B-Fairness and Structural B-Fairness in Petri Net Models of Concurrent Systems*

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Received September 18, 1987; revised March 1, 1990

Fairness properties are very important for the behavior characterization of distributed concurrent systems. This paper discusses in detail a bounded-fairness (or B-fairness) theory applied to Petri Net (PN) models. For a given initial marking two transitions in a Petri Net are said to be in a B-fair relation (BF-relation) if the number of times that either can fire before the other fires is bounded. Two transitions are in a structural B-fair relation (SF-relation) if they are in a B-fair relation for any initial marking. A (structural) B-fair net is a net in which every pair of transitions is in a (structural) B-fair relation. The above B-fairness concepts are further extended to groups (or subsets) of transitions, and are called group B-fairness. This paper presents complete characterizations of these B-fairness concepts. In addition, algorithms are given for determining B-fairness and structural B-fairness relations. It is shown that structural B-fairness relations can be computed in polynomial time.

1. INTRODUCTION

Petri Nets have been widely used to model and analyze concurrent systems. Among their main characteristics it is important to note the following properties: (1) graphical nature, (2) powerful theory to validate models, and (3) independence of particular implementations (they can be hardwired, microprogrammed, or programmed). It is assumed that the reader is familiar with the basic concepts of Petri Nets such as those in [1, 2]. Nevertheless, the main notations are briefly recalled in Section 2.

* This work was supported in part by the U.S.-Spain Joint Committee for Scientific and Technological Cooperation, Grant CCB-8409024.
There are many properties related to the behavior of Petri Net models. For example, by means of *liveness* it is possible to characterize system *deadlocks* (total or partial); by means of *reversibility*, overall cyclic behaviors are characterized (i.e., from any reachable marking the initial marking can be reached). In this paper several interrelated fairness concepts within the Petri Net (PN) theory framework are considered.

There exist many different definitions of *fairness* in the literature (see [6, 7, 8]), because “fairness is used as a generic name for a multitude of concepts” [8, p. 4]. In this paper, fairness is considered as an agreement between actors, valid for any possible behavior in the system that characterizes a symmetric finite delay property.

In PNs each transition represents an elementary action and basic activity is represented by the firing of a transition. Therefore we define the basic fairness concept, bounded-fairness (B-fairness), as a relation between the firing of two transitions. In the sequel it will be assumed that the firing of any transition (i.e., the execution of its associated action) takes a finite time.

Two transitions in a Petri Net are said to be a *B-fair relation* if the number of times that either can fire before the other fires is bounded. A *B-fair net* is a net in which every pair of transitions are in a B-fair relation. The delay between two consecutive firings of any transition in a B-fair net is always finite. Fortunately, B-fairness is an *equivalence* relation, leading to its nice characterization.

The two main generalizations of the basic B-fairness concept are the structural B-fairness concept and the group-B-fairness concept. Structural B-fairness characterizes the case in which B-fairness between two transitions holds independently of the initial marking. It is a sufficient condition for B-fairness, and its computation is only related to the net structure.

Group-B-fairness is introduced to consider the case in which the activity of an actor is better characterized by the firing of a group (a subset) of transitions. This last generalization allows us to deal with systems in which the important point is the global activity of a process or the global activity related with a resource, more than a single action (e.g., “a philosopher” thinks or eats). Structural group-B-fairness is introduced later in a similar way.

This paper reviews the fairness theory introduced in [9] by presenting new concepts and results, and efficient algorithms for their analysis. In particular, it is shown that structural B-fairness analysis can be carried out in polynomial space and time.

The paper is organized as follows: Section 2 introduces some basic concepts and properties, while Sections 3 and 4 are dedicated to B-fairness and structural B-fairness analysis, respectively. The analysis techniques introduced in Section 3 are based on the coverability graph (reachability graph for bounded nets), while those presented in Section 4 belong to the class of structural analysis techniques. In Section 5 group-B-fairness is considered. Finally, in Section 6, reduction rules are considered for the fairness analysis.
2. Basic Concepts and Properties

2.1. Preliminaries

Let $N$ be a Petri Net, $N = \langle P, T, \text{Pre}, \text{Post} \rangle$:

- $P$ is the set of places ($n = |P|$).
- $T$ is the set of transitions ($m = |T|$).
- $\text{Pre}$ (and $\text{Post}$) is the pre- (post-) incidence function (also called the input (output) function) where
  - $\text{Pre}: P \times T \rightarrow \mathbb{N}$ (it represents arcs going from places to transitions)
  - $\text{Post}: P \times T \rightarrow \mathbb{N}$ (it represents arcs going from transitions to places), where $\mathbb{N} = \{0, 1, 2, 3, \ldots \}$.

A marking of a net is a function $M: P \rightarrow \mathbb{N}$. The marking $M$ represents a distributed (through the places) state. The pair $\langle N, M_0 \rangle$ represent a marked Petri Net.

A marking $M$ enables a transition $t \in T$ iff $\forall p \in P$, $M(p) \geq \text{Pre}(p, t)$. If $t$ is enabled at $M_i$, then $t$ can be fired to yield a new marking $M_j$ defined as

$$M_j(p) = M_i(p) + \text{Post}(p, t) - \text{Pre}(p, t).$$

Obviously, if $p$ and $t$ are not directly related by any arc (i.e., $\text{Post}(p, t) = \text{Pre}(p, t) = 0$), then $M_j(p) = M_i(p)$.

A group of structural analysis techniques of PNs is based on the consideration of its state equation (see, for example, [2, 3]). If the firing of transition $t_j$ from marking $M_{k-1}$ allows marking $M_k$ to be reached, $M_{k-1}[t_j] > M_{k}$, then

$$M_k = M_{k-1} + C \cdot U_j = M_{k-1} + C(t_j) \quad \text{(state equation)},$$

where

- $C = [C_{ij}]$, $C_{ij} = \text{Post}(p_i, t_j) - \text{Pre}(p_i, t_j)$
- $U_j(g) = 0$, $\forall g \neq j$, $U_j(j) = 1$.

The $j$th column of $C$, $C(t_j)$, represents the flow of tokens caused by the firing of transition $t_j$. The $i$th row of $C$, $C(p_i)$, represents the flow of tokens through place $p_i$. Matrix $C$ is called [3] the token flow matrix of the PN (flow matrix, for short). If the net is pure (self-loop free, i.e., $\text{Pre}(p, t) \cdot \text{Post}(p, t) = 0$), the flow matrix may be interpreted as a place to transition incidence matrix.

Let

- $\sigma$ be a firing sequence;
- $\sigma$ be $\sigma$'s characteristic vector (i.e., the Parikh mapping whose $i$th component, $\sigma(i)$, is the number of occurrences of transition $t_i$ in $\sigma$);
• $\|V\|$ be the support of vector $V$ (i.e., the subset of elements corresponding to non-zero entries in vector $V$): $\|V\| = \{i \mid V(i) \neq 0\}$ (for example: $\|\sigma\|$ is the set of transitions that have been fired in the sequence $\sigma$);

• $M_i[\sigma > M_j$ express that $\sigma$ is applicable from $M_i$ and $M_j$ is the reached marking;

• $R(N, M_0)$ be the set of all reachable markings from $M_0$; and

• $L(N, M_0)$ be the set of all fireable sequences from $M_0$.

A marked net $\langle N, M_0 \rangle$ is said to be (behaviorally) bounded iff for any place $p_i$ there exists a $k_i \in \mathbb{N}$ such that for any reachable marking $M$, $M \in R(N, M_0)$, $M(p_i) \leq k_i$. Behavioral boundedness in PNs characterizes the finiteness of its state space. If $\langle N, M_0 \rangle$ is bounded for any $M_0$, it is said to be structurally bounded. A marked net $\langle N, M_0 \rangle$ is said to be live iff in any reachable marking $M$, there is an applicable sequence $\sigma$, $M[\sigma > M_i$, such that $M_i$ enables transition $t_i$, $\forall t_i \in T$. Liveness in PNs guarantees the possible (not actual) infinite activity of all the transitions in the net. If a net is live, it is deadlock free.

A firing sequence $\sigma$ in a net $\langle N, M_0 \rangle$ is said to be repetitive iff there exist two reachable markings $M$ and $M'$ such that $M[\sigma > M'$ and $M' > M$. A repetitive sequence can be repeated infinitely often. If the net is bounded, repetitive sequences lead back to the marking from which they are applied, i.e., $M[\sigma > M' = M$.

Integrating the state equation, $M_{k+1} = M_k + C \cdot U_j$, along a sequence $\sigma$ (i.e., summing for $k = 1, 2, \ldots$) we obtain

$$M_0[\sigma > M_q \Rightarrow M_q = M_0 + C \cdot \sigma.$$ 

Any vector $X$ (respectively $Y$) solution of $C \cdot \Sigma X = 0$ (respectively $Y^T \cdot C = 0$) is a right (respectively left) annihilator of $C$. In particular, any $X \in \mathbb{N}^n$ such that $C \cdot X = 0$ and $X \neq 0$ is a $t$-semiflow (also called a reproduction vector, because if $\sigma = X$ and $\sigma$ is fireable from the marking $M$, $M[\sigma > M$; i.e., $M$ is reproduced after firing sequence $\sigma$). A net is consistent iff all transitions belong to at least one $t$-semiflow (i.e., if there exists an $X \in \mathbb{N}^n$ such that $C \cdot X = 0$ and $\|X\| = T$). In a similar way, any $Y \in \mathbb{N}^n$ such that $Y^T \cdot C = 0$ and $Y \neq 0$ is a $p$-semiflow (also called a conservative component): The name comes from the token conservation law, $p$-invariant, obtained by pre-multiplying the state equation by $Y$: $Y^T \cdot M = Y^T \cdot M_0$. If there exists $Y$ such that $\|Y\| = P$, the net is said to be conservative (i.e., there exists a total token conservation law). Additionally it is particularly easy to check that conservative nets are structurally bounded [2].

2.2. Basic Fairness Concepts and Properties

DEFINITION 2.1 [9]. Two transitions in a marked Petri Net $\langle N, M_0 \rangle$ are said to be in a bounded-fair ($B$-fair) relation, denoted by $BF$, iff there exists a positive integer $k$ such that neither of them can fire more than $k$ times without firing the other, in any firing sequence starting at any marking $M$ reachable from $M_0$, $M \in R(N, M_0)$.
According to Definition 2.1, transitions $t_i, t_j \in T$ are in a B-fair relation (denoted by $t_i, t_j \in BF$ or $t_i BF t_j$) iff

$$\forall M \in R(N, M_0) \quad \{\sigma(t_i) = 0 \Rightarrow \sigma(t_j) \leq k$$
$$\forall \sigma \in L(N, M) \quad \{\sigma(t_i) = 0 \Rightarrow \sigma(t_j) \leq k.$$

If $t_i$ and $t_j$ are in a B-fair relation, it is possible for the total delay between two consecutive executions (firings) of one of them (for example, $t_i$) to be unbounded (because other transitions, $\{t_1, \ldots, t_q\}$, can fire infinitely often). Nevertheless, the number of executions of the other transition ($t_j$, in this case) will be bounded. B-fairness is a relative and symmetric finite delay property.

**Definition 2.2** [9]. A Petri Net $\langle N, M_0 \rangle$ is called a B-fair net iff every pair of transitions are in a B-fair relation.

According to Definition 2.2, a marked net $\langle N, M_0 \rangle$ is a B-fair net iff $\forall t_i, t_j \in T, t_i BF t_j$. If a marked net is B-fair, any transition is fired infinitely often in any infinitely large firing sequence. In other words, all the firing sequences (computations) in $\langle N, M_0 \rangle$ are impartial according to [6] (unconditionally fair, according to the terminology in [8]).

The Petri Nets shown in Figs. 1a and 1b are two B-fair nets, but the one shown in Fig. 1c is not. Transition $t_3$ in Fig. 1c is not in a B-fair relation with any other transition, because for the sequences $\sigma_1 = t_1 t_2 t_3 t_4$ and $\sigma_2 = t_3, \sigma = \sigma_1^k \cdot \sigma_2^k$ is firable from the initial marking $\forall k \in \mathbb{N}$.

Consideration of the Petri Net of Fig. 1a leads to the conclusion that for any finite initial marking this net will be B-fair. This property is characterized using the structural B-fairness concept.

![Fig. 1. B-fairness and structural B-fairness: (a) BF- and SF-net, (b) BF- and non SF-net, and (c) non-BF- and non-SF-net.](image_url)
**Definition 2.3** [9]. Two transitions in a Petri Net are said to be in a *structural B-fair relation*, denoted by \( SF \), iff for any initial marking \( M_0 \), the two transitions are in a B-fair relation.

**Definition 2.4** [9]. A Petri Net \( N \) is said to be *structurally B-fair* iff it is a B-fair net for any initial marking, \( M_0 \).

**Theorem 2.1.** *B-fair and structured B-fair relations on the set of transitions are equivalence relations.*

**Proof.** Let us consider first a B-fair relation in a net \( \langle N, M_0 \rangle \).

- Reflexivity obviously holds and symmetry is required in the definition.
- Transitivity can be shown as

\[
\begin{align*}
\forall M \in R(N, M_0) \quad & \forall \sigma \in L(N, M) \quad \left\{ \begin{array}{l}
\sigma(t_1) = 0 \Rightarrow \sigma(t_2) \leq k_1 \\
\sigma(t_2) = 0 \Rightarrow \sigma(t_1) \leq k_1
\end{array} \right. \\
\forall M \in R(N, M_0) \quad & \forall \sigma \in L(N, M) \quad \left\{ \begin{array}{l}
\sigma(t_2) = 0 \Rightarrow \sigma(t_3) \leq k_2 \\
\sigma(t_3) = 0 \Rightarrow \sigma(t_2) \leq k_2
\end{array} \right.
\end{align*}
\]

Then we can write

\[
\begin{align*}
\forall M \in R(N, M_0) \quad & \forall \sigma \in L(N, M) \quad \left\{ \begin{array}{l}
\sigma(t_1) = 0 \Rightarrow \sigma(t_3) \leq k \\
\sigma(t_3) = 0 \Rightarrow \sigma(t_1) \leq k,
\end{array} \right.
\end{align*}
\]

where \( k = (k_1 + 1)(k_2 + 1) \); i.e., \( t_1 SF t_3 \).

By considering now any finite \( M_0 \) we can conclude that \( t_1 SF t_3 \). 

According to Theorem 2.1 the set of transitions in any Petri Net can be *partitioned* by fairness relations into equivalence classes. This partition reflects the *fairness behavior* (B-fair relation) or the *fairness structure* (structural B-fair relation) of the net. If the partition has only one class (i.e., all the transitions belong to this class), the net will be B-fair or structurally B-fair, depending on the relation used.

Because (structural) B-fairness is an equivalence relation, in a (structural) B-fair net the delay between two consecutive firings of any transition is always finite. In other words, if a net is (structurally) B-fair, a total and symmetric finite delay property holds for any firable sequence (i.e., any computation).

According to Definitions 2.3 and 2.4 it is clear that structural B-fairness is a *sufficient condition* for B-fairness. It is obvious to trace the analog to boundedness properties: Structural boundedness is a sufficient condition for boundedness. The Petri Net of Fig. 1a is structurally B-fair, while the Petri Nets of Figs. 1b and 1c (same structure!) are not structurally B-fair.

A much less obvious example of the differences between bounded-fairness and structural-fairness is shown in Fig. 2.1 where the net is live. For the given initial
FIG. 2. All possible combinations of boundedness, liveness, and B-fairness.
marking this is a B-fair net: the only repetitive sequence is $\sigma = t_1 t_2 t_3 t_4 t_5 t_6$. If we add, for example, a token to $p_4$, the net will also be live, but not B-fair. The following are two of the possible repetitive sequences:

- $\sigma_1 = t_1 t_2 t_3 t_5$, where $M_0[t_2 > M_1[\sigma_1 > M_1$ (i.e., $\sigma_1$ can be applied infinitely often from $M_1$).
- $\sigma_2 = t_2 t_4 t_6$, where $M_0[\sigma_2 > M_0$.

From the information obtained from $\sigma_1$ and $\sigma_2$ we can say that any transition belonging to $\{t_1, t_3, t_5\}$ is in a B-unfair relation with any transition belonging to $\{t_2, t_4, t_6\}$, and vice versa. Figure 2 shows Petri Nets corresponding to all possible combinations of the three properties: boundedness (B), liveness (L), and bounded-fairness (F).

If two transitions, $t_1$ and $t_2$, are live, they can be in a B-fair relation (the net in Fig. 2.1 is bounded, while the net in Fig. 2.2 is not bounded) or in an unfair relation (the net in Fig. 2.5 is bounded, while the net in Fig. 2.6 is unbounded). If two transitions are not live, they can be in a B-fair relation (the net in Fig. 2.3 is bounded while the net in Fig. 2.4 is unbounded) or in an unfair relation (the net in Fig. 2.7 is bounded, while the net in Fig. 2.8 is unbounded).

The next theorem states the basic implication among the liveness of two transitions and its B-fairness relation.

**Theorem 2.2.** Let $t_1$ be a live transition in a marked net $\langle N, M_0 \rangle$, and $t_2$ be a non-live transition. Then $t_1$ and $t_2$ are not in a B-fair relation. (Thus, they are not in a structural B-fair relation.)

**Proof.** Since $t_2$ is a non-live transition, there exists at least a reachable marking $M \in R(N, M_0)$, from which $t_2$ cannot be fired any more. But since $t_1$ is live, it can be fired infinitely often. Thus $t_1$ and $t_2$ are not in a B-fair relation. 

**Corollary 2.1.** Let $\langle N, M_0 \rangle$ be a non-live net. If it is B-fair, then none of the transitions are live (i.e., there is a total deadlock, a state in which there are no successor states). The converse is not true.

![Fig. 3. A structurally bounded and structurally non-live net in which $t_1BFt_3$ and $t_2BFt_4$.](image)
It is easy to verify the above fact on the PNs of Figs. 2.3 (a bounded net) and 2.4 (a non-bounded net). Consideration of the PN of Fig. 3 leads to the conclusion that the converse is not true: if there exists a deadlock, then the net is not necessarily B-fair.

3. Analysis of B-Fairness

Using Definitions 2.1 and 2.2, this section discusses the results that lead to algorithms for B-fairness computation. For didactical reasons in Section 3.1 we only consider the case of (behaviorally) bounded PNs. Later this result is generalized to any PN in Section 3.2. In both cases the basic idea is to consider circuits (which are directed and elementary) in the reachability graph or the coverability graph, respectively. Theorems 3.1 and 3.2 present compact formal descriptions of a method for testing B-fairness. Section 3.3 deals with some algorithmic and complexity considerations.

3.1. The (Behaviorally) Bounded Case

Let \( \langle N, M_0 \rangle \) be a behaviorally bounded PN and \( RG(N, M_0) \) be its reachability graph (i.e., the graph in which each node represents a distinct reachable marking and each arc represents the firing of the transition that produces the corresponding marking evolution). Figure 4a represents the reachability graph of the PN of Fig. 2.5.

According to the definition (Section 2.1), repetitive sequences \( \sigma \) in \( \langle N, M_0 \rangle \) correspond to circuits of the reachability graph.

**Definition 3.1.** A repetitive sequence \( \sigma \) in \( \langle N, M_0 \rangle \) is elementary iff it corresponds to a circuit of \( RG(N, M_0) \).

![Fig. 4. Reachability and coverability graphs: (a) the reachability graph for the Petri Net shown in Fig. 2.5; (b) the coverability graph for the Petri Net shown in Fig. 2.6.](image)
Applying the above definition to the reachability graph of Fig. 4a we have

\[
\sigma_1 = t_4 t_1 t_2 t_1 \Rightarrow \sigma_1 = (2 \ 1 \ 0 \ 1)^T
\]

\[
\sigma_2 = t_4 t_1 t_3 \Rightarrow \sigma_2 = (1 \ 0 \ 1 \ 1)^T.
\]

At this point it is interesting to recall that the number of circuits of a graph can increase exponentially with the number of arcs. References [10, 11] give efficient algorithms for the computation of circuits. In [12] an algorithm to compute all the minimal t-invariants of a PN is presented: for a graph the minimal t-invariants define the circuits.

Let \( RS = (\sigma_1, \sigma_2, \ldots, \sigma_e) \) be the matrix in which the characteristic vectors of the circuits of \( RG(N, M_0) \) are the columns. \( RS \) can be partitioned into row vectors as

\[
RS = \begin{pmatrix}
  s_1 \\
  s_2 \\
  \vdots \\
  s_m
\end{pmatrix}, \quad \text{where } m = |T| \text{ (number of transitions)}.
\]

**Theorem 3.1.** Transitions \( t_i \) and \( t_j \) are in a B-fair relation in \( \langle N, M_0 \rangle \) iff the supports of \( s_i \) and \( s_j \) are the same: \( t_i, B F t_j \Leftrightarrow \|s_i\| = \|s_j\| \).

**Proof.** (a) Necessity: Let \( \|s_i\| \neq \|s_j\| \), e.g., \( s_i(q) = 0 \) and \( s_j(q) \neq 0 \) (i.e., \( \sigma_q(t_i) = 0, \sigma_q(t_j) \neq 0 \)).

Then in the circuit named \( q \) of \( RG(N, M_0) \) either \( t_i \) or \( t_j \) does not appear. If, for example, \( s_i(q) \neq 0 \) and \( s_j(q) = 0 \), repeating \( \omega \)-times the firing of the circuit \( q \), we have \( \sigma_q(t_j) = 0 \) and \( \sigma_q(t_j) > \omega \). Then \( t_i \) and \( t_j \) are not in a B-fair relation.

(b) Sufficiency: \( RG(N, M_0) \) is a finite graph. If \( v \) is the number of reachable markings, then any sequence of length \( |\sigma| > v \) must contain sequences associated with circuits in \( RG(N, M_0) \). If for any circuit \( q \) we have \( s_i(q) = s_j(q) \) or \( s_i(q) > 0 \) and \( s_j(q) > 0 \), then \( \forall o \in L(N, M) \) and \( \forall M \in R(N, M_0) \)

\[
\sigma(t_i) = 0 \Rightarrow \sigma(t_j) \leq v \quad \text{and} \quad \sigma(t_j) = 0 \Rightarrow \sigma(t_i) \leq v;
\]

i.e., \( t_i \) and \( t_j \) are in a B-fair relation.

According to Theorem 3.1, transitions \( t_i \) and \( t_4 \) are the only transitions in a B-fair relation in the PN of Fig. 2.5.

A t-semiflow \( X \) is said to be realizable in \( \langle N, M_0 \rangle \) if \( \exists M \in R(N, M_0) \) such that \( M[\sigma > M \land \sigma = X \text{ and } X^T = (1 \ 0 \ 1 \ 0 \ 1 \ 0) \) is not realizable in the live and bounded net of Fig. 2.1. Obviously, the characteristic vectors of the circuits of \( RG(N, M_0) \) are the realizable t-semiflows. Thus, Theorem 3.1 can be restated as follows: transitions \( t_i \) and \( t_j \) are in a B-fair relation in \( \langle N, M_0 \rangle \) iff for every realizable t-semiflow \( X \), \( \{t_i\} \cap \|X\| = \emptyset \Leftrightarrow \{t_j\} \cap \|X\| = \emptyset \).
3.2. The General Case

In the previous section, it is assumed that \( \langle N, M_0 \rangle \) is a bounded PN. Let us now consider the case where \( \langle N, M_0 \rangle \) is not necessarily bounded. \( CG(N, M_0) \) denotes its coverability graph \([2]\). The coverability graph of a net \( \langle N, M_0 \rangle \) can be obtained from the coverability tree \([2]\) by merging all the nodes having the same label. In \([13]\) the computation of the minimal coverability graph is considered. Figure 4b represents the coverability graph of the PN of Fig. 2.6. If the net is bounded, the coverability graph becomes the reachability graph.

DEFINITION 3.2. A circuit \( \sigma \) in \( CG(N, M_0) \) is non-decreasing iff \( C \cdot \sigma \geq 0 \), where \( \sigma \) is the characteristic vector of the firing sequences defined by the circuit and \( C \) the flow matrix of \( N \).

Because of the well-known relation \( C \cdot \sigma = M' - M \), non-decreasing circuits in \( CG(N, M_0) \) characterize elementary repetitive sequences in the PN.

In the coverability graph of Fig. 4b we have three (non-decreasing) circuits with \( (\sigma_1 = \sigma_2 = t_1 t_2 \text{ and } \sigma_3 = t_1 t_3) \):

\[
\begin{align*}
\sigma_1 &= (1 \ 1 \ 0)^T \Rightarrow M' - M = (0 \ 0 \ 0)^T \\
\sigma_2 &= (1 \ 0 \ 1)^T \Rightarrow M' - M = (0 \ 0 \ 1)^T \\
\sigma_3 &= (1 \ 0 \ 1)^T \Rightarrow M' - M = (0 \ 0 \ 1)^T.
\end{align*}
\]

LEMMA 3.1. Given an arbitrarily \( k \in \mathbb{N} \) and a circuit \( \mathcal{C} \) in \( CG \), there always exists a reachable marking \( M^k \), \( M_0 \mathcal{C} \sigma > M^k \), from which the firing of the transitions labeling arcs in \( \mathcal{C} \) can be repeated \( k \) times.

Proof. If the circuit \( \mathcal{C} \) is non-decreasing, the corresponding firing sequence \( \sigma \) can be repeated infinitely often.

Let \( \mathcal{C} \) be a circuit for which the non-decreasing property does not hold, because \( C(p) \cdot \sigma = -a < 0 \). Then for the place \( p \), the firing of each sequence firable in the circuit \( \mathcal{C} \) decreases the number of its tokens by "\( a \)." Then as \( \mathcal{C} \) is a circuit in the \( CG \), \( M(p) = \omega \) for all nodes (markings) in \( \mathcal{C} \). To be able to repeat \( \mathcal{C} \) \( k \) times it is sufficient to reach a marking \( M^k \) such that \( M^k(p) \geq ak \). The extension to the case in which several places \( p_i \) are such that \( C(p_i) \cdot \sigma < 0 \) is obvious. 

Let \( CS = (\sigma_1, \sigma_2, \ldots, \sigma_q) \), the matrix in which the characteristic vectors of the circuits of \( CG(N, M_0) \) are the columns. \( CS \) can be partitioned into row vectors as

\[
CS = \begin{pmatrix}
z_1 \\
z_2 \\
\vdots \\
z_m
\end{pmatrix}, \quad \text{where } m = |T|.
\]

THEOREM 3.2. Transitions \( t_i \) and \( t_j \) in \( \langle N, M_0 \rangle \) are in a B-fair relation iff the supports of \( z_i \) and \( z_j \) are the same: \( t_i \mathcal{B} t_j \Leftrightarrow \|z_i\| = \|z_j\| \).
Proof. It can be deduced as a direct generalization of Theorem 3.1, based on the finiteness of the coverability graph for any PN and Lemma 3.1. 

It is very simple to verify that Theorem 2.1 can be proved now directly from Theorem 3.2, because the equality of sets (supports of rows in CS(N, M₀)) is an equivalence relation. Additionally, it is obvious that 〈N, M₀〉 is a B-fair net iff all the rows have the same support: ∀i, j ∈ {1, 2, ..., m}, ∥z_i∥ = ∥z_j∥.

In the PN of Fig. 2.6 there exist no pairs of transitions that are in a B-fair relation.

3.3. Considerations on Algorithms and Complexity

Theorem 3.2 solves the problem of B-fairness analysis for PNs. We now consider some algorithmic and complexity issues.

Let CG_q(N, M₀) be the graph obtained by removing all arcs labelled t_q in CG(N, M₀). If Π_q means ∥z_i∥ ⊆ ∥z_j∥, the statement of Theorem 3.2 can be written as

\[ t_i \text{BFT}_j \Leftrightarrow \Pi_q \land \Pi_i, \]

but it is not difficult to realize that

\[ \Pi_q \Leftrightarrow \text{there are no circuits containing } t_j \text{ in } CG_q(N, M₀). \]

According to the above remarks it is immediate to accept the following restatement of Theorem 3.2.

Theorem 3.3. Transitions t_i and t_j in 〈N, M₀〉 are in a B-fair relation iff there is neither a circuit containing t_j in CG_q(N, M₀) nor a circuit containing t_i in CG_q(N, M₀).

Considering again the net in Fig. 2.5, RG_1(N, M₀) and RG_3(N, M₀) are acyclic graphs, thus t_i \text{BFT}_4. Nevertheless, RG_3(N, M₀) has a circuit containing \{t_1, t_2, t_4\}, therefore none of these transitions is in a BF-relation with t_3.

From a computational point of view, it must be recalled that the coverability graph is exponential-space-hard. For a very simple illustration, let us consider the net in Fig. 5. It is very easy to realize that it is live iff ∀i ∈ {1, 2, ..., k}, M₀(p_i) +

Fig. 5. A bounded net. If it is live, the reachability graph has an exponential number of nodes (markings).
$M_0(p_{i_2}) = a_i \geq 1$. If $v$ represents the number of nodes in the coverability graph (a reachability graph, because the net is structurally bounded), it is not difficult to check that $v = \prod_{i=1}^k (a_i + 1) \geq 2^k = 2^{n/2}$.

Let $v_q$ and $e_q$ be the number of nodes and edges in $CG_q(N, M_0)$, respectively. Obviously $v = v_q \leq e + 1$, $e \geq e_q$, and $e \geq e_q$. The computation of the condition in Theorem 3.3 is polynomial in $v$ and $e$. For example, it follows using the Johnson's algorithm \cite{11}.

**Theorem 3.4.** It can be decided in $O(v(v + e))$ running time and $O(v + e)$ space if two transitions $t_i$ and $t_j$ are in a $B$-fair relation.

**Proof.** According to Theorem 3.3, we only need to solve several analogous basic problems that consist of determining whether there exists a circuit through a given edge in a graph. But this problem is linear in $v + e$, if a slight modification of Johnson's algorithm \cite{11} is used (in particular, if we terminate when a first circuit appears). At most $v$ problems of the above need to be computed to check if there exists a circuit containing $t_j$ in $CG_j(N, M_0)$, because from each node only one of their output arcs can be labeled with $t_j$.  

Finally, it must be stated that, given a net $N$, all the above computations are valid just for a given initial marking, $M_0$. In the next section the structural $B$-fairness concept and structural analysis techniques are introduced. The analysis, of polynomial complexity, will be independent of $M_0$. In any case, it is also important to note that the practical complexity of $B$-fairness analysis may be greatly reduced by means of the net reduction techniques discussed in Section 6.

### 4. Analysis of Structural $B$-Fairness

This section presents basic results leading to full algebraic characterizations of structural $B$-fairness. It starts in a manner similar to that of the preceding section, just to emphasize the analogy between behavioral-fairness (BF) and structural-fairness (SF) characterizations. Proceeding in this way it is very simple to show that BF- and SF-relations are the same for live and bounded free-choice nets. It is not possible to extend the above result to asymmetric-choice (simple) nets (Fig. 2.1 is a counterexample). However, the practical experience on live net models of "real systems" shows that in "most cases" two transitions in a BF-relation are also in an SF-relation (the reverse is always true). This practical remark is particularly interesting because, in Section 4.3 it is shown that SF-relations can be computed in polynomial time.

#### 4.1. The Structurally Bounded PNs Case

A place is structurally bounded iff it is bounded for any $M_0$. A net is structurally bounded iff all of its places are structurally bounded.
The following theorem gives a full algebraic characterization of structural boundedness for $p$. Let $e_p$ be a characteristic vector such that

- $\text{dim}(e_p) = n = |P|
- e_p[i] := \text{if } i = p \text{ then } 1 \text{ else } 0.$

**Theorem 4.1 [14].** The following three statements are equivalent:

(a) $p$ is structurally bounded

(b) $\exists X \succcurlyeq 0$ such that $C \cdot X \succcurlyeq e_p$

(c) $\exists Y \succcurlyeq e_p$ such that $Y^T \cdot C \preceq 0$.

**Corollary 4.1 [14].** The following three statements are equivalent:

(a) $N$ is structurally bounded

(b) $\exists X \succcurlyeq 0$ such that $C \cdot X > 0$

(c) $\exists Y > 0$ such that $Y^T C \preceq 0$.

Theorem 4.1 and Corollary 4.1 show clearly that structural boundedness can be checked in polynomial time by proving that a system of linear inequalities has a solution or no solutions.

Let $R = (R_1 R_2 \cdots R_x)$, be the matrix in which the minimal $t$-semiflows are the columns. $R$ can be computed by the algorithm in [12] and can be partitioned into row vectors as

$$ R = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{pmatrix}, \quad \text{where } m = |T|. $$

**Theorem 4.2.** Transitions $t_i$ and $t_j$ are in a structural $B$-fair relation in a structurally bounded net $N$ iff the supports of $r_i$ and $r_j$ are the same: $t_i SF t_j \iff \|r_i\| = \|r_j\|$.

**Proof.** This theorem can be proved using Theorem 3.1 and taking into account the following:

(a) If $M_0$ is large enough, then it is possible to fire independently a sequence whose characteristic vector is a minimal $t$-semiflow, $R_i$.

(b) The characteristic vector $\sigma$ of any repetitive sequence $\sigma$ can be obtained by a non-negative linear combination of the minimal $t$-semiflows plus a bounded vector:

$$ \sigma = \sum_i \mu_i R_i + V. $$
For the PN of Fig. 2.1 we can write

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix}$$

It can be shown that any pair of transitions in the set \( \{t_1, t_3, t_5\} \) or \( \{t_2, t_4, t_6\} \) are in a structural B-fair relation.

An alternative statement of Theorem 4.2 is as follows: Transitions \( t_i \) and \( t_j \) are in a structural B-fair relation in \( N \) iff for every minimal \( t \)-semiflow \( R_k \),

\[ \{t_i\} \cap \|R_k\| = \emptyset \iff \{t_j\} \cap \|R_k\| = \emptyset. \]

**COROLLARY 4.2** [2, 9]. A structurally bounded PN is structurally B-fair iff:

(a) it is consistent and there is only one \( t \)-semiflow, or

(b) it is not consistent but there is no \( t \)-semiflow.

The second case corresponds to a case in which \( R = 0 \) (i.e., it is not possible to obtain an infinite sequence). Corollary 4.1 states that a structurally fair net \( N \) has at most one \( t \)-semiflow [9].

**COROLLARY 4.3** [9]. For consistent, structurally bounded and B-fair nets, \( \text{rank}(C) = m - 1 \).

**COROLLARY 4.4.** Strongly connected marked graphs are structurally B-fair nets.

**Proof.** Any connected marked graph is a consistent PN with a unique minimal \( t \)-semiflow. Any strongly connected marked graph is conservative [3]. Then it is structurally bounded. Finally, Corollary 4.4 is deduced from Corollary 4.2, case (a).

At this point the reader can easily check the strong analogy of the characterizations of BF- (Th. 3.1) and SF-relations (Th. 4.2). Clearly, if the set of the different characteristic vectors of the circuits of \( RG(N, M_0) \) equals the set of minimal \( t \)-semiflows (i.e., all minimal \( t \)-semiflow are realizable, \( RS \equiv R \)), then BF and SF collapse into the same relation. This happens for live and bounded free-choice (LBFC) nets.

**THEOREM 4.3.** Let \( \langle N, M_0 \rangle \) be a live and bounded free-choice (LBFC) net. Two transitions \( t_i \) and \( t_j \) are in a B-fair relation iff they are in structural B-fair relation: \( t_iBFt_j \Leftrightarrow t_iSFt_j \).
Proof (Sketch). It is based on two results of structure theory of LBCF (see [2, 15], where they are stated for live and safe free-choice (LSFC) nets). They can be informally restated as follows:

(a) Any LSFC net can be decomposed into strongly connected marked graph components and these cover the net.

(b) For any strongly connected marked graph component, there exist reachable markings, $M \in R(N, M_0)$, such that their restriction to the component makes it live (and safe).

The above two properties hold also for LBFC nets, because we can always freeze the activity of some tokens in such a way that a live and safe behavior is obtained for the FC net.

The only consideration to be taken into account now is that the subnet generated by a minimal $t$-semiflow is just one of the strongly connected marked graph components. The above theorem follows from Theorems 3.1 and 4.2.

The statement in the above theorem is not easy to generalize to a larger net class such as symmetric-choice nets (i.e., nets in which each transition has at most one shared input place). A counterexample is shown in the net of Fig. 2.1. Nevertheless, when considering live net models of "real systems" (not over all possible live net models), we find in practice that BF- and SF-relations coincide very frequently.

Theorem 4.2 gives a full algebraic characterization of structural B-fairness enumerating all minimal $t$-semiflows. Its number can be exponential, but bounded by $C^m_{m/2} = (\frac{m!}{(m/2)!})$, where $m = |T|$ (see [3]). In the sequel, we explore the utility of basis of right annihilators of $C$, instead of the set of minimal $t$-semiflows. An interesting point, probably not expected at a first glance, is that a basis does not explicitly provide full information to characterize the SF-relations. Nevertheless, polynomial-time-computable conditions allow us to decide on SF-relations for some particular cases. In Section 4.3, a more powerful result allows us to see that SF-relations can be characterized always in polynomial time.

Let $B = (B_1, B_2, \ldots, B_h)$ be a basis of right annihilators of $C$ (i.e., $C \cdot B_i = 0$), where $h = m - \text{rank}(C)$. The matrix $B$ can be partitioned into row vectors as

$$B = \begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{pmatrix} = (b_{we}), \quad \text{where } m = |T|.$$

**Theorem 4.4.** Given a structurally bounded net $N$, if there exists a $\lambda \neq 0$ such that $b_i = \lambda \cdot b_j$ (i.e., $b_i$ and $b_j$ are colinear), then $t_i$ and $t_j$ are in an SF-relation.

Proof. $b_i = \lambda \cdot b_j \Rightarrow r_i = \lambda \cdot r_j \Rightarrow ||r_i|| = ||r_j||$, and Theorem 4.2 applies. \[\square\]
The condition for a structural-fair relation in Theorem 4.4 is not a necessary condition as can be seen in the PN of Fig. 2.5 (a conservative and consistent net), where

\[
B = \begin{pmatrix}
1 & 2 \\
0 & 1 \\
1 & 0 \\
1 & 1
\end{pmatrix}
\]

and \( t_1 \mathcal{SF} t_4 \), but \( \exists \lambda \) such that \( b_1 = \lambda \cdot b_4 \).

Now consider the PN of Fig. 2.7 (a conservative but not consistent net). We have

\[
B = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 1 \\
0 & -1
\end{pmatrix}
\]

and \( t_1 \mathcal{SF} t_3 \), but \( \exists \lambda \) such that \( b_1 = \lambda \cdot b_3 \).

**COROLLARY 4.5.** If \( \forall u, v b_{uv} \in \{0, 1\} \), then \( b_i = b_o \Leftrightarrow t_i \mathcal{SF} t_o \).

**Proof.** (1) Sufficiency: This case is trivial because support equality is equivalent to colinearity and Theorem 4.2 applies:

\[
b_i = b_j \Rightarrow r_i = r_j \Rightarrow \|r_i\| = \|r_j\| \Leftrightarrow t_i \mathcal{SF} t_j.
\]

(2) Necessity: According to Theorem 4.2, \( t_i \mathcal{SF} t_j \Leftrightarrow \|r_i\| = \|r_j\| \). If \( B_i \) is defined over \( \{0, 1\} \) and all \( B_i \) are \( t \)-semiflows, then \( t_i \mathcal{SF} t_j \Rightarrow \|b_i\| = \|b_j\| \Leftrightarrow b_i = b_j \).

Corollary 4.5 is of some practical value in fairness analysis because many Petri net models found in practice satisfy the above conditions.

In the PN of Fig. 2.7 we have \( R = (1010)^T \). Thus \( \|b_i\| = \|b_j\| \) is not a necessary condition for structural B-fairness in non-consistent PNs. On the other hand, for consistent PNs, fairness between two transitions cannot be concluded from any non-negative basis of elementary annihilators of \( C \) (see Fig. 6).

**Fig. 6.** Support equality in \( B \) does not always imply SF-fairness. (\( \|b_3\| = \|b_4\| \) but \( \|r_5\| \neq \|r_4\| \Leftrightarrow t_3 \) and \( t_4 \) are not in an SF-relation.)
THEOREM 4.5. Let $N$ be a structurally bounded and consistent PN and $B^+$ a non-negative basis of right annihilers of $C$. If $B^+ = (p_j^0)$, where $D$ is a diagonal matrix, $t_i S F t_j$ iff $\|b_i^+\| = \|b_j^+\|$.

Proof. If $D$ is diagonal, then $R = B^+$ and Theorem 4.2 applies. $\square$

For the PN of Fig. 2.5 Theorem 4.5 makes it possible to determine the SF-relation in polynomial time, since $b_2$ and $b_3$ form an identity matrix.

By using Theorems 4.4 and 4.5 conclusions about structural-fairness can be obtained in polynomial time if the net under analysis satisfies one of the stated conditions. Of course (assuming $n \approx m$, where $n = |P|$), the computation of a basis or right annihilers of $C$ can be done in $O(m^3)$. On the other hand, consistency can be checked in $O(m^3)$ by finding if there exists a non-negative basis of right annihilers of $C$.

4.2. The General Case: Reduction to Structurally Bounded Nets

As in Section 3.2, we consider in this section PNs that are not necessarily bounded. Observing the progression between the statements in Theorems 3.1 and 3.2, and that in Theorem 4.2, it can be expected that the consideration of minimal solutions $X \in \mathbb{N}^m$ such that $C \cdot X \geq 0$ allows a general characterization of structural B-fairness. Nevertheless, this is not true as it is pointed out in Theorem 4.5. Using Theorem 4.6, structural B-fairness analysis can be done for structurally unbounded PNs by transforming the original net into another one that is structurally bounded.

The minimal support (elementary) non-negative solutions of the inequality system $C \cdot A_i \geq 0$ and $A_i \in \mathbb{N}^n$ will be the non-negative minimal support solutions of the equivalent system of equations,

$$(C | -I) \cdot \begin{pmatrix} A_i \\ V_i \end{pmatrix} = 0,$$

where $I$ is the identity matrix of dimension $n$ and $V_i$ represents $n$ slack variables, $V_i \in \mathbb{Z}_n$. The vectors, $A_i$'s, are called the elementary repetitive components of the net.

The net $N^*$ with flow matrix $C^* = (C | -I)$ is obtained simply by adding an output transition $t^*$ to each place, $p_i$:

(a) $Pre(p_i, t^*) = 1$, $Pre(p_j, t^*) = 0 \quad \forall j \neq i$

(b) $Post(p_k, t^*) = 0 \quad \forall p_k \in P$.

Let $A = (A_1 A_2 \cdots A_q)$ be the matrix in which the column vectors, $A_i$'s, are the non-negative minimal support solutions of $C \cdot A_i \geq 0$, $A_i \in \mathbb{N}^n$. Clearly the $t$-semiflows, $R_n$, appear as columns of $A$.

Matrix $A$ can be partitioned into row vectors as

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix},$$

where $m = |T|$. 

```
For the PN of Fig. 2.8 we can obtain [3]

\[ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \]

**Theorem 4.6.** Two transitions \( t_i \) and \( t_j \) are in a structural B-fair relation in \( N \) only if the supports of \( a_i \) and \( a_j \) are the same: \( t_i \mathcal{SF} t_j \Rightarrow \|a_i\| = \|a_j\| \). The converse is not true.

**Proof.** It is obvious that given an \( M_0 \) large enough, sequences such that \( t_i \in \|\sigma\| \) and \( t_i \notin \|\sigma\| \) (or vice versa) can be fired independently “ad infinitum.” Then equality of the support of \( \{a_i\} \) is a necessary condition. The converse is not true as is shown in Fig. 7, where \( \|a_1\| = \|a_4\| \) but \( t_1 \neq SF \). ■

The application of Theorem 4.6 to the PN of Fig. 2.8 shows that \( t_1 \neq SF \), \( t_1 \neq SF \), \( t_2 \neq SF \), \( t_3 \neq SF \). Then the only structural B-fairness possibilities are between \( t_1 \) and \( t_3 \), and between \( t_2 \) and \( t_4 \).

Let us now introduce a technique to obtain a complete characterization of structural B-fairness for unbounded nets. First, the structurally unbounded PN is transformed into a structurally bounded net by preserving the fairness relations. Then we apply the results for structurally bounded nets.

The following lemma provides the foundation for the surprising results of the next theorem.

**Lemma 4.1.** There exist finite initial markings and associated firing sequences such that every structurally unbounded place will contain an arbitrarily large number of tokens.

**Fig. 7.** Support equality in row vectors of \( A \) does not always imply structural B-fairness between transitions.
Proof. Let $X = \{X_i\}$ be the set of elementary non-negative minimal-support solutions of $C \cdot X_i \succeq 0$, and $X = \sum_i X_i$. By defining $M_0$ as

$$\forall p \in P \quad M_0(p) = \sum_{t \in P} X(t) \cdot Pre(p, t),$$

sequences $\sigma_i$ such that $\sigma_i = X_i$ can be executed independently, and the marking of structurally unbounded places will grow "ad infinitum."}

THEOREM 4.7. The removal of structurally unbounded places together with their incident arcs preserves the structural B-fairness relations.

Proof. By the transition firing rule, the input places of transition represent restrictions on the transition’s firing capabilities. But if a place contains an arbitrarily large number of tokens, it cannot restrict the firing of its output transitions. Because all structurally unbounded places can contain an arbitrarily large number of tokens (by Lemma 4.1), structural B-fairness will be preserved by removing structurally unbounded places.

COROLLARY 4.6. The removal of structurally unbounded places together with their incident arcs transforms any structurally unbounded PN, N, into a (unique) structurally bounded PN, $N^*$. According to Theorem 4.7 and Corollary 4.6, structural B-fairness can be studied by using Theorems 4.2 to 4.5 on the new PN, $N^*$. From a practical point of view it is interesting to define algorithms to transform $N$ into $N^*$. This transformation can be viewed as a reduction rule. In Section 6 some additional reduction rules are considered.

Algorithm for Obtaining $N^*$, from $N$

1. Let $\mathcal{L}$ be a list containing all places of $N$
2. $P^* := \emptyset$ (the set of places of $N^*$)
3. while $\mathcal{L}$ is not empty do
   begin
      3.1. $p_i := \text{first}(\mathcal{L})$;
      3.2. solve $C \cdot X \succeq e_i$, $X \succ 0$,
           where $e_i$ is a vector with $\text{dim}(e_i) = n = |P|$ and such that
           $e_i[j] := \text{if } j = i \text{ then } 1 \text{ else } 0$
      3.3. if $X_i$ is a solution (i.e., $C \cdot X_i \succeq e_i$, $X_i \succ 0$)
           then remove from $\mathcal{L}$ all $p_i$ such that $C(p_i) \cdot X_i > 0$
           else $P^* := P^* \cup p_i$  \{ $p_i$ is structurally bounded \}
      3.4. remove $p_i$ from $\mathcal{L}$
   end
4. $P^*$ is the set of places of $N^*$

The correctness of the algorithm trivially follows from the algebraic characterization of structurally bounded places in Theorem 4.1b. In the worst case, the “while”
is executed \( n = |P| \) times (when \( N \) is structurally bounded). Because obtaining a solution of \( C \cdot X \geq e_i \land X \geq 0 \) is of polynomial complexity, the algorithm is of polynomial complexity.

**Theorem 4.8.** The transformation of \( N \) into \( N^* \) by removing the structurally unbounded places can be done in \( O(n^{4.5}) \) running time, where \( n = |P| \).

**Proof.** Let us consider the following linear programming problem (LPP):

\[
\begin{align*}
\text{max} & \quad 0^T \cdot X \\
\text{s.t.} & \quad C \cdot X \geq e_i \\
& \quad X \geq 0.
\end{align*}
\]

The objective function is clearly constant, but this trick allows us to decide if there exists a solution to the set of constraints in \( O((n + m)^{\alpha}) \). The constant \( \alpha \) depends on the particular polynomial algorithm that is used to solve the problem (\( \alpha = 3.5 \) in Karmarkar's algorithm [163]). Because \( n \approx m \), an iteration will have \( O(n^{3.5}) \) complexity. The \( n \) iterations needed in the worst case analysis allow us to state the theorem.

**Remark 4.1.** Even if the well-known simplex method is theoretically of exponential complexity, in practice its revised version behaves better than the polynomial algorithms, with \( \alpha \approx 1 \) [17].

**Remark 4.2.** If \( N \) is expected to have none or very few structurally unbounded places, a more efficient and practical algorithm can be constructed using the algebraic characterization of Theorem 4.1c. The theoretical worst case analysis, assuming \( n \approx m \), is \( O(m^{4.5}) \). Thus it is analogous to that in the above theorem because \( n \approx m \).

**Examples.**

1. The application of the above method to the PN in Fig. 2.8 yields \( P^* = \{p_1, p_2, p_4\} \). The calculation of a basis \( B^* \) for the PN without place \( p_3 \) gives

\[
B^* = \begin{pmatrix}
1 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix}.
\]

Thus according to Theorem 4.3, \( t_1SFt_2 \) and \( t_3SFt_4 \).

2. The application of the above methods to the PN in Fig.2.6 gives \( P^* = \{p_1, p_2\} \). A basis \( B^* \) for \( N^* \) is found to be

\[
B^* = \begin{pmatrix}
1 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}.
\]
Thus, the new PN, \( N^* \), is consistent. By Theorem 4.2 or 4.5, there is no pair of transitions which are in a structural B-fair relation in \( N^* \).

3. In the PN of Fig. 2.2 the place \( p_4 \) is structurally unbounded. After removing place \( p_4 \) we obtain a circuit. It is easy to see that the elements of \( T = \{ t_1, t_2, t_3 \} \) are in a structural B-fair relation (i.e., the net is structurally B-fair).

### 4.3. A Polynomial Time Computation of Structural B-Fairness Relations

The basic results of the above two sections allow us to reduce the computations from any net to that of structurally bounded nets (Theorem 4.7) and to characterize the structural B-fairness relations using the set of minimal \( t \)-semiflows (Theorem 4.2).

Basically, this section introduces an alternative characterization to that discussed in Theorem 4.2, in which the minimal \( t \)-semiflows are not explicitly enumerated. The new characterization directly uses \( C \), the incidence, matrix of the net, as the generator.

Let \( N' \) denote the net obtained from \( N \) by removing \( t_q \) and let \( \Pi_{ij} \) denote \( \| r_i \| \leq \| r_j \| \). The condition on Theorem 4.2 can be rewritten as

\[
t_i . SF t_j \Leftrightarrow \Pi_{ij} \land \Pi_{ji}
\]

Now, the fundamental remark is the following:

\[
\Pi_{ij} \Leftrightarrow \forall X \geq 0 \text{ such that } C \cdot X = 0, e_i^T \cdot X > 0 \Rightarrow e_j^T \cdot X > 0
\]

\[
\Leftrightarrow \exists X \geq 0 \text{ such that } C \cdot X = 0, e_i^T \cdot X > 0 \text{ and } e_j^T \cdot X = 0
\]

\[
\Leftrightarrow \exists X^i \geq 0 \text{ such that } C^i \cdot X^i = 0, e_i^T \cdot X^i > 0.
\]

Therefore, Theorem 4.2 can be restated as follows:

**Theorem 4.9.** Two transitions \( t_i \) and \( t_j \) are in a structural B-fair relation in a structurally bounded net \( N \) iff

(a) \( \not\exists X^i \geq 0 \text{ such that } C^i \cdot X^i = 0, e_i^T \cdot X^i > 0 \), and

(b) \( \exists X^j \geq 0 \text{ such that } C^j \cdot X^j = 0, e_j^T \cdot X^j > 0 \).

The above result characterizes the SF-relations by means of the non-existence of solutions in two systems of linear inequalities. Because these problems are of polynomial complexity, the SF-relation problem is also of polynomial complexity in structurally bounded nets.

**Theorem 4.10.** Let \( N \) be a structurally bounded net. It can be decided in \( O(n^{3.5}) \) if \( t_i \) and \( t_j \) are in an SF-relation.

**Proof.** Use reasoning analogous to that in Theorem 4.8 (observe that now we have one more equation, \( e^T \cdot X^b > 0 \), but \( X^b \) has one less variable).
Remark 4.3. At the computational level, the basic difference between the characterization of the behavioral- and structural-fairness relations is the following: the former is based on explicit enumeration of circuits of the coverability graph, while the latter need not enumerate t-semiflows because the theory of linear inequations is used.

The removal of the structurally unbounded places in a net is also of polynomial complexity and preserves SF-relations. Thus the SF-relation problem in general has polynomial complexity.

5. Group Fairness

In this section the B-fairness concept is generalized in a manner similar to that in [25, 26]. We introduce a new concept called group-B-fairness with respect to a transition covering, $\theta = \{ \theta_i | T = \bigcup \theta_i \}$. In many Petri net models, processes or resources are represented by subsets of transitions. Thus, it is natural to define fairness among subsets of transitions.

Let $\theta_i$ be a subset of transitions, $\theta_i = \{ t_k \}$, and $\sigma(\theta_i) = \sum_{t_k \in \theta_i} \sigma(t_k)$.

**Definition 5.1.** Two subsets of transitions, $\theta_i$ and $\theta_j$, in a Petri Net $\langle N, M_0 \rangle$ are in a group-B-fair relation iff

$$\forall M \in R(N, M_0) \quad \{ \sigma(\theta_i) = 0 \Rightarrow \sigma(\theta_j) \leq k \}$$

$$\forall \sigma \in L(N, M) \quad \{ \sigma(\theta_i) = 0 \Rightarrow \sigma(\theta_j) \leq k \}.$$

**Definition 5.2.** Two subsets of transitions, $\theta_i$ and $\theta_j$, in a Petri Net $N$ are in a structural group-B-fair relation iff for any finite initial marking, $M_0$, the transition groups are in a group-B-fair relation.

As was the case for B-fairness, the above definitions can be applied to the concepts of group-B-fair nets and structurally group-B-fair nets, for a given transition covering.

We can now consider group-B-fairness analysis for the behavioral case or for the structural case. Let us consider structural group-B-fairness analysis. Analogous results can be obtained for group-B-fairness analysis for bounded nets.

In this section we assume that nets are structurally bounded. If the net under consideration is not structurally bounded, we can remove structurally unbounded places by Theorem 4.7. The analysis of group-B-fairness for structurally bounded nets can be based on the following theorems, which are generalizations of Theorems 4.2 and 4.9.
Let

- \( R = (R_1, R_2, \ldots, R_k) \) be the matrix, where \( R_k \) is the \( k \)th reproduction vector of a (structurally bounded) net \( N \).
- \( \theta = \{ \theta_i \mid T = \bigcup \theta_i \} \)
- \( r^i = \sum_{t_q \in \theta_i} r_q \), where \( r_q \) is the \( q \)th row of \( R \).

**Theorem 5.1.** In a structurally bounded net, two transition groups, \( \theta_i \) and \( \theta_j \), are in a structural group-B-fair relation iff \( \| r^i \| = \| r^j \| \).

**Proof.** The proof easily follows from that of Theorem 4.2 because the union of the support of non-negative vectors \( r_q \) is the same as the support of the sum of these vectors \( r_i \).

The next corollary is an interpretation of Theorem 5.1 for consistent and structurally bounded nets.

**Corollary 5.1.** A consistent and structurally bounded PN is structurally group-B-fair, for a given transition covering \( \theta \), iff at least one transition from each group \( \theta_i \) appears in every minimal \( t \)-semiflow.

The application of the above results to the PN of Fig. 2.5 leads to the conclusion that \( \theta = \{ \{ t_1 \}, \{ t_2, t_3 \}, \{ t_4 \} \} \) is a transition partition for which the net is structurally group-B-fair.

Theorem 5.1 is the direct generalization of Theorem 4.2. It is interesting and practical to generalize Theorem 4.8 because of the polynomial complexity. Let us define \( E_i \), the characteristic vector of \( \theta_i \subseteq T \), as

\[
E_i[j] := \begin{cases} 1 & \text{if } t_j \in \theta_i; \\ 0 & \text{else} \end{cases}
\]

Let \( N^k \) represent the net in which all transitions of \( \theta_k \subseteq T \) have been removed (\( C^k \) is its incidence matrix, and \( X^k \) a \( t \)-semiflow of \( N^k \)).

Using the above notation it is not difficult to accept the following compact and computationally efficient statement:

**Theorem 5.2.** In a structurally bounded net, two transitions groups, \( \theta_i \) and \( \theta_j \), are in a structural group-B-fair relation iff

(a) \( \exists X^i \geq 0 \) such that \( C^i \cdot X^i = 0 \), \( E^i_j \cdot X^i > 0 \), and

(b) \( \exists X^j \geq 0 \) such that \( C^j \cdot X^j = 0 \), \( E^j_i \cdot X^j > 0 \).

Combining the results in Section 4.2 (Theorem 4.7 and the algorithm to get \( N^* \) from \( N \)) with Theorem 5.2, it can be stated that the structural group-B-fairness problem is of polynomial complexity.
FAIRNESS ANALYSIS BY NET REDUCTION

The idea of net reduction is to transform the initial net into another net which is simpler to analyze, but which preserves some prescribed properties. Reduction techniques have proved to be very useful in the analysis of liveness and boundedness properties (see, for example, [3, 18, 19, 20]). In this section we consider some reduction rules that preserve fairness relations. As structural B-fairness is a sufficient condition for B-fairness, and live and B-fair nets are frequently structurally B-fair when considering "real systems net models," the following development is presented mainly for structural B-fairness preservation.

According to the reduction rule expressed by Theorem 4.7 all structurally unbounded places can be directly removed. Let us now consider another reduction rule by which a place can also be directly suppressed. It is based on the implicit place concept (see, for example, [3, 18, 19]).

Let \( C(p_i) \) be the row vector associated with \( p_i \) in the flow matrix \( C \) of the net \( N \). Let \( N', C', \) and \( M'_0 \) represent the net, flow matrix, and initial marking obtained by removing \( p_i \) in \( N, C, \) and \( M_0 \), respectively.

**Definition 6.1 [19].** In the marked net \( \langle N, M_0 \rangle \) place \( p_i \) is:

(a) **Firing implicit** if its removal does not change the set of firing sequences, i.e., \( L(N, M_0) = L(N', M'_0) \).

(b) **Marking implicit** if it is firing implicit and its marking is redundant (i.e., can be computed from the marking of other places).

According to the above definition, the elimination of implicit places preserves B-fairness. Thus it is a reduction rule for B-fairness analysis.

B-fairness is related only to transition firing. Thus in regard to fairness analysis it is clear that the firing implicit place concept is of more interest. For structurally bounded and live nets, the firing and marking implicit place concepts coincide.

Let us now consider the structural counterpart to the implicit place concept.

**Definition 6.2 [19].** A place \( p_j \) is **structurally implicit** in \( N \) iff for any \( M'_0 \), the initial marking on \( N' \), there exists an \( M_0(p_j) \) such that \( p_j \) is implicit in \( \langle N, M_0 \rangle \).

Places on a marked net represent constraints to the firing of its transitions. Thus taking a "large enough" initial marking for a structurally implicit place, it becomes implicit and does not at all constrain the behavior of the net. The following theorem summarizes the previous discussions on implicit places and fairness relations.

**Theorem 6.1.** The removal of an (structurally) implicit place, together with its incident arcs, preserves all the (structural) B-fairness relations.
The characterization of implicit places is not a polynomial problem. Nevertheless structurally implicit places and those that are effectively implicit for a given marking can be characterized in polynomial time.

**Theorem 6.2** [19]. (i) A place \( p_j \) is structurally implicit iff \( \exists Y \geq 0 \) such that \( Y^T \cdot C^i \preceq C(p_j) \).

(ii) Let \( \langle N, M_0 \rangle \) be a marked net and let \( p_j \) be a structurally implicit place of \( N \). The following linear programming problem defines the variable \( v \):

\[
v = \text{minimize } Y^T \cdot M'_0 + \mu
\]

subject to \( Y^T \cdot C^i \preceq C(p_j) \)

\[
Y^T \cdot \text{Pre}^i(t_k) + \mu \geq \text{Pre}^i(p_j, t_k) \quad \forall t_k \in p_j
\]

\[
Y \geq 0,
\]

where \( \text{Pre}^i(t_k) \) is the column vector that represents the previous incidence function on \( N^i \) for transition \( t_k \).

If \( M_0(p_j) \geq v \), then \( p_j \) is implicit in \( \langle N, M_0 \rangle \).

The next theorem gives a complementary rule that also preserves structural B-fairness by eliminating a place \( p \) after the fusion of transitions. The new net \( N^* \) will have \( m^* = m - 1 \) transitions.

Let a place \( p \) be

1. structurally bounded, and
2. have only one input or one output transition \((|p| = 1, \text{ or } |p'| = 1, \text{ where } p = \{t_{in}\} \text{ or } p' = \{t_{out}\})\).

By \( C(t_j) \) we denote the column vector associated with transition \( t_j \) in the flow matrix \( C \).

**Theorem 6.3.** The following reduction rule preserves (structural) B-fairness. A place \( p \) can be eliminated and the set of its input and output transitions can be replaced according to the following rule (Fig. 8):

1. If \( p = \{t_{in}\} \), the input transition, \( t_{in} \), will be simply eliminated. Each output transition, \( t_j \in p' \), will be replaced by \( t_j^* \) such that \( C^*(t_j^*) = C(t_j) + C(t_{in}) \).
2. If \( p' = \{t_{out}\} \), the output transition, \( t_{out} \), will be simply eliminated. Each input transition, \( t_k \in p \), will be replaced by \( t_k^* \) such that \( C^*(t_k^*) = C(t_k) + C(t_{out}) \).

**Proof** (Sketch). If \( p \) is structurally bounded in \( N \), then for each \( X_j \in \mathbb{N}^m \) such that \( V_j = C \cdot X_j \geq 0, V_j(p) = 0 \). The computation of vectors \( X_j \) can start with aiming at obtaining \( V_j(p) = 0 \). This theorem follows from the results in [12] for the annulation of the row vector associated with \( p \).
Remark 6.1. From a theoretical point of view, Theorem 6.3 can be generalized to the case in which the arcs have weights and the number of input and output transitions is larger than one. This generalization has not been performed above because it is not very practical for graphical reduction.

Corollary 6.1. By considering the special case of Theorem 6.3 represented in Fig. 9 and the structural implicit place reduction rule, all strongly connected marked graphs can be reduced to a single transition (i.e., strongly connected marked graphs are fully reducible).

If \( p \) is structurally bounded, then \( t_i \) and \( t_j \) can be fused into a transition \( t_{ij} \) and \( p \) can be eliminated. Structural fairness is preserved. In cases b and c, liveness is also preserved (see, for example, [3]).
As a consequence, from Corollary 6.1, it can be stated that strongly connected marked graphs are structurally B-fair nets (note that this result was established also in Corollary 4.3).

Finally, it is possible to state that in the above reduction process, global structural B-unfairness can also be inferred.

**THEOREM 6.4.** $N$ is structurally B-unfair if there exists

1. an identity transition (i.e., $\forall p \in P, \operatorname{Pre}(t, p) = \operatorname{Post}(t, p)$) and at least one more transition, or

2. two equivalent transitions $t_1$ and $t_2$ (i.e., $\forall p \in P, \operatorname{Pre}(t_1, p) = \operatorname{Pre}(t_2, p)$ and $\operatorname{Post}(t_1, p) = \operatorname{Post}(t_2, p)$) and transition $t_1$ (and then $t_2$) belongs to a $t$-semiflow (i.e., $t_1 \in \|X\|$).

**Proof.** In the first case, the identity transition can fire infinitely often for large enough $M_0$ without firing any other transition. In the second case $t_1$ and $t_2$ are clearly not in a B-fair relation, because for an $M_0$ large enough, $t_1$ (or $t_2$) can be fired infinitely often without firing $t_2$ (or $t_1$).

The above theorem can be stated for behavioral B-unfairness if $\langle N, M_0 \rangle$ is a marked net for which

1. the identity transition can be fired at least once,

2. transition $t_1$ (or $t_2$) belongs to an elementary repetitive sequence.

**EXAMPLES.** 1. In Fig. 2.1, the places $\{p_4, p_5, p_6\}$ are structurally implicit (e.g., $C(p_4) = C(p_2) + C(p_3) + C(p_8) + C(p_9), ...$). The elimination of $\{p_4, p_5, p_6\}$ leads to a net composed of two independent circuits with $\{t_1, t_3, t_5\}$ and $\{t_2, t_4, t_6\}$. Then, according to Theorems 6.1 and 6.2, both sets of transitions determine structural B-fairness equivalence classes for the original net.

2. The PN of Fig. 10 is reduced while preserving liveness, boundedness, and fairness. After obtaining $t_{35}$ and $t_{46}$, $EX$ is an implicit place and can be eliminated. Later $t_{135}$ and $t_{246}$ are obtained. Finally, the fusion of $t_{1357}$ and $t_{2468}$ leads to the conclusion that the net is live, 4-bounded, and structurally B-fair.

3. Place $p_3$ in Fig. 2.6 can be reduced because it is unbounded. The application of Theorem 4.7 allows the elimination of $p_3$, obtaining a state graph. There are no pairs of transitions that are in a B-fair relation in the net.

4. Places $p_1$ and $p_4$ in Fig. 2.8 can be reduced. After the elimination of the structurally implicit place $p_7^*$, the new net has a place $p_3$ with a source transition $t_{12}$ and a sink transition $t_{34}$. Then $\{t_1, t_2\}$ and $\{t_3, t_4\}$ are pairs of B-fair transitions.

5. The application of Theorem 6.3 part (1) to place $p_1$ in Fig. 2.5 gives a $p_3^*$ that is structurally implicit. Its elimination (Theorem 6.2) gives a net for which Theorem 6.4 allows a direct conclusion on structural B-unfairness.
By using reduction rules it is proved that the original net shown in (a) is structurally B-fair (therefore B-fair), since the reduced nets shown in (b) are structurally B-fair.

7. Conclusions

This paper has introduced several concepts for fairness characterization and techniques for its computation in the framework of Petri Net theory. Structural B-fairness is an easily computable property. It is a sufficient condition for B-fairness. Structural B-fairness is independent of the initial marking. For most live nets "found in the practical modelling of real systems," B-fairness and structural B-fairness coincide. In particular, this coincidence holds for all live and bounded free choice nets. Group-B-fairness allows the B-fairness characterization of actors whose activities are represented by subsets of transitions. Group-B-fairness computation appears to be a "natural" generalization of techniques developed for B-fairness computation. Applications of group-B-fairness are found in [25, 26].

From a theoretical point of view, the main results are Theorems 3.2, 4.7, and 5.2 from which many other results can be deduced. They give a very compact and general understanding of basic B-fairness properties. Some of their properties concerning bounded-fairness for structurally and behaviorally bounded PNs were presented in [9].

From a computational point of view, several results that lead to efficient algorithms have been introduced. In particular it has been proved that structural group-B-fairness computation can be done in polynomial time.
Finally, the techniques introduced for our fairness analysis use all three main methods for general Petri Net analysis: enumeration (coverability or reachability graphs), reduction rules, and linear algebraic structural analysis.

Relationships between the B-fairness concepts presented in this paper and other concepts of fairness [6, 7, 8] have been investigated in [21]. In particular, it is shown that

1. B-fairness and unconditional-fairness are equivalent concepts for bounded nets;
2. B-fairness, unconditional-fairness, and strong-fairness are equivalent concepts for bounded live asymmetric-choice (simple) nets; and
3. B-fairness, unconditional-fairness, strong-fairness, and weak-fairness are equivalent concepts for bounded live marked graphs.

Moreover, in [14, 22] some connections between B-fairness and other synchronic properties (see [3, 4, 23]) are established.

ACKNOWLEDGMENTS

The authors thank J. M. Colom, A. Elduque, D. J. Leu, and J. M. Jeffrey for their suggestions and help in preparation of this manuscript.

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