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Application of Rall's Theorem to Classical Elastodynamics

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1. INTRODUCTION REMARKS

Let H_u, H_ϕ by Hilbert Spaces of vector valued functions defined on $\Omega \times [0, a]$, where Ω denotes a compact, connected subset of the Euclidean space E^n with a smooth boundary.

The inner products on H_u, H_ϕ are defined as

$$(u_1, u_2) \quad \text{for all} \quad u_1, u_2 \in H_u,$$
$$\langle \phi_1, \phi_2 \rangle \quad \text{for all} \quad \phi_1, \phi_2 \in H_\phi.$$

A linear operator A maps H_u into H_ϕ. The domain of A is dense in H_u and consequently an operator A^* is uniquely defined on H_ϕ, mapping H_ϕ into H_u, where A^* is the adjoint of A.

T. Kato [7] has first considered equations of the type

$$AA^*f = g \qquad f, g \in H_\phi \tag{a}$$

and observed the fact that many equations of physics, or engineering may be written in the form (a).

By a simple relabelling process equation (a) may be written as a system:

$$\left.\begin{aligned} A^*f &= u \\ Au &= g \end{aligned}\right\} \qquad \begin{aligned} u &\in H_u \\ f, g &\in H_\phi . \end{aligned} \tag{b}$$

B. Noble has first observed the variational significance of systems of equations of the form (b) and demonstrated existence of complementary variational principles in various physical cases [11].

L. B. Rall has proved the sufficient and necessary conditions for the existence of variational principles associated with the expressions of the type (b), and in particular for the existence of minimal points [2], [13]. Following Rall's discussion the cases of

$$L(\phi_0, u_0) = \max \tilde{\phi}, \qquad \max \tilde{u}\,(L),$$
$$\min \tilde{\phi}, \qquad \min \tilde{u}\,(L)$$

for trial functions $\tilde{u}$ and $\tilde{\phi}$ can be obtained with no changes in the basic arguments ([13], pp. 6, 7 and pp. 10-13). Let us briefly review some of the results: Let us denote by H the direct sum of the Hilbert Space H_u, H_ϕ. The elements of H are the ordered pairs

$$h = \binom{u}{\phi} \qquad \begin{matrix} u \in H_u \\ \phi \in H_\phi \end{matrix} \qquad h \in H.$$

The inner product in H is defined as

$$\{h_1, h_2\} = (u_1, u_2) + \langle \phi_1, \phi_2 \rangle.$$

THEOREM 1 (Rall). *The function $F(h)$ whose domain is H is a Vainberg gradient function* ([17]) *if and only if $F(h)$ can be expressed in the form*

$$F(h) = \begin{matrix} A^*\phi - \dfrac{\partial W}{\partial u} \\ \\ Au - \dfrac{\partial W}{\partial \phi} \end{matrix} = f'(h), \tag{c}$$

where $W(h)$ is a functional which is twice Frechet differentiable on H, and A is a linear operator on H_u, whose adjoint is A^* (acting on H_ϕ).

Roughly speaking, the existence of a solution of a differential system which is of the form (c) coincides with a stationary behavior of a functional

$$L = W - (A^*\phi, u) = W - \langle \phi, Au \rangle. \tag{d}$$

In fact, the following theorem is true:

THEOREM 2. *If W is the required functional of the system* (c), *and $(\partial^2 W/\partial u^2)^{-1}$ and $(\partial^2 W/\partial \phi^2)^{-1}$ exist, and if there exists a point $(\phi_0, u_0) \in H$ for which we have $F(h_0) = 0$ in* Eq. (c), *where $h_0 = \binom{\phi_0}{u_0}$, and if in some neighborhood of (ϕ_0, u_0) in H we have constant sign of the quantities*

$$\left(-\frac{\partial^2 W}{\partial u^2} u_0, u\right), \qquad \left\langle -\frac{\partial^2 W}{\partial \phi^2} \phi_0, \phi \right\rangle, \tag{e}$$

and the norms $\| u - u_0, \phi - \phi_0 \|_{(H)} \| u_0, \phi \|_{(H)}, \| u, \phi_0 \|_{(H)}$ are all bounded in that neighborhood by some constant, then the functional

$$L = W - \langle \phi, Au \rangle = W - (A^*\phi, u)$$

attains a local extremum at the point (ϕ_0, u_0).

Furthermore if qualities

$$\left(-\frac{\partial^2 W}{\partial u^2} u_0\,, u\right), \qquad \left\langle -\frac{\partial^2 W}{\partial \phi^2} \phi_0\,, \phi\right\rangle$$

are of different sign in some neighborhood N_0 of the point $(\phi_0\,, u_0)$ then the point $(\phi_0\,, u_0)$ is a minimax point of $L(\phi, u)$.

Since the second Frechet derivatives of the Vainberg potential function

$$f(h) = \tfrac{1}{2}\langle \phi, Au\rangle + \tfrac{1}{2}(A^*\phi, u) - W$$

are completely analogous to the second variation taken in the classical sense, it is now easy to distinguish the cases when the point $(\phi_0\,, u_0)$ represents locally: [min u, max ϕ], [max u, max ϕ], [min u, min ϕ], [max u, min ϕ].

2. Application to the Linear Theory of Classical Elastodynamics

The linearalized equations of motion of an elastic solid can be written as

$$\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{xy} + \frac{\partial}{\partial z}\tau_{xz} + X - \rho\frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{yz} + Y - \rho\frac{\partial^2 v}{\partial t^2} = 0$$

$$\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz} + Z - \rho\frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where τ_{ij} is the stress tensor, u, v, w are components of the displacement vector $\mathbf{U}$, X, Y, Z are components of the body force which we shall assume to be equal to zero. For simplicity of the argument we shall assume for the time being that all components of τ_{ij}, $\mathbf{U}$ vanish outside a compact set $(\Omega \supseteq \Omega(t)) \subset E^3$ $(0 \leqslant t \leqslant a)$. This set has the properties of the set Ω discussed in the introductory remarks. (We could assume that τ_{ij} and $\mathbf{U}$ are periodic over the time period $[0, a]$.) We will show that boundary conditions of more difficult nature can be treated by our formalistic method. However we will first deal with conditions specified above because of their simplicity. We rewrite Eq. (1) in the operational form

$$T\tau = 0, \tag{1a}$$

where T is the linear operator

$$\begin{bmatrix} -\frac{\partial}{\partial x}, & -\frac{\partial}{\partial y}, & -\frac{\partial}{\partial z}, & 0, & 0, & 0, & j\sqrt{\rho}\frac{\partial}{\partial t}, & 0, & 0 \\ 0, & -\frac{\partial}{\partial x}, & 0, & -\frac{\partial}{\partial y}, & -\frac{\partial}{\partial z}, & 0, & 0, & j\sqrt{\rho}\frac{\partial}{\partial t}, & 0 \\ 0, & 0, & -\frac{\partial}{\partial x}, & 0, & -\frac{\partial}{\partial y}, & -\frac{\partial}{\partial z}, & 0, & 0, & j\sqrt{\rho}\frac{\partial}{\partial t} \end{bmatrix}, \tag{2}$$

and the "vector" τ is the transpose of

$$[\tau_{xx}, \tau_{xy}, \tau_{xz}, \tau_{yy}, \tau_{yz}, \tau_{zz}, p_u, p_v, p_w] \tag{3}$$

containing six stress components (τ_{yx}, τ_{yz}, τ_{zx} have been omitted on account of assumed symmetry of τ_{ij}) and three components of a quantity $\mathbf{p}$ closely related to the momentum density vector

$$p_u \overset{\text{def}}{=} j\sqrt{\rho}\frac{\partial u}{\partial t}$$

$$p_v \overset{\text{def}}{=} j\sqrt{\rho}\frac{\partial v}{\partial t}$$

$$p_w \overset{\text{def}}{=} j\sqrt{\rho}\frac{\partial w}{\partial t},$$

where ρ is the material density, which is assumed to be constant, and $j^2 = -1$. $T\tau = 0$ constitutes a system of three equations which can be easily put into the form (b) by introducing a functional $W(\tau)$ and an additional three-dimensional vector μ with components μ_x, μ_y, μ_z whose physical meaning is easily established.

We have

$$T\tau = \frac{\partial W}{\partial \mu} = 0 \tag{4a}$$

as the first equation of our variational formalism.

By Rall's theorem W is the generalized Hamiltonian function only if W obeys:

$$T^*\mu = \frac{\partial W}{\partial \tau}. \tag{4b}$$

Because of our assumptions concerning the boundary conditions, T^* (the adjoint of T) is simply the operator

$$T^* = \begin{bmatrix} \frac{\partial}{\partial x}, & 0, & 0 \\ \frac{\partial}{\partial y}, & \frac{\partial}{\partial x}, & 0 \\ \frac{\partial}{\partial z}, & 0, & \frac{\partial}{\partial x} \\ 0, & \frac{\partial}{\partial y}, & 0 \\ 0, & \frac{\partial}{\partial z}, & \frac{\partial}{\partial y} \\ 0, & 0, & \frac{\partial}{\partial z} \\ -j\sqrt{\rho}\frac{\partial}{\partial t}, & 0, & 0 \\ 0, & -j\sqrt{\rho}\frac{\partial}{\partial t}, & 0 \\ 0, & 0, & -j\sqrt{\rho}\frac{\partial}{\partial t} \end{bmatrix}. \tag{4c}$$

The system $T^*\mu = \partial W/\partial\tau$ ($\partial W/\partial\tau$ is a Frechet derivative) becomes

$$\begin{aligned} \frac{\partial\mu_x}{\partial x} &= \frac{\partial W}{\partial\tau_{xx}} \\ \frac{\partial\mu_x}{\partial y} + \frac{\partial\mu_y}{\partial x} &= \frac{\partial W}{\partial\tau_{xy}} \\ &\vdots \\ \frac{\partial\mu}{\partial z} &= \frac{\partial W}{\partial\tau_{zz}} \\ -j\sqrt{\rho}\frac{\partial\mu_x}{\partial t} &= \frac{\partial W}{\partial p_u}, \qquad \text{etc.} \end{aligned} \tag{5}$$

Upon identifying

$$\begin{aligned} \mu_x &= \frac{\partial u}{\partial t} \\ \mu_y &= \frac{\partial v}{\partial t} \\ \mu_z &= \frac{\partial w}{\partial t}, \end{aligned} \tag{6}$$

the first six equations state a dynamic version of Castigliano's theorem, while the last three equations relate the vector **p** to the rate of change of kinetic energy.

W is easily indentified as the functional

$$W = \frac{1}{2} \int_{\Omega_a} \left\{ \left(\sum_{ij=1}^{3} \tau_{ij} \dot{\epsilon}_{ij} \right) + \frac{\partial}{\partial t} (\rho \mid \mathbf{U} \mid^2) \right\} dv\, dt, \tag{7}$$

where the dot denotes differentiation with respect to time, and ϵ_{ij} is the linear strain tensor. $\dot{\epsilon}_{ij}$ are in fact the linear expressions on the left hand side of the first six equations of the system (5). The domain Ω_a is the product set $\Omega \times [0, a]$.

The functional to be minimized is

$$I = W - \langle T^* \mu, \tau \rangle, \tag{8}$$

the inner product $\langle\ \rangle$ denoting the usual inner product of two vector valued functions; the integral being taken over the domain Ω_a.

Checking on the conditions necessary for the existence of complementary variational principles, we see that one of them is violated, since $(\partial^2 W / \partial \mu^2)^{-1}$ does not exist. Consequently, following our formulation only one variational principle is valid, and it can be stated as follows: Of all stress distributions satisfying d'Alambert's equations (1), the one which satisfies also the system of equations (5) will cause the functional I of Eq. (8) to have stationary behavior. And in particular if a constitutive relationship of the form

$$\dot{\epsilon}_{ij} = C_{ijk\ell}(\tau)\, \tau_{k\ell}$$

is given, and if the matrix $C_{ijk\ell}(\tau)$ is positive definite for all τ, then the functional I will be minimized. Of course this result can be obtained by the classical use of Lagrangian multipliers. By seeking stationary conditions for the functional $I = \int_{\Omega_a} \bar{L}\, dv\, dt$ where $\bar{L}$ is the Lagrangian density function, subject to auxiliary conditions expressed by Eq. (1), we have

$$\int_{\Omega_a} \bar{L}\, dv\, dt + \int_{\Omega_a} \mu_1 \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} - \rho \frac{\partial^2 u}{\partial t^2} \right) + \mu_2(\cdots) + \mu_3(\cdots) +$$

boundary terms. Due to our assumptions the boundary terms are all equal to zero, and we have

$$\delta \bar{L} = \int_{\Omega_a} \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} - \rho \frac{\partial^2 u}{\partial t^2} \right) \delta\mu_1 + (\cdots)\, \delta\mu_2 + (\cdots)\, \delta\mu_3 +$$

$$+ \int_{\Omega_a} \left(\dot{\epsilon}_{xx} - \frac{\partial \mu_1}{\partial x} \right) \delta\tau_{xx} + \left(\dot{\epsilon}_{xy} - \frac{\partial \mu_1}{\partial y} - \frac{\partial \mu_2}{\partial x} \right) \delta\tau_{xy} + \cdots$$

$$+ \left(\dot{\epsilon}_{zz} - \frac{\partial \mu_3}{\partial z} \right) \delta\tau_{zz}, \quad \epsilon_{xx} = \frac{\partial W}{\partial \tau_{xx}} \cdots \qquad \text{etc.}$$

(Frechet derivatives of W) completely agreeing with our formalistic results upon examination of the condition: $\delta\bar{L} = 0$. So far the advantage over the classical variational methods seem to lie in a more compact presentation, and in a much greater ease of determining the necessary conditions for an extremal point, provided the boundary conditions do not enter the picture. We shall now show that quite complex boundary conditions can be easily incorporated into this formalistic method.

3. Treatment of the Boundary Condition

We consider the "Hamiltonian" system

$$\left.\begin{aligned} \frac{\partial W}{\partial u} - A^*\phi = 0 \quad (10a) \\ \frac{\partial W}{\partial \phi} - Au = 0 \quad (10b) \end{aligned}\right\} \quad \text{in } \Omega_a$$

where W, A, A^* have the same meaning as in the introductory remarks, except that A and A^* are now only formal adjoints of each other. We wish to find a functional $I(u, \phi)$ whose stationary behavior will coincide with the solution of the system (10a), (10b) subject to the boundary condition

$$g(\phi, u) = 0 \qquad \text{on } \partial\Omega. \tag{11}$$

The inner products $(A^*\phi, u)$ and $\langle \phi, Au \rangle$ are connected by the relationship

$$(A^*\phi, u)_{(\Omega)} - \langle \phi, Au \rangle_{(\Omega)} = f(\phi, u)_{\partial\Omega}, \tag{12}$$

where $f(\phi, u)$ is a bounded functional defined with respect to the set $\partial\Omega$. (ϕ, u are elements of a Hilbert space $\hat{H}$ with domain $\partial\Omega$ of ϕ, u. Inner product is defined independently of the inner products (,) or $\langle , \rangle$.) To formulate a lemma answering the problem, we shall consider a functional I on $\Omega + \partial\Omega$:

$$I = \{W - (A^*\phi, u)\}_\Omega + \Lambda\{G\}_{\partial\Omega}, \tag{13}$$

where Λ is a real constant, and $G(\phi, u)$ is a bounded functional of u, ϕ, where u, ϕ are restricted to the set $\partial\Omega$. We assume the existence of an operator A^*, which is related to A by the formula (12). The Frechet first and second derivatives of I are assumed to exist.

$$\frac{\partial I}{\partial u} = \left(\frac{\partial W}{\partial u} - A^*\phi\right)_\Omega + \Lambda\left(\frac{\partial G}{\partial u}\right)_{\partial\Omega}. \tag{14a}$$

Substituting relationship (12), we also have

$$\frac{\partial I}{\partial \phi} = \left(\frac{\partial W}{\partial \phi} - Au\right)_{\Omega} + \frac{\partial}{\partial \phi}(\Lambda G - f)_{\partial\Omega} . \tag{14b}$$

A necessary condition for a stationary behavior of $I(u)$ for a fixed ϕ is

$$\frac{\partial I}{\partial u} = 0,$$

or of $I(\phi)$ for a fixed u,

$$\frac{\partial I}{\partial \phi} = 0.$$

The first of these conditions implies that we must satisfy simultaneously

$$\frac{\partial W}{\partial u} - A^*\phi = 0 \qquad \text{in } \Omega$$

and

$$\frac{\partial G}{\partial u} = g(\phi, u) = 0 \qquad \text{in } \partial\Omega. \tag{15}$$

The second implies that we must satisfy simultaneously

$$\frac{\partial W}{\partial \phi} - A\phi = 0 \qquad \text{in } \Omega$$

and

$$\Lambda \frac{\partial G}{\partial \phi} - \frac{\partial f}{\partial \phi} = 0 \qquad \text{on } \partial\Omega. \tag{16}$$

To conclude more about the functional I, we consider the second Frechet derivative:

$$I'' = \begin{bmatrix} \left(-\frac{\partial^2 W}{\partial u^2}\right)_{\Omega} - \left(\Lambda \frac{\partial^2 G}{\partial u^2}\right)_{\partial\Omega}, & \left(A - \frac{\partial^2 W}{\partial u\, \partial \phi}\right)_{\Omega} - \left(\Lambda \frac{\partial^2 G}{\partial u\, \partial \phi}\right)_{\partial\Omega} \\ \left(A - \frac{\partial^2 W}{\partial \phi\, \partial u}\right)_{\Omega} - \left(\Lambda \frac{\partial^2 G}{\partial \phi\, \partial u}\right)_{\partial\Omega}, & \left(-\frac{\partial^2 W}{\partial \phi^2}\right)_{\Omega} - \left(\Lambda \frac{\partial^2 G}{\partial \phi^2}\right)_{\partial\Omega} \end{bmatrix} . \tag{17}$$

(The Frechet derivatives are assumed to coincide with the Gateaux derivatives through out this article.) The following conditions are necessary for an extremal behavior of $I(\phi, u)$:

(a) The matrix (17) must have commutative property in the following sense:

$$\{I''h_1 , h_2\} = \{h_1 , I''h_2\},$$

where { } is the inner product defined in the introductory remarks, and h_1, h_2 are vectors of the space $H_u \times H_\phi$, with u, ϕ defined on $\Omega + \partial\Omega$. (For proof see [2] page 157.)

(b) The signs of the diagonal terms of (17) must remain constant in some neighborhood of the extremal point $(u_0, \phi_0) \in H_u \times H_\phi$.
Condition (a) is satisfied if

$$\frac{\partial^2 W}{\partial u\, \partial\phi} = \frac{\partial^2 W}{\partial\phi\, \partial u} \qquad \text{in } \Omega$$

and

$$\frac{\partial^2 G}{\partial u\, \partial\phi} = \frac{\partial^2 G}{\partial\phi\, \partial u} \qquad \text{in } \partial\Omega.$$

Condition (b):

Here the consistency of sign rule necessitates that the trial functions must be chosen in a neighborhood of (u_0, ϕ_0) in which the expressions

$$\left[-\frac{\partial^2 W}{\partial u^2}_{(\Omega)} - \Lambda \frac{\partial^2 G}{\partial u^2}_{(\partial\Omega)}\right] \tag{18a}$$

and

$$\left[-\frac{\partial^2 W}{\partial \phi^2}_{(\Omega)} - \frac{\partial^2}{\partial\phi^2}(f - \Lambda G)_{(\partial\Omega)}\right] \tag{18b}$$

are of constant sign in the same sense as expressions (e) in the introductory remarks. The additional assumptions of the theorem (2) then constitute sufficient conditions for the existence of an extremum of $I(u, \phi)$ at the point (u_0, ϕ_0). (See [13].) In particular if the signs of (18a) and (18b) differ in a neighborhood of (u_0, ϕ_0) then (u_0, ϕ_0) is a minimax point of I.

A remark on a special case. Let the boundary condition be of the form

$$g(\phi) = 0 \qquad \text{on } \partial\Omega. \tag{19}$$

Then the associated functional on $\partial\Omega$ is

$$(u, g(\phi))_{\partial\Omega} = G(u, \phi). \tag{20}$$

If $G(u, \phi)$ satisfies

$$\frac{\partial}{\partial\phi}(f - \Lambda G) = 0 \qquad \text{for some constant } \Lambda,$$

then for a fixed u, $G(u, \phi)$ does not contribute anything in the minimization process, and $g(\phi) = 0$ is a natural boundary condition. Otherwise $g(\phi) = 0$ is an essential boundary condition.

4. Application to Elastodynamics

Let the boundary conditions be of the form

$$\left.\begin{aligned} X_\nu &= f_1(t, \mathbf{x}) \\ Y_\nu &= f_2(t, \mathbf{x}) \\ Z_\nu &= f_3(t, \mathbf{x}) \end{aligned}\right\} \quad \text{on } \partial\Omega$$

$$0 \leqslant t \leqslant a, \qquad \mathbf{x} \in \partial\Omega,$$

where

$$X_\nu = X \cos(x, \nu)$$
$$Y_\nu = Y \cos(y, \nu)$$
$$Z_\nu = Z \cos(z, \nu),$$

X, Y, Z are components of the externally applied boundary pressure, and ν is the outward normal direction.

The boundary condition of the form $g(\tau, \mu) = 0$ on $\partial\Omega$ becomes

$$\left.\begin{aligned} X_\nu - \tau_{x\nu} - \rho \frac{\partial^2 u}{\partial t^2} &= 0 \\ Y_\nu - \tau_{y\nu} - \rho \frac{\partial^2 v}{\partial t^2} &= 0 \\ Z_\nu - \tau_{z\nu} - \rho \frac{\partial^2 w}{\partial t^2} &= 0 \end{aligned}\right\} \quad \text{on } \partial\Omega. \tag{21}$$

$$(\tau_{x\nu} = \tau_{xx} \cos(\nu x + \tau_{xy} \cos(\nu y) + \tau_{xz} \operatorname{cox}(\nu z), \text{etc.})$$

This is of the form $g(\tau) = 0$ on $\partial\Omega$, and the associated functional $G(\mu, \tau)$ is

$$\int_{\partial\Omega_a} \mu_x(X_\nu - \tau_{x\nu} - \rho\ddot{u})\, ds\, dt$$

$$\int_{\partial\Omega_a} \mu_y(Y_\nu - \tau_{y\nu} - \rho\ddot{v})\, ds\, dt$$

$$\int_{\partial\Omega_a} \mu_z(Z_\nu - \tau_{z\nu} - \rho\ddot{w})\, ds\, dt,$$

where μ has the same meaning as in (6):

$$\mu_x = \frac{\partial u}{\partial t}, \qquad \mu_y = \frac{\partial v}{\partial t}, \qquad \mu_z = \frac{\partial w}{\partial t},$$

while the function $f(\mu, \tau)$ is immediately supplied by integrating by parts $(T\tau, \mu)$, since in this case T is a simple matrix differential operator of order one. As in (6) μ's could be regarded as Lagrangian multipliers following a classical calculus of variation treatment (such as given for example in [18]) for the auxiliary boundary conditions (21) imposed on the problem of extremizing the functional I.

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