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Numerical study of partial slip on the MHD flow of an Oldroyd 8-constant fluid

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Abstract

The steady flow of an Oldroyd 8-constant magnetohydrodynamic (MHD) fluid is considered for a cylindrical geometry when the no-slip condition between the cylinders and the fluid is no longer valid. The inclusion of the partial slip at boundaries modifies the governing boundary conditions, changing from a linear to a non-linear situation. The non-linear differential equation along with non-linear boundary conditions governing the flow has been solved numerically using a finite-difference scheme in combination with an iterative technique. The solution for the no-slip condition is a special case of the presented analysis. A critical assessment is made for the cases of partial slip and no-slip conditions.

Keywords: Partial slip; Rheological parameters; Oldroyd 8-constant fluid

1. Introduction

We recall that fluids in which the shear stress is a multiple of the shear strain are called Newtonian fluids. The proportionality coefficient is the viscosity. Other fluids are known as non-Newtonian fluids. Examples of Newtonian fluids are: water, alcohol, benzene, kerosene and glycerine. Examples of non-Newtonian fluids are: blood plasma, chocolate, tomato sauce, mustard, mayonnaise, toothpaste, asphalt, some greases and sewage.

The governing equation that describes the flow of a Newtonian fluid is the Navier–Stokes equation. During the past several years, generalizations of the Navier–Stokes model to highly non-linear constitutive laws have been proposed and studied because of their interest in applications. There is not a single governing equation which exhibits all the properties of non-Newtonian fluids and these fluids cannot be described simply as Newtonian fluids. Moreover, there are very few cases in which the exact analytic solution of Navier–Stokes equations can be obtained. These are even rare if the constitutive equations for the non-Newtonian fluids are considered. One of the popular models for non-Newtonian fluids is the model that is called the Oldroyd 8-constant fluid. It is reasonable to use the Oldroyd 8-constant fluid model to see the rheological effects even for unidirectional and steady flow. It is pertinent to mention here that unidirectional flows of an Oldroyd 3-constant fluid (Rajagopal and Bhatnagar [1], Hayat et al. [2] and Fetecau and Fetecau [3–5], Tan and Masuoka [6] and Chen et al. [7]) take into account the rheological effects in an

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unsteady situation only and lack the features of viscoelasticity for the steady state situation. Baris et al. [8] considered an Oldroyd 8-constant model to discuss the steady flow in a convergent channel. In continuation, Hayat et al. [9] discussed the Couette, Poiseuille and generalized Couette flows of an Oldroyd 8-constant magnetohydrodynamic fluid.

In all the mentioned studies, the effect of the slip condition is not considered. Navier [10] proposed a slip boundary condition when the slip velocity depends on the shear stress. He developed slip boundary conditions based on molecular calculations. There is much rigorous work [11–20] concerning the flow of a Navier–Stokes-slip, threshold-slip, etc. Since the equations for non-Newtonian fluids are of higher order than the Navier–Stokes equations, additional boundary conditions are necessary in order to obtain the unique solution. The adherence boundary conditions are insufficient to determine a unique solution. Rajagopal and Gupta [21] and Rajagopal and Kaloni [22] gave examples of non-uniqueness in domains with porous boundaries. This implies that additional boundary conditions are necessary to ensure the well-posedness, but it remains an open question what boundary conditions should be imposed. Moreover, non-Newtonian fluids such as polymer melts often exhibit wall slip. The fluids exhibiting boundary slip have important technological applications. For example, the polishing of artificial heart valves and internal cavities in a variety of manufactured parts is achieved by imbedding such fluids with abrasives [23]. Several attempts have been made to explain slip phenomena [24–27]. Examples of well-posedness results for the Navier–Stokes equations with Navier slip, and more references, are given in [28–31]. Rao and Rajagopal [32] also examined the effect of the slip boundary condition on the flow of fluids in a channel. Roux [33] studied in detail the existence and uniqueness of the flow of second grade fluids with slip boundary conditions. Non-Newtonian flows with wall slip have been studied numerically in Refs. [34–42]. The effect of the slip condition at the wall for Couette flow for steady and unsteady state conditions has been studied respectively by Jha [43] and Marques et al. [44], and for Stokes and Couette flows by Khaled and Vafai [45].

The object of the present analysis is to examine the partial slip effects on an MHD Oldroyd 8-constant fluid between coaxial cylinders. The conducting fluid is permeated by an imposed uniform magnetic field when the no-slip condition at the boundaries is invalid. The inclusion of the partial slip at boundaries modifies the governing boundary conditions, changing from a linear to a non-linear situation. The highly non-linear problem has been solved numerically, and the results have been discussed in detail. The considered Hartman flow of an electrically conducting fluid in the presence of a transverse magnetic field has applications in many devices such as MHD power generators, MHD pumps, and accelerators; in processes such as aerodynamics heating, electrostatic precipitation, polymer technology; and in the purification of molten metals from nonmetallic inclusions and fluid-droplet sprays.

2. Governing equations

Consider the flow of an incompressible magnetohydrodynamic (MHD) fluid. The magnetic field is applied transversely to the flow. The following set of pertinent field equations governing the unsteady motion of the conducting Oldroyd 8-constant fluid is given by

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\,\mathbf{V}\right]=\operatorname{div}\mathbf{T}+\mathbf{J}\times\mathbf{B}, \tag{1}$$

$$\operatorname{div}\mathbf{V}=0, \tag{2}$$

$$\nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{B}=\mu_m\mathbf{J},\quad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{3}$$

$$\mathbf{J}=\sigma\left(\mathbf{E}+\mathbf{V}\times\mathbf{B}\right), \tag{4}$$

$$\mathbf{T}=-p\mathbf{I}+\mathbf{S}, \tag{5}$$

$$\mathbf{S}+\lambda_1\frac{D\mathbf{S}}{Dt}+\frac{\lambda_3}{2}\left(\mathbf{S}\mathbf{A}_1+\mathbf{A}_1\mathbf{S}\right)+\frac{\lambda_5}{2}\left(\operatorname{tr}\mathbf{S}\right)\mathbf{A}_1+\frac{\lambda_6}{2}\left[\operatorname{tr}\left(\mathbf{S}\mathbf{A}_1\right)\right]\mathbf{I}$$
$$=\mu\left[\mathbf{A}_1+\lambda_2\frac{D\mathbf{A}_1}{Dt}+\lambda_4\mathbf{A}_1^2+\frac{\lambda_7}{2}\left[\operatorname{tr}\left(\mathbf{A}_1^2\right)\right]\mathbf{I}\right], \tag{6}$$

$$\mathbf{A}_1=\mathbf{L}+\mathbf{L}^{\mathsf{T}},\quad \mathbf{L}=\operatorname{grad}\mathbf{V}. \tag{7}$$

In above equations $\mathbf{V} = (u, v, w)$ is the velocity vector, ρ is the density, $\mathbf{T}$ is the Cauchy stress tensor, $-p\mathbf{I}$ is the constitutively indeterminate part of the stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra stress tensor, $\mathbf{A}_1$ is the first Rivlin–Ericksen tensor, and μ and λ_i $(i = 1, 2, \ldots, 7)$ are the material constants of the fluid. The contravariant convected derivative D/Dt is defined as

$$\frac{D\mathbf{S}}{Dt} = \frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} - \mathbf{LS} - \mathbf{SL}^{\mathsf{T}}, \tag{8}$$

where $\mathrm{d}/\mathrm{d}t$ is the material derivative. Moreover, $\mathbf{J}$, μ_m, $\mathbf{E}$ and σ are the current density, magnetic permeability, electric field, and electric conductivity, respectively, and $\mathbf{B}$ is the total magnetic field, so that $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, $\mathbf{b}$ is the induced magnetic field.

We further consider the following:

- The quantities ρ, μ_m and σ are all constants throughout the flow field.
- The magnetic field $\mathbf{B}$ is perpendicular to the velocity field $\mathbf{V}$ and the induced magnetic field is negligible compared with the imposed magnetic field so that the magnetic Reynolds number is small [46].
- The electric field $\mathbf{E}$ is assumed to be zero.

Based on these considerations particularly of small magnetic Reynolds number, the magnetohydrodynamic force involved in Eq. (1) becomes

$$\mathbf{J} \times \mathbf{B} = -\sigma B_0^2 \mathbf{V}. \tag{9}$$

For the motion under consideration, we take the stress and velocity field of the form

$$\mathbf{S}(r, t) = \begin{pmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{z\theta} & S_{zz} \end{pmatrix}, \qquad \mathbf{V}(r, t) = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}. \tag{10}$$

Using Eq. (10), the continuity equation is identically satisfied and the remaining field equations now reduce to

$$0 = -\frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(r S_{rr}) - \frac{S_{\theta\theta}}{r}, \tag{11}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 S_{r\theta}\right), \tag{12}$$

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r S_{rz}) - \sigma B_0^2 u, \tag{13}$$

$$\left(1 + \lambda_1\frac{\partial}{\partial t}\right) S_{rr} + (\lambda_3 + \lambda_6) S_{rz}\frac{\partial u}{\partial r} = \mu(\lambda_4 + \lambda_7)\left(\frac{\partial u}{\partial r}\right)^2, \tag{14}$$

$$\left(1 + \lambda_1\frac{\partial}{\partial t}\right) S_{r\theta} + \frac{\lambda_3}{2} S_{z\theta}\frac{\partial u}{\partial r} = 0, \tag{15}$$

$$\left(1 + \lambda_1\frac{\partial}{\partial t}\right) S_{rz} + \left(\frac{\lambda_3 + \lambda_5}{2}\right)(S_{rr} + S_{zz})\frac{\partial u}{\partial r} + \frac{\lambda_5}{2} S_{\theta\theta}\frac{\partial u}{\partial r} - \lambda_1 S_{rr}\frac{\partial u}{\partial r} = \mu\left(1 + \lambda_2\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial r}, \tag{16}$$

$$\left(1 + \lambda_1\frac{\partial}{\partial t}\right) S_{\theta\theta} + \lambda_6 S_{rz}\frac{\partial u}{\partial r} = \mu\lambda_7\left(\frac{\partial u}{\partial r}\right)^2, \tag{17}$$

$$\left(1 + \lambda_1\frac{\partial}{\partial t}\right) S_{\theta z} + \left(\frac{\lambda_3 - 2\lambda_1}{2}\right) S_{\theta r}\frac{\partial u}{\partial r} = 0, \tag{18}$$

$$\left(1 + \lambda_1\frac{\partial}{\partial t}\right) S_{zz} + (\lambda_3 + \lambda_6 - 2\lambda_1) S_{rz}\frac{\partial u}{\partial r} = \mu(\lambda_4 + \lambda_7 - 2\lambda_2)\left(\frac{\partial u}{\partial r}\right)^2. \tag{19}$$

The above equations for steady flow become

$$\frac{\partial p}{\partial r} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r S_{rr}) - \frac{S_{\theta\theta}}{r}, \tag{20}$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 S_{r\theta}\right), \tag{21}$$

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r S_{rz}\right) - \sigma B_0^2 u, \tag{22}$$

$$S_{rr} + (\lambda_3 + \lambda_6) S_{rz}\frac{\mathrm{d}u}{\mathrm{d}r} = \mu(\lambda_4 + \lambda_7)\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2, \tag{23}$$

$$S_{r\theta} + \frac{\lambda_3}{2} S_{z\theta}\frac{\mathrm{d}u}{\mathrm{d}r} = 0, \tag{24}$$

$$S_{rz} + \left(\frac{\lambda_3 + \lambda_5}{2}\right)(S_{rr} + S_{zz})\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{\lambda_5}{2} S_{\theta\theta}\frac{\mathrm{d}u}{\mathrm{d}r} - \lambda_1 S_{rr}\frac{\mathrm{d}u}{\mathrm{d}r} = \mu\frac{\mathrm{d}u}{\mathrm{d}r}, \tag{25}$$

$$S_{\theta\theta} + \lambda_6 S_{rz}\frac{\mathrm{d}u}{\mathrm{d}r} = \mu\lambda_7\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2, \tag{26}$$

$$S_{\theta z} + \left(\frac{\lambda_3 - 2\lambda_1}{2}\right) S_{\theta r}\frac{\mathrm{d}u}{\mathrm{d}r} = 0, \tag{27}$$

$$S_{zz} + (\lambda_3 + \lambda_6 - 2\lambda_1) S_{rz}\frac{\mathrm{d}u}{\mathrm{d}r} = \mu(\lambda_4 + \lambda_7 - 2\lambda_2)\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2. \tag{28}$$

From Eq. (23) to (28) we have

$$S_{r\theta} = S_{\theta z} = 0, \tag{29}$$

$$S_{rr} = \frac{\mu(\lambda_4 + \lambda_7 - \lambda_3 - \lambda_6)\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 + \mu\{(\lambda_4 + \lambda_7)\alpha_2 - (\lambda_3 + \lambda_6)\alpha_1\}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^4}{1 + \alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}, \tag{30}$$

$$S_{rz} = \frac{\mu\frac{\mathrm{d}u}{\mathrm{d}r} + \mu\alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^3}{1 + \alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}, \tag{31}$$

$$S_{\theta\theta} = \frac{\mu(\lambda_7 - \lambda_6)\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 + \mu(\lambda_7\alpha_2 - \lambda_6\alpha_1)\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^4}{1 + \alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}, \tag{32}$$

where

$$\alpha_1 = \lambda_1(\lambda_4 + \lambda_7) - (\lambda_3 + \lambda_5)(\lambda_4 + \lambda_7 - \lambda_2) - \frac{\lambda_5\lambda_7}{2},$$
$$\alpha_2 = \lambda_1(\lambda_3 + \lambda_6) - (\lambda_3 + \lambda_5)(\lambda_3 + \lambda_6 - \lambda_1) - \frac{\lambda_5\lambda_6}{2}.$$

3. Physical model and numerical method

Let us consider the steady flow of an incompressible electrically conducting Oldroyd 8-constant fluid between two infinite coaxial cylinders. The motion starts suddenly due to a constant pressure gradient and by the motion of the inner cylinder parallel to its length while the outer cylinder keeps stationary. Using Eq. (29), Eqs. (20)–(22) can be written as

$$\frac{\partial p}{\partial r} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r S_{rr}\right) - \frac{S_{\theta\theta}}{r}, \tag{33}$$

$$\frac{\partial p}{\partial \theta} = 0, \tag{34}$$

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r S_{rz}\right) - \sigma B_0^2 u. \tag{35}$$

In the derivation of the above equations, the transverse magnetic field is taken to be a body force. Moreover, the z-differential of the pressure is constant since the flow is due to a prescribed pressure gradient and the motion of the inner cylinder. From Eq. (35) the velocity field is determined. Next, the pressure field is determined from Eq. (33).

Using the value of S_{rz} from Eq. (31) in Eq. (35), we arrive at

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left\{\frac{\mu\frac{\mathrm{d}u}{\mathrm{d}r} + \mu\alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^3}{1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right\}\right] - \sigma B_0^2 u = \frac{\mathrm{d}p}{\mathrm{d}z}. \tag{36}$$

Here, we also consider the existence of partial slip between the velocity of the fluid and of the cylinders. The relative velocity between the fluid and the walls of the cylinders is assumed to be proportional to the shear rate at the wall. The equation which is to be solved is Eq. (36) subject to the following partial slip conditions:

$$\begin{aligned} &u(r) - \gamma\left\{\frac{1+\alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right\}\frac{\mathrm{d}u}{\mathrm{d}r} = U_0 \quad \text{at } r = R_0,\\ &u(r) + \gamma\left\{\frac{1+\alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right\}\frac{\mathrm{d}u}{\mathrm{d}r} = 0 \quad \text{at } r = R_1, \end{aligned} \tag{37}$$

where γ (≥ 0) is the slip coefficient having the dimension of length, R_0 is the radius of the inner cylinder, R_1 is that of the outer cylinder, and U_0 is the velocity of the inner cylinder.

Defining the non-dimensional variables

$$\overline{u} = \frac{u}{U_0},\quad \overline{r} = \frac{r}{R_0},\quad \overline{z} = \frac{z}{R_0},\quad \overline{p} = \frac{p}{\mu U_0/R_0},\quad \overline{\gamma} = \frac{\gamma}{R_0},\quad \overline{\alpha}_1 = \frac{\alpha_1}{(R_0/U_0)^2},\quad \overline{\alpha}_2 = \frac{\alpha_2}{(R_0/U_0)^2} \tag{38}$$

the governing boundary value problem in dimensionless variables after dropping the bars is

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left\{\frac{\frac{\mathrm{d}u}{\mathrm{d}r} + \alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^3}{1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right\}\right] - M^2 u = \frac{\mathrm{d}p}{\mathrm{d}z}, \tag{39}$$

$$\begin{aligned} &u(r) - \gamma\left\{\frac{1+\alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right\}\frac{\mathrm{d}u}{\mathrm{d}r} = 1 \quad \text{at } r = 1,\\ &u(r) + \gamma\left\{\frac{1+\alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right\}\frac{\mathrm{d}u}{\mathrm{d}r} = 0 \quad \text{at } r = b, \end{aligned} \tag{40}$$

where $M^2 = \sigma B_0^2/\left(\mu/R_0^2\right)$ and $b = R_1/R_0$.

In the following, we are interested in obtaining the numerical solution for above boundary value problem. To obtain the numerical solution we rewrite Eq. (39) in the form

$$\left\{\frac{1+(3\alpha_1-\alpha_2)\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 + \alpha_1\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^4}{\left(1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2\right)^2}\right\}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\left\{\frac{1+\alpha_1\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}{1+\alpha_2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2}\right\}\frac{\mathrm{d}u}{\mathrm{d}r} - M^2 u = \frac{\mathrm{d}p}{\mathrm{d}z}. \tag{41}$$

The above equation and the partial slip conditions (40) are non-linear, so the solution cannot be obtained directly by the finite-difference method. In the present case, an iterative procedure is employed.

We construct an iterative procedure for the non-linear boundary value problem (41) and (40) as

$$\left\{\frac{1+(3\alpha_1-\alpha_2)\left(\frac{du^{(n)}}{dr}\right)^2+\alpha_1\alpha_2\left(\frac{du^{(n)}}{dr}\right)^4}{\left(1+\alpha_2\left(\frac{du^{(n)}}{dr}\right)^2\right)^2}\right\}\frac{d^2u^{(n+1)}}{dr^2}$$
$$+\frac{1}{r}\left\{\frac{1+\alpha_1\left(\frac{du^{(n)}}{dr}\right)^2}{1+\alpha_2\left(\frac{du^{(n)}}{dr}\right)^2}\right\}\frac{du^{(n+1)}}{dr}-M^2u^{(n+1)}=\frac{dp}{dz}, \tag{42}$$

$$u^{(n+1)}-\gamma\left\{\frac{1+\alpha_1\left(\frac{du^{(n)}}{dr}\right)^2}{1+\alpha_2\left(\frac{du^{(n)}}{dr}\right)^2}\right\}\frac{du^{(n+1)}}{dr}=1 \quad \text{at } r=1,$$
$$u^{(n+1)}+\gamma\left\{\frac{1+\alpha_1\left(\frac{du^{(n)}}{dr}\right)^2}{1+\alpha_2\left(\frac{du^{(n)}}{dr}\right)^2}\right\}\frac{du^{(n+1)}}{dr}=0 \quad \text{at } r=b, \tag{43}$$

where the index (n) indicates the iterative step.

Eq. (42) subject to partial slip conditions (43) defines a differential boundary value problem for unknown $u^{(n+1)}$. By means of the finite-difference method a linear algebraic system can be deduced and solved for each iterative step. Thus, a sequence of functions $u^{(0)}(r), u^{(1)}(r), u^{(2)}(r), \ldots$ is determined as follows: for a given estimated $u^{(0)}(r)$; $u^{(1)}(r), u^{(2)}(r), \ldots$ are calculated successively as the solutions of the boundary value problem (42) and (43).

For a better convergence, the so-called method of under-relaxation is used. We solve the boundary value problem (42) and (43) for the iterative step $n+1$ to obtain an estimated value of $u^{(n+1)}$: $\widetilde{u}^{(n+1)}$, then $u^{(n+1)}$ is defined by the formula

$$u^{(n+1)}=u^{(n)}+\tau\left(\widetilde{u}^{(n+1)}-u^{(n)}\right), \qquad \tau\in(0,1], \tag{44}$$

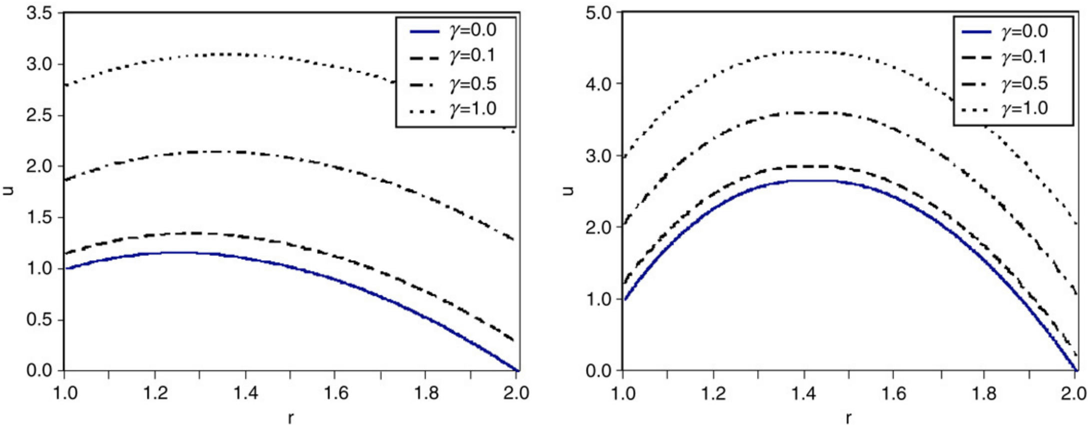
where $\tau \in (0,1]$ is an under-relaxation parameter. We choose τ so small that convergent iteration is reached. The iteration should be carried out until the relative difference of the computed $u^{(n+1)}$ and $u^{(n)}$ between two iterative steps are smaller than a given error chosen to be 10^{-16}.

4. Numerical results and discussion

The differential equation (36) subject to partial slip conditions (37) is solved numerically. A comparison is made between the profiles of the velocity for two kinds of fluids: a Newtonian fluid, for which $\alpha_1=\alpha_2$, and an Oldroyd 8-constant fluid. Of particular interest here are the effects of pressure gradient, partial slip, rheological and magnetic parameters.

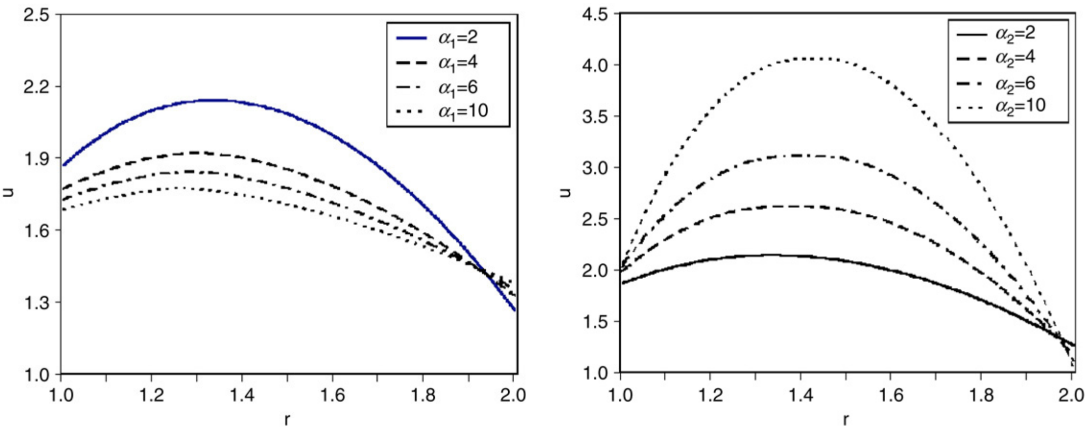
Fig. 1 show the effect of the partial slip on the velocity profiles. It is obvious from these figures that the velocity gradient for an Oldroyd 8-constant fluid (panel b) is much larger than that of a Newtonian fluid (panel a) for both no-slip ($\gamma=0$) and partial slip ($\gamma>0$) conditions. Moreover, it is noted that with the increase of the partial slip parameter γ, the velocity gradient for both fluid decreases, and hence the velocity profiles become flatter due to the decreasing shearing force from the slip boundary. Furthermore, by the increase of slip parameter γ, the main effect seems to be that the reduced boundary resistance leads to the increase of the velocity over the whole cross section.

The influence of the rheological parameters α_1 and α_2 of an Oldroyd 8-constant fluid on the velocity profiles are shown in Fig. 2 in the presence of partial slip at the boundaries. From these figures we can see that the velocity profiles are strongly influenced by the material parameters α_1 and α_2 even in the presence of partial slip. It can be seen from Fig. 2(a) that for an Oldroyd 8-constant fluid, when the material parameter α_1 increases from $\alpha_1=2$ to 10 while



(a) Newtonian fluid ($\alpha_1 = 2, \alpha_2 = 2$). (b) Oldroyd fluid ($\alpha_1 = 2, \alpha_2 = 8$).

Fig. 1. Profiles of the dimensionless velocity $u(r)$ for various values of partial slip parameter γ when $M = 0.5$, $\mathrm{d}p/\mathrm{d}z = -5$ and $b = 2$ are fixed.

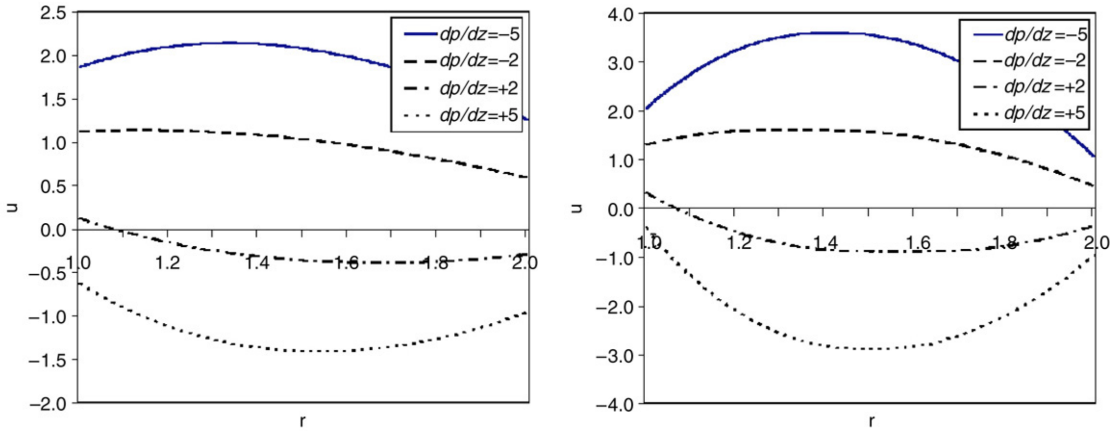


(a) $\alpha_1 = 2$. (b) $\alpha_2 = 2$.

Fig. 2. Profiles of the dimensionless velocity $u(r)$ for various values of rheological parameters α_1 and α_2 when $\gamma = 0.5$, $M = 0.5$, $\mathrm{d}p/\mathrm{d}z = -5$ and $b = 2$ are fixed.

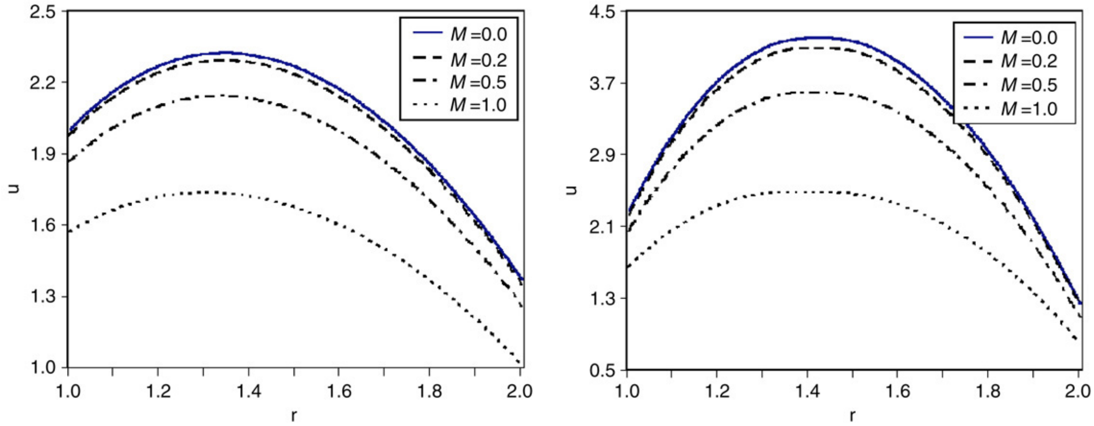
α_2 maintains a constant value $\alpha_2 = 2$, the flow profiles tend to approach a linear distribution; thus, the shearing can extend without attenuation to the whole flow domain from the boundaries, which corresponds to a shear-thickening phenomenon. On the other hand, the variation of α_2 is given in Fig. 2(b) when α_1 is fixed. From this figure, it is noted that with the increase of α_2 from $\alpha_2 = 2$ to 10 for fixed value of $\alpha_1 = 2$, an opposite phenomenon can be observed in which the velocity near the boundary becomes sharper, which corresponds to shear-thinning behavior of the examined non-Newtonian fluid.

Fig. 3 show the effects of pressure gradient on the velocity profile in the presence of partial slip. For both a Newtonian fluid (panel a) and an Oldroyd 8-constant fluid (panel b), symmetric parabolic flow profiles are found. Their amplitudes depend on the magnitude of the pressure gradient and the flow directions are against the direction of pressure gradient. From these figures we can see that for an Oldroyd 8-constant fluid the flow velocities are obviously much larger than those of a Newtonian fluid. Actually, it can be seen from the governing equation (36) that if $\alpha_1 < \alpha_2$ ($\alpha_1 > \alpha_2$) the flow velocity of an Oldroyd 8-constant fluid is larger (smaller) than that of a Newtonian fluid. Thus, the effects of the pressure gradient are similar to those for the no-slip case. The slip parameter γ tends to increase the flow velocity; therefore, if there is partial slip at the boundary/boundaries the velocity is higher than that of the no-slip case.



(a) Newtonian fluid ($\alpha_1 = 2, \alpha_2 = 2$). (b) Oldroyd fluid ($\alpha_1 = 2, \alpha_2 = 8$).

Fig. 3. Profiles of the dimensionless velocity $u(r)$ for various values of pressure gradient $\mathrm{d}p/\mathrm{d}z$ when $\gamma = 0.5$, $M = 0.5$ and $b = 2$ are fixed.



(a) Newtonian fluid ($\alpha_1 = 2, \alpha_2 = 2$). (b) Oldroyd fluid ($\alpha_1 = 2, \alpha_2 = 8$).

Fig. 4. Profiles of the dimensionless velocity $u(r)$ for various values of magnetic parameter M when $\gamma = 0.5$, $\mathrm{d}p/\mathrm{d}z = -5$ and $b = 2$ are fixed.

Fig. 4(a) and (b) provide the effects of the magnetic parameter M for a Newtonian fluid and an Oldroyd 8-constant fluid, respectively. As expected, increasing the magnitude of the magnetic parameter M reduces the velocity monotonically due to the effect of the magnetic force against the direction of the flow.

5. Concluding remarks

In this paper, a non-linear analysis of an Oldroyd 8-constant magnetohydrodynamic fluid between coaxial cylinders subject to partial slip at the boundaries is discussed numerically. The partial slip condition might also explain the flow of polymers that has been observed through various experiments [47–50]. However, the governing equation and the slip boundary conditions here are rather daunting, as they lead to a highly non-linear system. The pertinent controlling parameters are pressure gradient, partial slip, rheological and magnetic parameters. The main conclusions are:

- The presented analysis includes the Oldroyd-B fluid ($\lambda_i = 0; i = 3, 4, \ldots, 7$), second grade fluid ($\lambda_i = 0; i = 1, 3, 4, \ldots, 7$), Maxwell fluid ($\lambda_i = 0; i = 2, 3, \ldots, 7$) and the Newtonian fluid ($\lambda_i = 0; i = 1, 2, \ldots, 7$) as special cases.
- With partial slip at the walls, the reduced shearing force from the boundaries causes the velocity profiles to become flatter.

- When the flow caused by the moving wall dominates over that induced by the axial pressure gradient, near the moving wall where the shearing force from the boundary is a driving force, the velocity decreases with the partial slip condition, while near the fixed wall where the shearing force from the boundary is a resistance, the velocity increases.
- If the flow induced by the pressure gradient dominates over that caused by the moving wall, the shearing force from the boundary acts as a resistance force to the flow. In such a case, the partial slip condition leads the velocity to increase over the whole cross-section.
- The variation of α_1 leads to a shear-thickening property when α_2 is fixed while the variation in α_2 gives a shear-thinning behavior for fixed α_1.
- Keeping the pressure gradient fixed, the velocity in the case of an Oldroyd 8-constant fluid is larger (smaller) than that of a Newtonian fluid for $\alpha_1 < \alpha_2$ ($\alpha_1 > \alpha_2$).
- Increasing the applied magnetic field will generally reduces the flow velocity even in the presence of partial slip.

References

[1] K.R. Rajagopal, R.K. Bhatnagar, Exact solutions for some simple flows of an Oldroyd-B fluid, Acta Mech. 113 (1995) 233–239.

[2] T. Hayat, A.M. Siddiqui, S. Asghar, Some simple flows of an Oldroyd B fluid, Internat. J. Engrg. Sci. 39 (2001) 135–147.

[3] C. Fetecau, C. Fetecau, The first problem of Stokes for an Oldroyd-B fluid, Internat. J. Non-Linear Mech. 38 (2003) 1539–1544.

[4] C. Fetecau, C. Fetecau, Decay of a potential vortex in an Oldroyd-B fluid, Internat. J. Engrg. Sci. 43 (2005) 340–351.

[5] C. Fetecau, C. Fetecau, Unsteady flows of Oldroyd-B fluids in a channel of rectangular cross-section, Internat. J. Non-Linear Mech. 40 (2005) 1214–1219.

[6] W.C. Tan, T. Masuoka, Stokes' first problem for an Oldroyd-B fluid in a porous half space, Phys. Fluids 17 (2005) 023101–023107.

[7] C.I. Chen, C.K. Chen, Y.T. Yang, Unsteady unidirectional flow of an Oldroyd-B fluid in a circular duct with different given volume flow rate conditions, Heat Mass Transfer 40 (2004) 203–209.

[8] S. Baris, Flow of an Oldroyd 8-constant fluid in a convergent channel, Acta Mech. 148 (2001) 117–127.

[9] T. Hayat, M. Khan, S. Asghar, Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid, Acta Mech. 168 (2004) 213–232.

[10] M. Navier, Memoire sur les lois du mouvement des fluides, Mem. Acad. Sci. Inst. France 6 (1823) 389–440.

[11] H. Fujita, A mathematical analysis of motions of viscous incompressible fluid under leak or slip boundary conditions, Surikaisekikenkyusho Kokyuroko 888 (1994) 199–216.

[12] E.B. Dussan, S.H. Davis, On the motion of a fluid–fluid interface along a solid surface, J. Fluid Mech. 65 (1974) 71–95.

[13] R.I. Tanner, X. Huang, Stress singularities in non-Newtonian stick-slip and edge flows, J. Non-Newtonian Fluid Mech. 50 (1993) 135–160.

[14] S. Goldstein, Modern developments in fluid dynamics, vol. II, Oxford University Press, Oxford, 1938.

[15] M.A. Day, The no-slip condition of fluid dynamics, Erkenntnis 33 (1990) 285–296.

[16] G.V. Vinogradov, L.I. Ivanova, Wall slippage and elastic turbulence of polymers in the rubbery state, Rheol. Acta 7 (1968) 243–254.

[17] J.L. White, M.H. Han, N. Nakajima, R. Brzoskowski, The influence of materials of construction on biconical rotor and considerations of slippage, J. Rheol. 35 (1991) 167–189.

[18] S.G. Hatzikiriakos, J.M. Dealy, Wall slip of molten high density polyethylenes. II. Capillary rhiometer studies, J. Rheol. 36 (1992) 703–741.

[19] R.I. Tanner, Partial wall slip in polymer flow, Ind. Eng. Chem. Res. 33 (1994) 2434–2436.

[20] V.A. Solonnikov, V.E. Scadilov, On a boundary value problem for a stationary system of Navier–Stokes equations, Proc. Steklov Inst. Math. 125 (1973) 186–199.

[21] K.R. Rajagopal, A.S. Gupta, An exact solution for the flow of a non-Newtonian fluid past an infinite plate, Meccanica 19 (1984) 158–160.

[22] K.R. Rajagopal, P.N. Kaloni, Some remarks on boundary conditions for fluids of the differential type, in: G.A.C. Graham, S.K. Malik (Eds.), Continuum Mechanics and its application, Hemisphere, New York, 1989, pp. 935–942.

[23] L.J. Rhoades, R. Resnick, T. O'Bradovich, S. Stegman, Abrasive Flow Machining of Cylinder Heads and its Positive Effects on Performance and Cost Characteristics, Presented at the Motorsports Engineering Conference and Exposition, Dearborn, Michigan, 1996.

[24] A.I. Leonov, On the dependence of friction force on sliding velocity in the theory of adhesive friction of elastometers, Wear 141 (1990) 137–145.

[25] M.A. Valder, J. Tejero, Hydrodynamic interactions of dilute polymer solutions under shear flow in a narrow channel, Rheol. Acta 33 (1994) 125–135.

[26] S.G. Hatzikiriakos, A multimode interfacial constitutive equation for molten polymers, J. Rheol. 39 (1995) 61–71.

[27] R.L. Schiek, E.S. Shaqfeh, A nonlocal theory for stress in bound, Brownian suspensions of slender, rigid fibers, J. Fluid Mech. 296 (1995) 271–324.

[28] R. Farwig, Stationary solutions of compressible Navier–Stokes equations with slip boundary condition, Comm. Partial Differential Equations 14 (1989) 1579–1606.

[29] A. Tani, The initial value problem for the equations of motion of general fluids with general slip boundary condition, Surikaisekikenkyusko Kokyuroko 734 (1990) 123–142.

[30] P. Secchi, On a stationary problem for the compressible Navier–Stokes equations: the self-gravitating equilibrium solutions, Differential Integral Equations 7 (1994) 463–482.

[31] A. Tani, S. Itoh, N. Tanaka, The initial value problem for the Navier–Stokes equations with general slip boundary condition, Adv. Math. Sci. Appl. 4 (1994) 51–69.

[32] I.J. Rao, K.R. Rajagopal, The effect of the slip condition on the flow of fluids in a channel, Acta Mech. 135 (1999) 113–126.
[33] C.L. Roux, Existence and uniqueness of the flow of second-grade fluids with slip boundary conditions, Arch. Ration. Mech. Anal. 148 (1999) 309–356.
[34] E.L. Tarunin, Flow of a viscous fluid in a closed cavity in the presence of slip effects, Fluid Dynam. 15 (1980) 6–11.
[35] B. Debbaut, On the development of secondary motion induced by the free surface in the rod climbing flow, J. Non-Newtonian Fluid Mech. 48 (1993) 357–364.
[36] R.E. Khayat, Perturbation solution to planar flow of a viscoelastic fluid with two moving free boundaries, Quart. J. Mech. Appl. Math. 47 (1994) 341–365.
[37] A. Lawal, D.H. Kalyon, Nonisothermal model of single screw extrusion of generalized Newtonian fluids, Numer. Heat Transfer Part A 26 (1994) 103–121.
[38] Y.P. Shih, C.C. Huang, S.Y. Tsay, Extended leveque solution for laminar heat transfer to power law fluids in pipes with wall slip, Internat. J. Heat Mass Transfer 38 (1995) 403–408.
[39] A.C. Fowler, Waves on glaciers, J. Fluid Mech. 120 (1982) 283–321.
[40] X. Dong Chen, Slip and no-slip squeezing flow of liquid in a wedge, Rheol. Acta 32 (1993) 477–482.
[41] A. Torres, et al., Boundary conditions for contact lines in coextrusion flows, Rheol. Acta 32 (1993) 513–525.
[42] R.T. Fenner, On local solutions to non-Newtonian slow viscous flows, Internat. J. Non-Linear Mech. 10 (1975) 207–214.
[43] B.K. Jha, Natural convection in unsteady MHD Couette flow, Heat Mass Transfer 37 (2001) 329–331.
[44] W. Marques Jr., G.M. Kremer, F.M. Sharipov, Couette flow with slip and jump boundary conditions, Contin. Mech. Thermodyn. 12 (2002) 379–386.
[45] A.R.A. Khaled, K. Vafai, The effect of slip condition on Stokes and Couette flows due to an oscillating wall: exact solutions, Internat. J. Non-Linear Mech. 39 (2004) 795–804.
[46] V.J. Rossow, On flow of electrically conducting fluids over a flat plate in the presence of a transverse magnetic field, NASA, Report No. 1358, 489, 1958.
[47] A.V. Ramamurthy, Wall slip in viscous fluids and influence of material of construction, J. Rheol. 30 (1986) 337–357.
[48] D.S. Kalika, M.M. Denn, Wall slip and extrudate distortion in linear low-density polyethylene, J. Rheol. 31 (1987) 815–834.
[49] A.M. Kraynik, W.R. Schowater, Slip at the wall and extrudate roughness with aqueous solutions of polyvinyl alcohol and sodium borate, J. Rheol. 25 (1981) 95–114.
[50] F.J. Lim, W.R. Schowater, Wall slip of narrow molecular weight distribution polybutadienes, J. Rheol. 33 (1989) 1359–1385.