \end{bmatrix} (\det V^{(0)})^{-1},$$

where $V^{(0)}$ *stands for the ordinary Vandermonde matrix constructed from* $\lambda_1, \lambda_2, \dots, \lambda_l$.

For the *proof* it suffices to observe that the Jordan pair (Q_j, J_j) associated with the zero λ_j of $L(\lambda)$ with multiplicity r_j is given by the formula

$$Q_j = [1 \quad 0 \quad \cdots \quad 0], \qquad J_j = \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_j & 1 \\ & & & \lambda_j \end{bmatrix},$$

where $Q_j \in \mathbb{C}^{1 \times r_j}$, $J_j \in \mathbb{C}^{r_j \times r_j}$. Hence

$$Q = [Q_1 \quad Q_2 \quad \cdots \quad Q_s], \qquad J = [\delta_{jk} J_j]_{j,k=1}^{s}$$

is a Jordan pair for $L(\lambda)$ in the general case. If all the zeros of $L(\lambda)$ are simple, then

$$Q = [1 \quad 1 \quad \cdots \quad 1], \qquad J = [\delta_{jk} \lambda_j]_{j,k=1}^{l}$$

form a Jordan pair for $L(\lambda)$.

Now apply Theorem 4 with $X = Q$, $T = J$. ■

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