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Random analytic solution of coupled differential models with uncertain initial condition and source term

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Abstract

This paper deals with the construction of random power series solution of vector initial value problems containing uncertainty in both initial condition and source term. Under appropriate hypothesis on the data, we prove that the random series solution constructed by a random Fröbenius method is convergent in the mean square sense. Also, the main statistical functions of the approximating stochastic process solution generated by truncation of the exact series solution are given. Finally, we apply the proposed technique to several illustrative examples.

Keywords: Random differential equation; Power series solution; Mean square

1. Introduction

It is well known that deterministic differential equations are powerful tools to model a large class of important problems appearing in a wide variety of scientific disciplines as mechanics, control systems, neural networks, ecosystem dynamics, population genetics or economic models, for instance. However, in the last few years, in modeling, analyzing, and predicting the behaviors of physical and natural phenomena, greater emphasis has been placed upon probabilistic methods. This is due to combinations of complexity, uncertainties, and ignorance which are present in the formulation of a great number of these problems. As a consequence, differential equations containing uncertainty in their formulation are rapidly becoming an extensive way to formulate mathematical models.

In this paper we consider random coupled differential systems of the form

$$\left.\begin{array}{r}\dot{\vec{X}}(t)=\mathbf{A}(t)\vec{X}(t)+\vec{B}(t),\ |t|\leq c\\ \vec{X}(0)=\vec{X}_0,\end{array}\right\}, \tag{1.1}$$

where the coefficient $\mathbf{A}(t)$ is $\mathbb{R}^{r\times r}$ valued analytic function in $|t|\leq c$, the source term $\vec{B}(t)$ is a vectorial stochastic process of size $r\times 1$ which is also analytic in a stochastic sense that will be specified later, and the initial condition $\vec{X}_0$ is a random vector of the same size.

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Random linear matrix differential equations of type (1.1) appear in the study of the response of a mass-spring linear oscillator to a random excitation [1, p. 158], when determining the effect on a earthbound structure of an earthquake-type disturbance [2,3], in constructing compartmental models of pharmacokinetics [1, p. 223] or in formulating epidemiologic models [4]. Modeling population growth requires the estimation of some parameters such as the birth rate and the death rate which are difficult to obtain in practice because they are influenced by many circumstances like sanitation and healthcare, wars, pollution, medicines, diet, psychological stress and anxiety. Other factors that have to be considered are migration, living space restrictions, availability of food and water, climatology and epidemics. The huge complexity of such factors recommends to introduce them into mathematical models through random variables or stochastic processes instead of deterministic parameters or functions, respectively [1,4,5]. Then a special stochastic calculus (namely mean square calculus) for handling these uncertainty magnitudes will be required in this paper.

On the other hand, the study of random problems of type (1.1) provides a first stage to consider non-linear random models by means of linearization techniques. As the exact solution process of (1.1) is not available, in general, in [7, 8] one develops random numerical methods for constructing the numerical solutions of random differential systems both linear and nonlinear that can be considered as an alternative to the present work for approximating the process solution.

The aim of this paper is to construct random power series solutions of problem (1.1) and its organization is as follows. Section 2 contains stochastic results, definitions, and examples related to mean square convergence of vector random variables and vector stochastic processes that will play an important role in the following sections, particularly providing a random matrix version of the Mertens' theorem related to product matrix series. Section 3 deals with the construction of a series solution of (1.1) by a random Fröbenius method, as well as to prove its convergence in the mean square sense under appropriate conditions. This allows us to construct analytic–numerical finite series process solution by means of truncation whose main statistic properties (mean and covariance matrix functions) are computed in Section 4. Finally, several illustrative examples are given in Section 5.

2. Mean square calculus

For the sake of clarity in the presentation, we begin this section by introducing some concepts, notations and results that may be found in [1, chap. 4]. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a second-order random real variable (2-r.v.), that is,

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\mathrm{d}x < \infty, \tag{2.1}$$

where $f_X(x)$ denotes the probability density function of X and $E[\cdot]$ is the expectation operator. The set of all 2-r.v.'s defined on $(\Omega, \mathcal{F}, P)$ is denoted by L_2 and endowed with the norm $\|X\|_2 = \left(E\left[X^2\right]\right)^{1/2}$ has a Banach space structure. An important fact is that the 2-norm in L_2 does not provide a Banach algebra structure, i.e., it is not submultiplicative. In general, the property $\|XY\|_2 \le \|X\|_2 \|Y\|_2$ is not true.

Given a positive integer r, a second-order random vector of size r is a vector $\vec{X}$ whose entries X_i lie in L_2 for $1 \le i \le r$. The set L_2^r of all these vectors with the norm

$$\left\|\vec{X}\right\|_r = \max_{1\le i\le r} \|X_i\|_2 , \tag{2.2}$$

provides a Banach structure to L_2^r. It is clear from (2.2) that mean square convergence in L_2^r is equivalent to the componentwise mean square convergence. We say that $\left\{\vec{X}(t)\right\}_{t\in\mathcal{T}}$ is a second-order vector stochastic process (2-v.s.p.) in L_2^r, if $\vec{X}(t) \in L_2^r$ for each $t \in \mathcal{T}$, being $\mathcal{T}$ the so-called space of times. The expectation function of $\vec{X}(t)$ is the deterministic vector function $E\left[\vec{X}(t)\right] = (E\left[X_i(t)\right])_{r\times 1}$, note that by definition one gets $E\left[\vec{X}^{\mathrm{T}}(t)\right] = \left(E\left[\vec{X}(t)\right]\right)^{\mathrm{T}}$ where $\vec{X}^{\mathrm{T}}(t)$ denotes the transposed vector of $\vec{X}(t)$. The covariance matrix function of $\left\{\vec{X}(t)\right\}_{t\in\mathcal{T}}$ is defined by

$$\boldsymbol{\Lambda}_{\vec{X}(t)} = E\left[\left(\vec{X}(t) - E\left[\vec{X}(t)\right]\right)\left(\vec{X}(t) - E\left[\vec{X}(t)\right]\right)^{\mathrm{T}}\right] = \left(v_{ij}(t)\right)_{r\times r}, \quad t \in \mathcal{T}, \tag{2.3}$$

where

$$\begin{aligned} v_{ij}(t) &= E\left[\left(X_i(t) - E\left[X_i(t)\right]\right)\left(X_j(t) - E\left[X_j(t)\right]\right)\right] \\ &= E\left[X_i(t)X_j(t)\right] - E\left[X_i(t)\right]E\left[X_j(t)\right], \quad 1 \le i, j \le r, t \in \mathcal{T}. \end{aligned} \tag{2.4}$$

Note that $v_{ii}(t)$ denoted by $V\left[X_i(t)\right]$ is the variance of the r.v. $X_i(t)$, $1 \le i \le r$. Given two 2-v.s.p.'s in L_2^r, say $\left\{\vec{X}(t)\right\}_{t\in\mathcal{T}} = \left\{(X_1(t), \ldots, X_r(t))^{\mathrm{T}}\right\}_{t\in\mathcal{T}}$ and $\left\{\vec{Y}(t)\right\}_{t\in\mathcal{T}} = \left\{(Y_1(t), \ldots, Y_r(t))^{\mathrm{T}}\right\}_{t\in\mathcal{T}}$, one defines their covariance matrix function by

$$\boldsymbol{\Lambda}_{\vec{X}(t),\vec{Y}(t)} = E\left[\left(\vec{X}(t) - E\left[\vec{X}(t)\right]\right)\left(\vec{Y}(t) - E\left[\vec{Y}(t)\right]\right)^{\mathrm{T}}\right] = \left(v_{ij}(t)\right)_{r\times r}, \quad t \in \mathcal{T}, \tag{2.5}$$

where

$$\begin{aligned} v_{ij}(t) &= E\left[\left(X_i(t) - E\left[X_i(t)\right]\right)\left(Y_j(t) - E\left[Y_j(t)\right]\right)\right] \\ &= E\left[X_i(t)Y_j(t)\right] - E\left[X_i(t)\right]E\left[Y_j(t)\right], \quad 1 \le i, j \le r, t \in \mathcal{T}. \end{aligned} \tag{2.6}$$

From the above definitions, one gets $\boldsymbol{\Lambda}_{\vec{X}(t),\vec{X}(t)} = \boldsymbol{\Lambda}_{\vec{X}(t)}$. For $t \in \mathcal{T}$ fixed, note that if $\{X_i(t)\}_{i=1}^r$ and $\{Y_i(t)\}_{i=1}^r$ are pairwise independent r.v.'s, then $\boldsymbol{\Lambda}_{\vec{X}(t),\vec{Y}(t)}$ is the null matrix of size $r \times r$. In the particular case where $\vec{X}(t) = \vec{Y}(t)$, for $t \in \mathcal{T}$ fixed, being $\{X_i(t)\}_{i=1}^r$ pairwise independent r.v.'s, $\boldsymbol{\Lambda}_{\vec{X}(t)}$ is the diagonal matrix $Diag\left(V\left[X_i(t)\right]\right)$, $1 \le i \le r$. Finally, note that the following property holds: $\boldsymbol{\Lambda}_{\vec{Y}(t),\vec{X}(t)} = \boldsymbol{\Lambda}^{\mathrm{T}}_{\vec{X}(t),\vec{Y}(t)}$.

We say that a sequence of second-order random vectors $\left\{\vec{X}_n\right\}_{n\ge 0}$ is mean square (m.s.) convergent to $\vec{X} \in L_2^r$ if

$$\lim_{n\to\infty} \left\|\vec{X}_n - \vec{X}\right\|_r = 0. \tag{2.7}$$

From the corresponding properties of its components, see [1, p. 88], if $\left\{\vec{X}_n\right\}$ is a sequence of random vectors in L_2^r m.s. convergent to $\vec{X}$, then

$$E\left[\vec{X}_n\right] \xrightarrow[n\to\infty]{} E\left[\vec{X}\right], \qquad \boldsymbol{\Lambda}_{\vec{X}_n} \xrightarrow[n\to\infty]{} \boldsymbol{\Lambda}_{\vec{X}}. \tag{2.8}$$

Given a matrix $\mathbf{A} = \left(a_{ij}\right)$ in $\mathbb{R}^{r\times r}$, we denote by $\|\mathbf{A}\|$ the norm defined as in [9, p. 57]

$$\|\mathbf{A}\| = \max_{1\le i\le r} \sum_{j=1}^{r} \left|a_{ij}\right|. \tag{2.9}$$

Now we establish a result that will be crucial in the following:

Lemma 2.1. *Let* $\mathbf{A}$ *be a matrix in* $\mathbb{R}^{r\times r}$ *and* $\vec{X} \in L_2^r$, *then*

$$\left\|\mathbf{A}\vec{X}\right\|_r \le \|\mathbf{A}\| \left\|\vec{X}\right\|_r. \tag{2.10}$$

Proof. By (2.2) and (2.9) it follows that

$$\begin{aligned} \left\|\mathbf{A}\vec{X}\right\|_r &= \max_{1\le i\le r} \left\|\sum_{k=1}^{r} a_{ik}X_k\right\|_2 \le \max_{1\le i\le r} \sum_{k=1}^{r} \|a_{ik}X_k\|_2 \\ &= \max_{1\le i\le r} \sum_{k=1}^{r} |a_{ik}| \, \|X_k\|_2 \le \left(\max_{1\le i\le r} \sum_{k=1}^{r} |a_{ik}|\right) \left\|\vec{X}\right\|_r \\ &= \|\mathbf{A}\| \left\|\vec{X}\right\|_r. \quad \square \end{aligned}$$

The following lemma is a direct consequence of Lemma 2.1:

Lemma 2.2. *Let* $\{\mathbf{B}_n\}_{n\geq 0}$ *be a sequence of matrices in* $\mathbb{R}^{r\times r}$ *and assume that* $\lim_{n\to\infty}\mathbf{B}_n = \mathbf{B}\in\mathbb{R}^{r\times r}$. *Let* $\vec{X}$ *be a second-order random vector in* L_2^r. *Then,* $\left\{\mathbf{B}_n\vec{X}\right\}_{n\geq 0}$ *is m.s. convergent in* L_2^r *to* $\mathbf{B}\vec{X}\in L_2^r$.

Proof. Note that by (2.10) one gets

$$\left\|\mathbf{B}_n\vec{X}-\mathbf{B}\vec{X}\right\|_r = \left\|(\mathbf{B}_n-\mathbf{B})\,\vec{X}\right\|_r \leq \|\mathbf{B}_n-\mathbf{B}\|\left\|\vec{X}\right\|_r \xrightarrow[n\to\infty]{} 0.$$

Given a sequence of 2-v.s.p.'s $\left\{\vec{X}_n(t)\right\}_{t\in\mathcal{T}}$ in L_2^r for each $t\in\mathcal{T}$, we say that it is uniformly mean square convergent to $\left\{\vec{X}(t)\right\}_{t\in\mathcal{T}}\in L_2^r$, if for every $\epsilon>0$, there exists $N>0$ such that for all $n\geq N$ (independent of t) one satisfies

$$\left\|\vec{X}_n(t)-\vec{X}(t)\right\|_r \leq \epsilon,\quad \forall t\in\mathcal{T}.\quad\square$$

The following result is a vector random version of Mertens' theorem.

Proposition 2.3. *Let* $\{\mathbf{A}_n\}_{n\geq 0}$, $\mathbf{A}$ *be matrices in* $\mathbb{R}^{r\times r}$ *such that the series* $\sum_{n\geq 0}\mathbf{A}_n$ *is absolutely convergent to* $\mathbf{A}$. *Let* $\vec{X}_n$ *be a second random vector in* L_2^r *such that* $\sum_{n\geq 0}\vec{X}_n$ *is mean square convergent to* $\vec{X}\in L_2^r$. *If* $\vec{C}_n=\sum_{k=0}^{n}\mathbf{A}_{n-k}\vec{X}_k$, *then* $\sum_{n\geq 0}\vec{C}_n$ *is m.s. convergent to* $\mathbf{A}\vec{X}$.

Proof. Let us introduce

$$\mathbf{A}'_n=\sum_{k=0}^{n}\mathbf{A}_k,\qquad \vec{X}'_n=\sum_{k=0}^{n}\vec{X}_k,\qquad \vec{C}'_n=\sum_{k=0}^{n}\vec{C}_k,\qquad \vec{Y}'_n=\vec{X}'_n-\vec{X},$$

and note that we wish to prove that $\left\{\vec{C}'_n\right\}$ is m.s. convergent to $\mathbf{A}\vec{X}$ in L_2^r. By definition of $\vec{C}'_n$ we have

$$\begin{aligned}
\vec{C}'_n &= \vec{C}_0+\vec{C}_1+\cdots+\vec{C}_n = \mathbf{A}_0\vec{X}_0+\left(\mathbf{A}_0\vec{X}_1+\mathbf{A}_1\vec{X}_0\right)+\cdots+\left(\mathbf{A}_0\vec{X}_n+\cdots+\mathbf{A}_n\vec{X}_0\right)\\
&= \mathbf{A}_0\left(\vec{X}_0+\cdots+\vec{X}_n\right)+\mathbf{A}_1\left(\vec{X}_0+\cdots+\vec{X}_{n-1}\right)+\cdots+\mathbf{A}_n\vec{X}_0\\
&= \mathbf{A}_0\vec{X}'_n+\mathbf{A}_1\vec{X}'_{n-1}+\cdots+\mathbf{A}_n\vec{X}'_0\\
&= \mathbf{A}_0\left(\vec{Y}'_n+\vec{X}\right)+\mathbf{A}_1\left(\vec{Y}'_{n-1}+\vec{X}\right)+\cdots+\mathbf{A}_n\left(\vec{Y}'_0+\vec{X}\right)\\
&= (\mathbf{A}_0+\mathbf{A}_1+\cdots+\mathbf{A}_n)\,\vec{X}+\mathbf{A}_0\vec{Y}'_n+\mathbf{A}_1\vec{Y}'_{n-1}+\cdots+\mathbf{A}_n\vec{Y}'_0\\
&= \mathbf{A}'_n\vec{X}+\left(\mathbf{A}_0\vec{Y}'_n+\mathbf{A}_1\vec{Y}'_{n-1}+\cdots+\mathbf{A}_n\vec{Y}'_0\right).
\end{aligned}$$

If we denote $\vec{Z}'_n=\mathbf{A}_0\vec{Y}'_n+\mathbf{A}_1\vec{Y}'_{n-1}+\cdots+\mathbf{A}_n\vec{Y}'_0$, then one gets

$$\vec{C}'_n=\mathbf{A}'_n\vec{X}+\vec{Z}'_n. \tag{2.11}$$

Note that by Lemma 2.2, $\left\{\mathbf{A}'_n\vec{X}\right\}$ is m.s. convergent to $\mathbf{A}\vec{X}$, and by (2.11) in order to prove the result it is sufficient to show that $\left\{\vec{Z}'_n\right\}$ is m.s. convergent to the vector process $\vec{0}$ in L_2^r. As $\left\{\vec{Y}'_n\right\}$ is m.s. convergent to the zero process in L_2^r, given $\epsilon>0$, there exists $N_0>0$ such that

$$\left\|\vec{Y}'_n\right\|_r<\frac{\epsilon}{2\alpha},\quad n\geq N_0, \tag{2.12}$$

being

$$\alpha=\sum_{n\geq 0}\|\mathbf{A}_n\|<+\infty. \tag{2.13}$$

Let $\beta_{N_0} > 0$ be defined as

$$\beta_{N_0} = \max \left\{ \left\| \vec{Y}'_0 \right\|_r, \ldots, \left\| \vec{Y}'_{N_0} \right\|_r \right\}, \tag{2.14}$$

and as sequences $\{\mathbf{A}_n\}_{n\geq 0}, \{\mathbf{A}_{n-1}\}_{n\geq 1}, \ldots, \left\{\mathbf{A}_{n-N_0}\right\}_{n\geq N_0}$ are convergent to the zero matrix in $\mathbb{R}^{r\times r}$, we may also assume that

$$\left\| \mathbf{A}_{n-j} \right\| \leq \frac{\epsilon}{2(N_0+1)\beta_{N_0}}, \quad j = 0, 1, \ldots, N_0. \tag{2.15}$$

Let $n > N_0$, then by Lemma 2.1, (2.13)–(2.15) one gets

$$\begin{aligned}
\left\| \vec{Z}'_n \right\|_r &\leq \left\| \mathbf{A}_n \vec{Y}'_0 + \cdots + \mathbf{A}_{n-N_0} \vec{Y}'_{N_0} \right\|_r + \left\| \mathbf{A}_{n-N_0-1} \vec{Y}'_{N_0+1} + \cdots + \mathbf{A}_0 \vec{Y}'_n \right\|_r \\
&\leq \|\mathbf{A}_n\| \left\| \vec{Y}'_0 \right\|_r + \cdots + \|\mathbf{A}_{n-N_0}\| \left\| \vec{Y}'_{N_0} \right\|_r + \|\mathbf{A}_{n-N_0-1}\| \left\| \vec{Y}'_{N_0+1} \right\|_r + \cdots + \|\mathbf{A}_0\| \left\| \vec{Y}'_n \right\|_r \\
&\leq \beta_{N_0}(N_0+1) \frac{\epsilon}{2(N_0+1)\beta_{N_0}} + \left(\left\| \mathbf{A}_{n-N_0-1} \right\| + \cdots + \|\mathbf{A}_0\| \right) \frac{\epsilon}{2\alpha} \\
&\leq \frac{\epsilon}{2} + \alpha \frac{\epsilon}{2\alpha} = \epsilon.
\end{aligned}$$

Hence the result is established. □

We say that the 2-v.s.p. $\left\{ \vec{X}(t) \right\}_{t\in\mathcal{T}}$ is mean square continuous in $\mathcal{T}$ if

$$\lim_{\tau\to 0} \left\| \vec{X}(t+\tau) - \vec{X}(t) \right\|_r = 0,$$

for each $t \in \mathcal{T}$ with $t + \tau \in \mathcal{T}$.

Example 2.4. Let $\left\{ \vec{X}_n \right\}_{n\geq 1}$ be a sequence of second random vectors in L_2^r and let $t \in \mathcal{T}$ being $\mathcal{T}$ an interval of $\mathbb{R}$, then the 2-v.s.p. $\left\{ n_0 \vec{X}_{n_0} t^{n_0-1} \right\}_{t\in\mathcal{T}}$ for each $n_0 \in \mathbb{N}$ is m.s. continuous for each $t \in \mathcal{T}$. In fact,

$$\left\| n_0 \vec{X}_{n_0} (t+\tau)^{n_0-1} - n_0 \vec{X}_{n_0} t^{n_0-1} \right\|_r = \left| n_0 \left((t+\tau)^{n_0-1} - t^{n_0-1} \right) \right| \left\| \vec{X}_{n_0} \right\|_r \xrightarrow[\tau\to 0]{} 0,$$

because $\left\| \vec{X}_{n_0} \right\|_r < \infty$ as $\vec{X}_{n_0} \in L_2^r$ for each $n_0 \in \mathbb{N}$ and the continuity of the deterministic function $f(t) = t^{n_0-1}$ with respect to t.

If $\left\{ \vec{X}(t) \right\}_{t\in T}$ is a 2-v.s.p., we say that $\left\{ \vec{X}(t) \right\}_{t\in T}$ is m.s. differentiable at $t \in \mathcal{T}$ if

$$\lim_{\tau\to 0} \left\| \frac{\vec{X}(t+\tau) - \vec{X}(t)}{\tau} - \dot{\vec{X}}(t) \right\|_r = 0,$$

for all $t \in \mathcal{T}$ with $t + \tau \in \mathcal{T}$. In this case, $\left\{ \dot{\vec{X}}(t) \right\}_{t\in\mathcal{T}}$ is called the m.s. derivative of $\left\{ \vec{X}(t) \right\}_{t\in\mathcal{T}}$.

Example 2.5. With the notation of Example 2.4, the vector process $\left\{ \vec{X}_{n_0} t^{n_0} \right\}_{t\in\mathcal{T}}$ is m.s. differentiable in $\mathcal{T}$ and its derivative is $\left\{ n_0 \vec{X}_{n_0} t^{n_0-1} \right\}_{t\in\mathcal{T}}$. Note that,

$$\left\| \frac{\vec{X}_{n_0}(t+\tau)^{n_0} - \vec{X}_{n_0} t^{n_0}}{\tau} - n_0 \vec{X}_{n_0} t^{n_0-1} \right\|_r = \left| \frac{(t+\tau)^{n_0} - t^{n_0}}{\tau} - n_0 t^{n_0-1} \right| \left\| \vec{X}_{n_0} \right\|_r \xrightarrow[\tau \to 0]{} 0.$$

A 2-v.s.p. $\left\{\vec{X}(t)\right\}_{t\in\mathcal{T}}$ in L_2^r for each $t \in \mathcal{T}$ is said to be mean square analytic in t_0 in the interval $\mathcal{T}$, if the vector series

$$\sum_{n\geq 0} \frac{\vec{X}^{(n)}(t_0)}{n!} (t - t_0)^n, \quad t, t_0 \in \mathcal{T},$$

where $\vec{X}^{(n)}(t_0)$ denotes the m.s. derivative of order n at $t = t_0$ of $\vec{X}(t)$, is m.s. convergent to the 2-v.s.p. $\left\{\vec{X}(t)\right\}_{t\in\mathcal{T}}$. In connection with the m.s. analyticity, one can extend the corresponding scalar characterization given in terms of the correlation function associated to the process (see [1, p. 99]) to vectorial case by means of the covariance matrix function as follows:

Proposition 2.6. *A second-order vectorial stochastic process* $\left\{\vec{X}(t)\right\}_{t\in\mathcal{T}}$ *is m.s. analytic on* $\mathcal{T}$ *if, and only if, its covariance matrix function* $\boldsymbol{\Lambda}_{\vec{X}(t),\vec{X}(t)}$ *is analytic at point t for every* $t \in \mathcal{T}$.

The following result is a direct consequence of the corresponding scalar version (see Theorem 3.1. of [10]), because the m.s. convergence of a process in L_2^r is equivalent to the componentwise m.s. convergence of each of its entries.

Proposition 2.7. *Assume that* $\left\{\vec{X}_n(t)\right\}_{t\in\mathcal{T}}$ *is a 2-v.s.p. in* L_2^r *for each n, where* $\mathcal{T}$ *is an interval, and satisfies:*

- $\vec{X}(t) = \sum_{n\geq 0} \vec{X}_n(t)$ *is m.s. convergent in* L_2^r.
- $\vec{X}_n(t)$ *is m.s. differentiable and* $\dot{\vec{X}}_n(t)$ *is m.s. continuous for each* $t \in \mathcal{T}$, $n \in \mathbb{N}$.
- $\sum_{n\geq 0} \dot{\vec{X}}_n(t)$ *is locally uniformly m.s. convergent in* L_2^r.

Then for each $t \in \mathcal{T}$, $\vec{X}(t)$ *is m.s. differentiable and*

$$\dot{\vec{X}}(t) = \sum_{n\geq 0} \dot{\vec{X}}_n(t).$$

3. Random power series solutions of linear systems

Consider the random differential system (1.1) where $\mathbf{A}(t)$ is a matrix valued deterministic analytic function about $t = 0$, i.e.,

$$\mathbf{A}(t) = \sum_{n\geq 0} \mathbf{A}_n t^n, \quad |t| \leq c, \tag{3.1}$$

where $\mathbf{A}_n \in \mathbb{R}^{r\times r}$, for each $n \geq 0$, and $\vec{B}(t)$ is an analytic 2-v.s.p. about $t = 0$

$$\vec{B}(t) = \sum_{n\geq 0} \vec{B}_n t^n, \quad |t| \leq c, \tag{3.2}$$

where $\vec{B}_n \in L_2^r$ and $\vec{X}_0 \in L_2^r$.

Let $F(\vec{X}, t) = \mathbf{A}(t)\vec{X} + \vec{B}(t)$, where $\vec{X} \in L_2^r$ and $|t| \leq c$. Then, by Lemma 2.1, it follows that

$$\left\| F(\vec{Y}, t) - F(\vec{X}, t) \right\|_r = \left\| \mathbf{A}(t)(\vec{Y} - \vec{X}) \right\|_r \leq \|\mathbf{A}(t)\| \left\| \vec{Y} - \vec{X} \right\|_r \leq k \left\| \vec{Y} - \vec{X} \right\|_r \tag{3.3}$$

where, from the continuity of $\mathbf{A}(t)$, one gets

$$\widetilde{a}_{ij} = \max_{|t|\le c} \left|a_{ij}(t)\right|; \qquad k = \max_{1\le i\le r} \sum_{j=1}^{r} \widetilde{a}_{ij} < +\infty. \tag{3.4}$$

From Theorem 5.1.2. [1, p. 118], condition (3.3) guarantees the existence of a unique m.s. solution of (1.1).

Let us seek a formal solution process of problem (1.1) of the form

$$\vec{X}(t) = \sum_{n\ge 0} \vec{X}_n t^n, \tag{3.5}$$

where coefficients $\vec{X}_n$ are second-order random vectors of size r to be determined. Assuming that $\vec{X}(t)$ is termwise m.s. differentiable and applying Propositions 2.3 and 2.7, it follows that

$$\dot{\vec{X}}(t) = \sum_{n\ge 1} n\vec{X}_n t^{n-1} = \sum_{n\ge 0} (n+1)\vec{X}_{n+1} t^n, \tag{3.6}$$

$$\mathbf{A}(t)\vec{X}(t) = \left(\sum_{n\ge 0} \mathbf{A}_n t^n\right)\left(\sum_{n\ge 0} \vec{X}_n t^n\right) = \sum_{n\ge 0}\left(\sum_{k=0}^{n} \mathbf{A}_{n-k}\vec{X}_k\right) t^n, \tag{3.7}$$

and by imposing that $\vec{X}(t)$ given by (3.5) satisfies (1.1), one gets

$$\sum_{n\ge 0}(n+1)\vec{X}_{n+1}t^n = \sum_{n\ge 0}\left\{\left(\sum_{k=0}^{n} \mathbf{A}_{n-k}\vec{X}_k\right) + \vec{B}_n\right\} t^n.$$

Hence

$$(n+1)\vec{X}_{n+1} = \vec{B}_n + \sum_{k=0}^{n} \mathbf{A}_{n-k}\vec{X}_k, \tag{3.8}$$

with the initial condition $\vec{X}_0 \in L_2^r$, or

$$\vec{X}_{n+1} = \frac{1}{n+1}\left\{\vec{B}_n + \sum_{k=0}^{n} \mathbf{A}_{n-k}\vec{X}_k\right\}, \quad n \ge 0. \tag{3.9}$$

Now, we prove that the stochastic process $\vec{X}(t)$ defined by (3.5) and (3.9) is m.s. convergent, termwise differentiable and then Proposition 2.3 is applicable. Let $\widehat{T} = \left|\widehat{t}\right| < c$, we show firstly that

$$\sum_{n\ge 0} \left\|\vec{X}_n\right\|_r |t|^n < +\infty, \quad |t| < \widehat{T}. \tag{3.10}$$

As $\mathbf{A}(t)$ and $\vec{B}(t)$ are analytic, by Cauchy inequalities, [11, p. 222], there exists $M > 0$ such that

$$\|\mathbf{A}_n\| \le \frac{M}{\widehat{T}^n}, \qquad \left\|\vec{B}_n\right\|_r \le \frac{M}{\widehat{T}^n}, \quad n \ge 0. \tag{3.11}$$

From (3.8), taking into account (3.11) and Lemma 2.1, it follows that

$$\begin{aligned}(n+1)\left\|\vec{X}_{n+1}\right\|_r &\le \left\|\vec{B}_n\right\|_r + \sum_{k=0}^{n} \|\mathbf{A}_{n-k}\| \left\|\vec{X}_k\right\|_r \\ &\le \frac{M}{\widehat{T}^n} + \sum_{k=0}^{n} \frac{M}{\widehat{T}^{n-k}}\left\|\vec{X}_k\right\|_r \\ &= \frac{M}{\widehat{T}^n}\left\{1 + \sum_{k=0}^{n} \widehat{T}^k \left\|\vec{X}_k\right\|_r\right\}.\end{aligned} \tag{3.12}$$

Let us introduce positive numbers y_n, such that

$$y_0 = \left\| \vec{X}_0 \right\|_r , \tag{3.13}$$

and inductively y_n is defined by the equation

$$(n+1) y_{n+1} = \frac{M}{\widehat{T}^n} \left\{ 1 + \sum_{k=0}^{n} \widehat{T}^k y_k \right\}, \quad n \geq 0. \tag{3.14}$$

By definition and (3.12) it follows that

$$\left\| \vec{X}_n \right\|_r \leq y_n, \quad n \geq 0. \tag{3.15}$$

From (3.14) it follows that

$$y_{n+1} = \frac{M + n\widehat{T}^{-1}}{n+1} y_n,$$

and for $|t| < \widehat{T}$ one gets

$$\lim_{n \to \infty} \frac{y_{n+1} \, |t|^{n+1}}{y_n \, |t|^n} = \lim_{n \to \infty} \frac{M\widehat{T} + n}{(n+1)\widehat{T}} \, |t| = \frac{|t|}{\widehat{T}} < 1. \tag{3.16}$$

As $\widehat{T} = |t| < c$ is arbitrary, by (3.16) one gets the absolutely norm convergence in L_2^r of the series defined by (3.5) and (3.9) for all t with $|t| < c$. Let us prove that $\vec{X}(t)$ defined by (3.5) and (3.9) is termwise m.s. differentiable at t with $0 < |t| < c$. Take ρ with $0 < |t| < \rho < c$, and since

$$\sum_{n \geq 0} \left\| \vec{X}_n \right\|_r \rho^n < +\infty, \tag{3.17}$$

given $\epsilon > 0$, there exists N_0 such that

$$\left\| \vec{X}_{n+1} \right\|_r \rho^{n+1} < \epsilon, \quad \forall n \geq N_0.$$

Note that for $n \geq N_0$, $|t| < \left|t'\right| < \rho$,

$$\begin{aligned} \left\| (n+1) \vec{X}_{n+1} \left(t'\right)^n \right\|_r &\leq (n+1) \left\| \vec{X}_{n+1} \right\|_r \left|t'\right|^n = (n+1) \left\| \vec{X}_{n+1} \right\|_r \rho^{n+1} \frac{1}{\rho} \left(\frac{\left|t'\right|}{\rho} \right)^n \\ &< \frac{n+1}{\rho} \epsilon \left(\frac{\left|t'\right|}{\rho} \right)^n = c_n, \end{aligned}$$

with

$$\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = \lim_{n \to \infty} \frac{n+2}{n+1} \frac{\left|t'\right|}{\rho} = \frac{\left|t'\right|}{\rho} < 1.$$

Hence, the series $\sum_{n \geq 0} (n+1) \vec{X}_{n+1} \tau^n$ is m.s. uniformly absolutely convergent in $|\tau| \leq \left|t'\right|$. Thus hypotheses of Proposition 2.7 are satisfied and series (3.5) and (3.9) are termwise m.s. differentiable.

Remark 1. If $\vec{B}(t)$ is the null vector stochastic process and $\vec{X}_0$ is a random vector in L_2^r, then applying the above random Fröbenius method one gets that the m.s. solution of (1.1) is given by $\vec{X}(t) = \exp(\mathbf{A}t)\vec{X}_0$,as it is well known, see [1, chap. 6]. However this result does not hold, in general, if coefficient $\mathbf{A}(t)$ is a matrix stochastic process or even a matrix random variable without imposing additional conditions on $\mathbf{A}(t)$, see [12].

Remark 2. From a practical point of view it is important to point out that processes such as the Brownian motion or white noise are not allowed to play the role of $\vec{B}(t)$ into (1.1) because they are not m.s. differentiable anywhere, but in the last section we shall show some interesting examples where the developing work can be applied successfully.

4. Statistic properties of the approximating process

From a practical point of view, it is interesting to compute the expectation vector and the covariance matrix of the approximating solution stochastic process. Using the property for scalar 2-r.v.'s, [1, p. 97], in the vector case one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E\left[\vec{X}(t)\right]\right) = E\left[\dot{\vec{X}}(t)\right], \tag{4.1}$$

where the derivative of the left-hand side of (4.1) is the deterministic one, and the one of the right-hand side is the m.s. derivative. Then taking expectations in (1.1), one gets the deterministic analytic system

$$\left.\begin{array}{l} \frac{\mathrm{d}}{\mathrm{d}t}\left(E\left[\vec{X}(t)\right]\right) = \mathbf{A}(t)E\left[\vec{X}(t)\right] + E\left[\vec{B}(t)\right], \quad |t| \le c \\ E\left[\vec{X}(0)\right] = E\left[\vec{X}_0\right]. \end{array}\right\}. \tag{4.2}$$

Problem (4.2) admits a series solution whose construction is the same as the one of above section, although (4.2) is a deterministic problem. This allows to construct an approximating process by the truncation of the series defined by (3.5) and (3.9)

$$\vec{X}_N(t) = \sum_{n=0}^{N} \vec{X}_n t^n. \tag{4.3}$$

In order to compute its expectation, note that

$$E\left[\vec{X}_N(t)\right] = \sum_{n=0}^{N} E\left[\vec{X}_n\right] t^n, \tag{4.4}$$

where taking into account (3.9), series coefficients are given by

$$E\left[\vec{X}_n\right] = \frac{1}{n}\left\{E\left[\vec{B}_{n-1}\right] + \sum_{k=0}^{n-1} \mathbf{A}_{n-k-1} E\left[\vec{X}_k\right]\right\}, \quad n \ge 1, \tag{4.5}$$

where $E\left[\vec{X}_0\right]$ as well as $E\left[\vec{B}_n\right]$ for $n \ge 0$ are known. In a similar way, from (3.9) one gets

$$E\left[\vec{X}_n^{\mathrm{T}}\right] = \frac{1}{n}\left\{E\left[\vec{B}_{n-1}^{\mathrm{T}}\right] + \sum_{k=0}^{n-1} E\left[\vec{X}_k^{\mathrm{T}}\right] \mathbf{A}_{n-k-1}^{\mathrm{T}}\right\}, \quad n \ge 1. \tag{4.6}$$

It is worth pointing out that in [6] one develops an analytic–numerical procedure for constructing approximations of (4.2) with a prefixed accuracy.

Let us compute the covariance matrix of the approximating process $\vec{X}_N(t)$ defined by (4.3). For that we will assume that $\boldsymbol{\Lambda}_{\vec{X}_0,\vec{X}_0} = \boldsymbol{\Lambda}_{\vec{X}_0}$, $\boldsymbol{\Lambda}_{\vec{B}_i,\vec{X}_0} = \boldsymbol{\Lambda}^{\mathrm{T}}_{\vec{X}_0,\vec{B}_i}$ and $\boldsymbol{\Lambda}_{\vec{B}_i,\vec{B}_j} = \boldsymbol{\Lambda}^{\mathrm{T}}_{\vec{B}_j,\vec{B}_i}$, $0 \le i, j \le N$ are data. Considering (4.3) and the corresponding definition, the covariance matrix of $\vec{X}_N(t)$ is given by

$$\begin{aligned}
\boldsymbol{\Lambda}_{\vec{X}_N(t)} &= E\left[\left(\vec{X}_N(t) - E\left[\vec{X}_N(t)\right]\right)\left(\vec{X}_N(t) - E\left[\vec{X}_N(t)\right]\right)^{\mathrm{T}}\right] \\
&= E\left[\vec{X}_N(t)\vec{X}_N(t)^{\mathrm{T}}\right] - E\left[\vec{X}_N(t)\right] E\left[\vec{X}_N(t)\right]^{\mathrm{T}} \\
&= E\left[\left(\sum_{n=0}^{N} \vec{X}_n t^n\right)\left(\sum_{n=0}^{N} \vec{X}_n^{\mathrm{T}} t^n\right)\right] - \left(\sum_{n=0}^{N} E\left[\vec{X}_n\right] t^n\right)\left(\sum_{n=0}^{N} E\left[\vec{X}_n^{\mathrm{T}}\right] t^n\right) \\
&= \sum_{i=0}^{N}\sum_{j=0}^{N} E\left[\vec{X}_i \vec{X}_j^{\mathrm{T}}\right] t^{i+j} - \sum_{i=0}^{N}\sum_{j=0}^{N} E\left[\vec{X}_i\right] E\left[\vec{X}_j^{\mathrm{T}}\right] t^{i+j} \\
&= \sum_{i=0}^{N}\sum_{j=0}^{N} \left(E\left[\vec{X}_i \vec{X}_j^{\mathrm{T}}\right] - E\left[\vec{X}_i\right] E\left[\vec{X}_j^{\mathrm{T}}\right]\right) t^{i+j} \\
&= \sum_{i=0}^{N}\sum_{j=0}^{N} \boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_j} t^{i+j}.
\end{aligned} \tag{4.7}$$

In order to save computations, considering in (4.7) the property $\boldsymbol{\Lambda}_{\vec{X}_j,\vec{X}_i} = \boldsymbol{\Lambda}^{\mathrm{T}}_{\vec{X}_i,\vec{X}_j}$, then

$$\boldsymbol{\Lambda}_{\vec{X}_N(t)} = \sum_{i=0}^{N} \boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_i} t^{2i} + \sum_{i=1}^{N}\sum_{j=0}^{i-1} \left(\boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_j} + \boldsymbol{\Lambda}^{\mathrm{T}}_{\vec{X}_i,\vec{X}_j}\right) t^{i+j}.$$

For the sake of clarity in the next computations, let us write the above expression in the form:

$$\boldsymbol{\Lambda}_{\vec{X}_N(t)} = \boldsymbol{\Lambda}_{\vec{X}_0,\vec{X}_0} + \sum_{i=1}^{N} \boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_i} t^{2i} + \sum_{i=1}^{N} \left(\boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_0} + \boldsymbol{\Lambda}^{\mathrm{T}}_{\vec{X}_i,\vec{X}_0}\right) t^{i} + \sum_{i=2}^{N}\sum_{j=1}^{i-1} \left(\boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_j} + \boldsymbol{\Lambda}^{\mathrm{T}}_{\vec{X}_i,\vec{X}_j}\right) t^{i+j}. \tag{4.8}$$

Now we show that the computations involved in (4.8) can be handled in practice. From (3.9) and (4.5) one gets

$$\begin{aligned}
\boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_0} &= E\left[\left(\vec{X}_i - E\left[\vec{X}_i\right]\right)\left(\vec{X}_0 - E\left[\vec{X}_0\right]\right)^{\mathrm{T}}\right] \\
&= E\left[\left(\frac{1}{i}\left\{\vec{B}_{i-1} + \sum_{k=0}^{i-1} \mathbf{A}_{i-k-1}\vec{X}_k\right\} - \frac{1}{i}\left\{E\left[\vec{B}_{i-1}\right] + \sum_{k=0}^{i-1} \mathbf{A}_{i-k-1} E\left[\vec{X}_k\right]\right\}\right)\left(\vec{X}_0 - E\left[\vec{X}_0\right]\right)^{\mathrm{T}}\right] \\
&= \frac{1}{i} E\left[\left(\left(\vec{B}_{i-1} - E\left[\vec{B}_{i-1}\right]\right) + \sum_{k=0}^{i-1} \mathbf{A}_{i-k-1}\left(\vec{X}_k - E\left[\vec{X}_k\right]\right)\right)\left(\vec{X}_0 - E\left[\vec{X}_0\right]\right)^{\mathrm{T}}\right] \\
&= \frac{1}{i} \boldsymbol{\Lambda}_{\vec{B}_{i-1},\vec{X}_0} + \frac{1}{i} \sum_{k=0}^{i-1} \mathbf{A}_{i-k-1} \boldsymbol{\Lambda}_{\vec{X}_k,\vec{X}_0}, \quad i \geq 1,
\end{aligned} \tag{4.9}$$

thus, the terms $\boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_0}$ appearing in (4.8) can be computed from the data $\boldsymbol{\Lambda}_{\vec{X}_0,\vec{X}_0}$ and $\boldsymbol{\Lambda}_{\vec{B}_{i-1},\vec{X}_0}$.

In (4.8), it only remains to explain the computation of matrices $\boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_j}$ for $i \geq 1$, $1 \leq j \leq i$. For this goal, note that

$$\begin{aligned}
\boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_j} &= E\left[\left(\vec{X}_i - E\left[\vec{X}_i\right]\right)\left(\vec{X}_j - E\left[\vec{X}_j\right]\right)^{\mathrm{T}}\right] \\
&= E\left[\left(\vec{X}_i - E\left[\vec{X}_i\right]\right)\left(\frac{1}{j}\left\{\left(\vec{B}_{j-1} - E\left[\vec{B}_{j-1}\right]\right)^{\mathrm{T}} + \sum_{l=0}^{j-1}\left(\vec{X}_l - E\left[\vec{X}_l\right]\right)^{\mathrm{T}} \mathbf{A}^{\mathrm{T}}_{j-l-1}\right\}\right)\right] \\
&= \frac{1}{j} \boldsymbol{\Lambda}_{\vec{X}_i,\vec{B}_{j-1}} + \frac{1}{j} \sum_{l=0}^{j-1} \boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_l} \mathbf{A}^{\mathrm{T}}_{j-l-1}, \quad i \geq 1, 1 \leq j \leq i,
\end{aligned} \tag{4.10}$$

then, one requires to know the matrices $\boldsymbol{\Lambda}_{\vec{X}_i,\vec{B}_{j-1}}$ and $\boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_l}$ for $0 \le l \le j-1$. Following the same procedure as the one used to obtain (4.10), one gets

$$\boldsymbol{\Lambda}_{\vec{X}_i,\vec{B}_{j-1}} = \frac{1}{i}\boldsymbol{\Lambda}_{\vec{B}_{i-1},\vec{B}_{j-1}} + \frac{1}{i}\sum_{k=0}^{i-1}\mathbf{A}_{i-k-1}\boldsymbol{\Lambda}_{\vec{X}_k,\vec{B}_{j-1}}. \tag{4.11}$$

Thus, $\boldsymbol{\Lambda}_{\vec{X}_i,\vec{B}_{j-1}}$ one computes from the data $\boldsymbol{\Lambda}_{\vec{B}_{i-1},\vec{B}_{j-1}}$ and $\boldsymbol{\Lambda}_{\vec{X}_0,\vec{B}_{j-1}}$. On the other hand, for $0 \le l \le j-1$ fixed, in order to compute $\boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_l}$ by (4.10) it is necessary to know $\boldsymbol{\Lambda}_{\vec{X}_i,\vec{B}_{l-1}}$ which is computable by (4.11) from the data $\boldsymbol{\Lambda}_{\vec{X}_0,\vec{B}_{l-1}}$ and $\boldsymbol{\Lambda}_{\vec{X}_i,\vec{X}_0}$.

5. Examples

Finally, we show two examples of application of model (1.1). In the first one, the solution of the deterministic Eq. (4.2) is available, then we can compare the results obtained by the proposed method in order to approximate the expectation of the stochastic process solution of problem (1.1).

Example 5.1. Let us consider a random prey–predator model based upon (1.1), where

$$\mathbf{A}(t) = \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \qquad \vec{B}(t) = \begin{bmatrix} 0 \\ Ze^t \end{bmatrix}, \tag{5.1}$$

being Z an exponential r.v. of parameter $\lambda = 0.5$, $Z \sim \text{Exp}(\lambda = 0.5)$, then $E[Z] = 2$. Note that, species S_1 is predator of S_2 with null *per capita* growth rate and, since its *per capita* growth rate is negative, species S_2 becomes extinct. Moreover, we are assuming that incorporation of foreign prey individuals is allowed. Finally, let us suppose that initial conditions are described by $X_1(0) = U_1$ and $X_2(0) = U_2$, where U_1 and U_2 are uniform r.v.'s on intervals [0, 2] and [0, 10], respectively. In this case, coefficients $\mathbf{A}_n$ of (3.1) are given by

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \qquad \mathbf{A}_n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad n \ge 1. \tag{5.2}$$

The covariance matrix function of $\vec{B}(t)$ in the diagonal points is the matrix,

$$\boldsymbol{\Lambda}_{\vec{B}(t),\vec{B}(t)} = E\left[\left(\vec{B}(t) - E\left[\vec{B}(t)\right]\right)\left(\vec{B}(t) - E\left[\vec{B}(t)\right]\right)^{\mathrm{T}}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 4e^{2t} \end{bmatrix}, \tag{5.3}$$

then applying Proposition 2.6 one deduces that $\vec{B}(t)$ is m.s. analytic. Moreover, coefficients $\vec{B}_n$ of (3.2) are given by

$$\vec{B}_n = \begin{bmatrix} 0 \\ Z\dfrac{1}{n!} \end{bmatrix}, \quad n \ge 0. \tag{5.4}$$

Table 1 compares the expectation of the approximating stochastic process $\vec{X}_N(t)$ given by (4.3)–(4.5) for different orders of truncation N on the interval $t \in [0, 2]$ and the exact solution of Eq. (4.2), which is given by

$$E\left[\vec{X}(t)\right] = \begin{bmatrix} \dfrac{1}{2}e^{-t}\left(1 + e^{2t} + 10t\right) \\ \dfrac{1}{2}e^{-t}\left(9 + e^{2t} - 10t\right) \end{bmatrix}. \tag{5.5}$$

For that, we compute the componentwise absolute difference of vectors $E\left[\vec{X}(t)\right] - E\left[\vec{X}_N(t)\right]$, that will be denoted by $\left|E\left[\vec{X}(t)\right] - E\left[\vec{X}_N(t)\right]\right|$. From Table 1 one observes that for t fixed, the approximation is better when N increases, and for a fixed value of N, the approximations are worse when t separates from the origin, where the problem is proposed. It is worth pointing out after computing that every value of the corresponding Table 1 for $N = 30$ is of the order 10^{-15} at least.

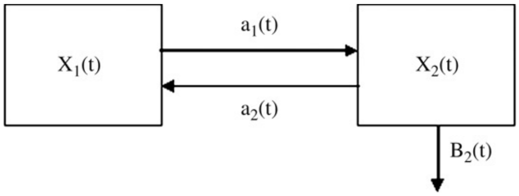

Fig. 1. A two-compartment model in chemical kinetics.

Table 1
Comparison between exact and approximate expectations for Example 5.1

t	$\left\lvert E[\vec{X}(t)] - E[\vec{X}_5(t)]\right\rvert$	$\left\lvert E[\vec{X}(t)] - E[\vec{X}_{10}(t)]\right\rvert$	$\left\lvert E[\vec{X}(t)] - E[\vec{X}_{15}(t)]\right\rvert$
0	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$
0.25	$\begin{pmatrix}0.00019\\0.00022\end{pmatrix}$	$\begin{pmatrix}3.21521 \times 10^{-13}\\3.44169 \times 10^{-13}\end{pmatrix}$	$\begin{pmatrix}2.978644 \times 10^{-20}\\6.889865 \times 10^{-20}\end{pmatrix}$
0.50	$\begin{pmatrix}0.00593\\0.00685\end{pmatrix}$	$\begin{pmatrix}6.43942 \times 10^{-10}\\6.89888 \times 10^{-13}\end{pmatrix}$	$\begin{pmatrix}4.44089 \times 10^{-16}\\4.44089 \times 10^{-16}\end{pmatrix}$
0.75	$\begin{pmatrix}0.04311\\0.04963\end{pmatrix}$	$\begin{pmatrix}5.45273 \times 10^{-8}\\5.83769 \times 10^{-8}\end{pmatrix}$	$\begin{pmatrix}3.64153 \times 10^{-14}\\3.86358 \times 10^{-14}\end{pmatrix}$
1.00	$\begin{pmatrix}0.17415\\0.19980\end{pmatrix}$	$\begin{pmatrix}1.26439 \times 10^{-6}\\1.35265 \times 10^{-6}\end{pmatrix}$	$\begin{pmatrix}3.55005 \times 10^{-12}\\3.82716 \times 10^{-12}\end{pmatrix}$
1.25	$\begin{pmatrix}0.51030\\0.58320\end{pmatrix}$	$\begin{pmatrix}0.000014422\\0.000015416\end{pmatrix}$	$\begin{pmatrix}1.24251 \times 10^{-10}\\1.33990 \times 10^{-10}\end{pmatrix}$
1.50	$\begin{pmatrix}1.22120\\1.38948\end{pmatrix}$	$\begin{pmatrix}0.000105024\\0.000112173\end{pmatrix}$	$\begin{pmatrix}2.26358 \times 10^{-9}\\2.44218 \times 10^{-9}\end{pmatrix}$
1.75	$\begin{pmatrix}2.54249\\2.87833\end{pmatrix}$	$\begin{pmatrix}0.000561249\\0.000598921\end{pmatrix}$	$\begin{pmatrix}2.62795 \times 10^{-8}\\2.83672 \times 10^{-8}\end{pmatrix}$
2.00	$\begin{pmatrix}4.78222\\5.38315\end{pmatrix}$	$\begin{pmatrix}0.002391560\\0.00254969\end{pmatrix}$	$\begin{pmatrix}2.19401 \times 10^{-7}\\2.36951 \times 10^{-7}\end{pmatrix}$

Note that coefficients of $E\left[\vec{X}_N(t)\right]$ defined by (4.5), in our case are given by

$$E\left[\vec{X}_0\right] = \begin{bmatrix}1\\5\end{bmatrix}, \tag{5.6}$$

$$E\left[\vec{X}_n\right] = \frac{1}{n}\left\{\begin{bmatrix}0\\ \dfrac{2}{(n-1)!}\end{bmatrix} + \begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix} E\left[\vec{X}_{n-1}\right]\right\}, \quad n \geq 1. \tag{5.7}$$

Example 5.2. Let us consider a two-compartment model in chemical kinetics or pharmacokinetics as shown in Fig. 1. The quantities $X_1(t)$ and $X_2(t)$ represent the chemical concentrations in the compartments; $a_1(t) = \mathrm{e}^{-t}$ and $a_2(t) = \mathrm{e}^{-2t}$ represent the time varying deterministic rate; $B_2(t) = E_2 \sim \mathrm{Exp}\,(\lambda_2 = 1000)$ is an exponential r.v. with parameter $\lambda_2 = 1000$, so $E\,[E_2] = 0.001$. Let us assume that initial conditions $X_{10} = U_1$ and $X_{20} = U_2$ are uniform r.v.'s on the interval [0, 1], then $E\,[U_1] = E\,[U_2] = 0.5$. This problem is modeled by the system (1.1) where

$$\mathbf{A}(t) = \begin{bmatrix}-a_1(t) & a_2(t)\\ a_1(t) & -a_2(t)\end{bmatrix} = \begin{bmatrix}-\mathrm{e}^{-t} & \mathrm{e}^{-2t}\\ \mathrm{e}^{-t} & -\mathrm{e}^{-2t}\end{bmatrix}, \tag{5.8}$$

$$\vec{B}(t) = \begin{bmatrix}0\\ -B_2(t)\end{bmatrix} = \begin{bmatrix}0\\ -E_2\end{bmatrix}, \qquad \vec{X}(0) = \begin{bmatrix}X_{10}\\ X_{20}\end{bmatrix} = \begin{bmatrix}U_1\\ U_2\end{bmatrix}. \tag{5.9}$$

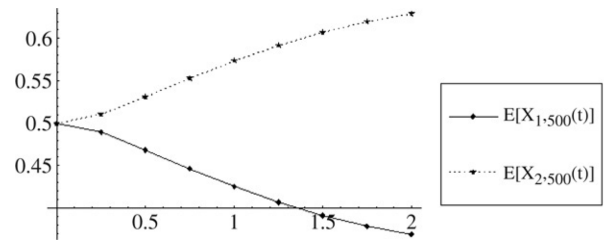

Fig. 2. Plotting $E\left[\vec{X}_N(t)\right]$ for $N = 500$ on the interval $t \in [0, 2]$, for Example 5.2.

Now, coefficients $\mathbf{A}_n$ of matrix (3.1) are given by

$$\mathbf{A}_n = \frac{(-1)^n}{n!}\begin{bmatrix} -1 & 2^n \\ 1 & -2^n \end{bmatrix}, \quad n \geq 0. \tag{5.10}$$

In this case, the covariance matrix function of $\vec{B}(t)$ in the diagonal points is given by

$$\Lambda_{\vec{B}(t),\vec{B}(t)} = \begin{bmatrix} 0 & 0 \\ 0 & 10^{-6} \end{bmatrix}, \tag{5.11}$$

then Proposition 2.6 assures that $\vec{B}(t)$ is m.s. analytic. The coefficients $\vec{B}_n$ of (3.2) are given by

$$\vec{B}_0 = \begin{bmatrix} 0 \\ -E_2 \end{bmatrix}, \qquad \vec{B}_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad n \geq 1. \tag{5.12}$$

Fig. 2 shows the expectation of the components of the approximating stochastic process $\vec{X}_N(t)$ given by (4.3) for the order of truncation $N = 500$ on the interval $t \in [0, 2]$. Note that coefficients of $E\left[\vec{X}_N(t)\right]$ defined by (4.5), in our case are given by

$$E\left[\vec{X}_0\right] = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \qquad E\left[\vec{X}_1\right] = \begin{bmatrix} 0 \\ -0.001 \end{bmatrix}, \tag{5.13}$$

$$E\left[\vec{X}_n\right] = \frac{1}{n}\sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{(n-k-1)!}\begin{bmatrix} -1 & 2^{n-k-1} \\ 1 & -2^{n-k-1} \end{bmatrix} E\left[X_k\right], \quad n \geq 2. \tag{5.14}$$

In order to compute the covariance matrix function, note that from a practical point of view, it is realistic to assume that U_1, U_2 and E_2 are pairwise independent r.v.'s, then one gets

$$\Lambda_{\vec{X}_0,\vec{X}_0} = \frac{1}{12}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Lambda_{\vec{X}_0,\vec{B}_j} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad j \geq 0, \tag{5.15}$$

$$\Lambda_{\vec{B}_0,\vec{B}_0} = \begin{bmatrix} 0 & 0 \\ 0 & 10^{-6} \end{bmatrix} \quad \text{and} \quad \Lambda_{\vec{B}_i,\vec{B}_j} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{otherwise}. \tag{5.16}$$

The matrices $\Lambda_{\vec{X}_i,\vec{B}_j}$ for $i \geq 1$ and $j \geq 0$ can be obtained by (4.11). Note that for each $j \geq 1$, $\Lambda_{\vec{X}_i,\vec{B}_j}$ is the null matrix of size 2×2. Table 2 shows covariance matrices of $\vec{X}_N(t)$ for several values of N and t. From its diagonal elements one deduces that for N fixed, standard deviation decreases as t increases. On the other hand, the negative values of secondary diagonal for $N = 10$ and $N = 15$ are in accordance with the results shown in Fig. 2. Although convergence of covariance matrix is not evident from values contained in Table 2, this is guaranteed because of the m.s. convergence of $\vec{X}_n(t)$ and the property (2.8).

Remark 3. It is possible to extend the work developed in Sections 2 and 3 to matrix random models of the form (1.1), that is, where the unknown, the initial condition as well as the source term are not vectors but matrix stochastic processes of suitable sizes. This task requires to introduce the set, say, $L_2^{r\times s}$ of all these matrix stochastic processes $\mathbf{X}$

Table 2
Covariance matrices of approximating processes for Example 5.2

t	$\Lambda_{\vec{X}_5(t)}$	$\Lambda_{\vec{X}_{10}(t)}$	$\Lambda_{\vec{X}_{15}(t)}$
0	$\begin{pmatrix}0.08333 & 0\\ 0 & 0.08333\end{pmatrix}$	$\begin{pmatrix}0.08333 & 0\\ 0 & 0.08333\end{pmatrix}$	$\begin{pmatrix}0.08333 & 0\\ 0 & 0.08333\end{pmatrix}$
0.10	$\begin{pmatrix}0.07005 & 0.01293\\ 0.01293 & 0.07076\end{pmatrix}$	$\begin{pmatrix}0.06512 & -0.01455\\ -0.01455 & 0.06814\end{pmatrix}$	$\begin{pmatrix}0.06432 & -0.01532\\ -0.01532 & 0.06514\end{pmatrix}$
0.20	$\begin{pmatrix}0.06128 & 0.02083\\ 0.02083 & 0.06372\end{pmatrix}$	$\begin{pmatrix}0.05888 & -0.01556\\ -0.01556 & 0.06012\end{pmatrix}$	$\begin{pmatrix}0.05789 & -0.01876\\ -0.01876 & 0.05998\end{pmatrix}$
0.30	$\begin{pmatrix}0.05489 & 0.02610\\ 0.02610 & 0.05960\end{pmatrix}$	$\begin{pmatrix}0.05232 & -0.01989\\ -0.01989 & 0.05774\end{pmatrix}$	$\begin{pmatrix}0.05012 & -0.02134\\ -0.02134 & 0.05532\end{pmatrix}$
0.40	$\begin{pmatrix}0.04953 & 0.03039\\ 0.03039 & 0.05637\end{pmatrix}$	$\begin{pmatrix}0.04766 & -0.02343\\ -0.02343 & 0.05564\end{pmatrix}$	$\begin{pmatrix}0.04667 & -0.02445\\ -0.02445 & 0.04989\end{pmatrix}$
0.50	$\begin{pmatrix}0.04422 & 0.03514\\ 0.03514 & 0.05216\end{pmatrix}$	$\begin{pmatrix}0.04232 & -0.02565\\ -0.02565 & 0.04932\end{pmatrix}$	$\begin{pmatrix}0.04144 & -0.02674\\ -0.02674 & 0.04821\end{pmatrix}$

whose entries X_{ij} lie in L_2 for $1 \le i \le r$, $1 \le j \le s$. This set $L_2^{r\times s}$ endowed with the norm

$$\|\mathbf{X}\|_{r\times s} = \max_{1\le i\le r} \sum_{j=1}^{s} \left\| X_{ij} \right\|_2 ,$$

is a Banach space. Moreover, one can establish the crucial property $\|\mathbf{AX}\|_{r\times s} \le \|\mathbf{A}\|\, \|\mathbf{X}\|_{r\times s}$, and just to introduce the concept of expectation of the matrix stochastic process as well as its corresponding relationships given by (4.1)–(4.6). However, as $\mathbf{X}(t) = \left(X_{ij}(t)\right)_{r\times s}$ is a matrix stochastic process, in dealing with covariance computation it is necessary to introduce the so-called vectorization of a matrix

$$\mathrm{vec}(\mathbf{X}(t))^{\mathrm{T}} = \begin{bmatrix} X_{11}(t) & \cdots & X_{r1}(t) & X_{12}(t) & \cdots & X_{r2}(t) & X_{1s}(t) & \cdots & X_{rs}(t)\end{bmatrix},$$

and define the covariance matrix function of stochastic matrix $\mathbf{X}(t)$ by means of

$$\Lambda_{\mathbf{X}(t)} = \Lambda_{\mathrm{vec}(\mathbf{X}(t))} = E\left[(\mathrm{vec}(\mathbf{X}(t)) - E\left[\mathrm{vec}(\mathbf{X}(t))\right])\,(\mathrm{vec}(\mathbf{X}(t)) - E\left[\mathrm{vec}(\mathbf{X}(t))\right])^{\mathrm{T}}\right].$$

For this reason, dealing with random matricial models of type (1.1), it is more appropriate to consider the corresponding vector problem resulting after taking the operator $\mathrm{vec}(\cdot)$ on the original matrix equation, and then applying the vectorial technique developed in this paper.

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