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Subregular Spreads of $\mathscr{PG}(2n+1, q)$

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In this paper, we develop some of the theory of spreads of projective spaces with an eye towards generalizing the results of R. H. Bruck (1969, *in* "Combinatorial Mathematics and Its Applications," Chap. 27, pp. 426–514, Univ. of North Carolina Press, Chapel Hill). In particular, we wish to generalize the notion of a *subregular* spread to the higher dimensional case. Most of the theory here was anticipated by Bruck in later papers; however, he never provided a detailed formulation. We fill this gap here by developing the connections between a regular spread of $(2n+1)$-dimensional projective space and an n-dimensional circle geometry, which is the appropriate generalization of the Miquelian inversive plane. After developing this theory, we provide a fairly general method for constructing subregular spreads of $\mathscr{PG}(5, q)$. Finally, we explore a special case of this construction, which yields several examples of three-dimensional subregular translation planes which are not André planes. © 1998 Academic Press

1. INTRODUCTION

Let q be a prime power, and n a positive integer. A *spread* of $\mathscr{PG}(2n+1, q)$ is a partition of this space into $q^{n+1}+1$ pairwise disjoint projective subspaces of dimension n. The study of spreads of projective spaces is motivated by the following construction of Bose and Bruck [6]. We note that essentially the same construction was given by André [1], but we follow the presentation of Bose and Bruck here.

Let $\Sigma = \mathscr{PG}(2n+2, q)$, and let Σ^* be a hyperplane of Σ, i.e., a projective subspace of dimension $2n+1$. Let $\mathscr{S}$ be a spread of Σ^*. Define an incidence

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structure Π as follows. The points of Π are the points of $\Sigma \backslash \Sigma^*$, together with the n-spaces in $\mathscr{S}$. The lines of Π are the $(n+1)$-dimensional subspaces of Σ which meet Σ^* in an element of $\mathscr{S}$, together with the spread $\mathscr{S}$ itself. Incidence is given by inclusion. Bruck and Bose proved that Π is a translation plane, and further that every finite translation plane can be obtained in this manner for some prime power q and some positive integer n.

By this result, the problem of finding new translation planes is equivalent to that of finding new spreads; further, André's theory readily implies that inequivalent spreads give rise to nonisomorphic planes. Much work has been done on this problem in the case where $n = 1$, i.e., the construction of spreads of $\mathscr{PG}(3, q)$. One of the earliest efforts in this area was Bruck's work [2] on subregular spreads, which we now describe.

Let q be a prime power, and n a positive integer. A *regulus* in $\mathscr{PG}(2n+1, q)$ is a set of $q+1$ pairwise disjoint n-spaces with the property that any line of $\mathscr{PG}(2n+1, q)$ which meets three spaces of the regulus must meet all of the spaces of the regulus. One can easily prove the following well-known proposition:

PROPOSITION 1.1. *Let q be a prime power and n a positive integer. Then any set of three pairwise disjoint n-spaces in $\mathscr{PG}(2n+1, q)$ determines a unique regulus.*

A spread of $\mathscr{PG}(2n+1, q)$ is said to be *regular* if the regulus determined by any three spaces of the spread is fully contained in the spread. Regular spreads are significant because if one performs the construction of Bose and Bruck using a regular spread, a Desarguesian plane is obtained (at least in the finite case). Further, regular spreads are frequently used as starting points for the construction of other spreads. In fact, this is the idea behind subregular spreads.

Consider a regulus $\mathscr{R}$ in $\mathscr{PG}(3, q)$. It is a set of $q+1$ lines with the property that any line which meets three lines of $\mathscr{R}$ must meet every line in $\mathscr{R}$. It turns out that those lines which meet every line of $\mathscr{R}$ themselves form a new regulus, which is called the *opposite regulus* of $\mathscr{R}$, denoted $\mathscr{R}'$.

Now, suppose $\mathscr{R}$ is contained in a spread $\mathscr{S}$. Since the lines of $\mathscr{R}'$ cover the exact same points as the lines of $\mathscr{R}$, we can construct a new spread $\mathscr{S}'$ by removing the lines of $\mathscr{R}$ from $\mathscr{S}$, and replacing them with the lines of $\mathscr{R}'$. This procedure is called *reversing* the regulus $\mathscr{R}$.

This procedure can ostensibly be repeated. If we start with a spread $\mathscr{S}$ which contains a regulus $\mathscr{R}$, we can reverse $\mathscr{R}$ to get a new spread $\mathscr{S}'$. If we are so fortunate that $\mathscr{S}'$ contains a new regulus, then we may reverse this regulus to obtain a new spread, and so on. A spread $\mathscr{S}'$ which can be obtained from a regular spread $\mathscr{S}$ by such a sequence of regulus reversals is called *subregular*. It was shown by Orr [10] that any subregular spread of $\mathscr{PG}(3, q)$ can be obtained from a regular spread $\mathscr{S}$ by starting with a set of pairwise disjoint reguli in $\mathscr{S}$ and simultaneously reversing them.

Our goal in this paper is to generalize this notion of a subregular spread to spreads of $\mathscr{PG}(2n+1, q)$. Unfortunately, the idea of reversing reguli does not generalize directly to higher dimensions. However, Ostrom [11] has defined a more general concept of net replacement, of which regulus reversal is the simplest example. Indeed, the procedure we discuss, due in its geometric form to Bruck [3], is another special case of Ostrom's work.

2. A MODEL FOR A REGULAR SPREAD AND BRUCK'S NORM-SURFACES

Let q be a prime power, and let n be a positive integer. We wish to describe a model, introduced by Bruck [3], for a regular spread of $\Gamma = \mathscr{PG}(2n+1, q)$. Let $F = \mathscr{GF}(q)$ be the finite field of order q, and let $K = \mathscr{GF}(q^{n+1})$ be an $(n+1)$-dimensional extension of F. Let V be the vector space over F whose vectors are $\{(x, y): x, y \in K\}$. It is easy to see, by considering K as an $(n+1)$-dimensional vector space over F and expanding, that V is a $(2n+2)$-dimensional vector space over F. Thus, by taking our points to be the one-dimensional subspaces of V, our lines to be the two-dimensional subspaces of V, etc., we obtain a model of Γ.

We define a spread of Γ using the sets

$$J(\infty) = \{(0, y): y \in K\}$$

and

$$J(k) = \{(x, kx): x \in K\}, \tag{1}$$

as k varies over all the elements in K. The spread $\mathscr{S}$ is then given by

$$\mathscr{S} = \{J(\infty)\} \cup \{J(k): k \in K\}. \tag{2}$$

It is straightforward to show the following proposition:

PROPOSITION 2.1. *Let q be a prime power and n a positive integer. Let $\mathscr{S}$ be defined as in Eq.* (2). *Then $\mathscr{S}$ is a regular spread of $\Gamma = \mathscr{PG}(2n+1, q)$.*

Now that we have a model for a regular spread, we would like to describe our higher-dimensional analog of the regulus. This construction was given by Bruck [3], and he called the resulting structures *norm-surfaces.*

Let $\mathscr{S}$ be a regular spread as defined in Eq. (2). Let N be the norm function of $K = \mathscr{GF}(q^{n+1})$ over F, i.e., $N(x)$ is the product of all the algebraic conjugates of x over F for all $x \in K$. A *norm-surface* is any set projectively equivalent to the set

$$\mathscr{N} = \bigcup_{N(x)=1} J(x).$$

It is obvious that $\mathcal{N}$ contains one set of $\frac{q^{n+1}-1}{q-1}$ pairwise disjoint n-dimensional spaces which cover its points, namely $\{J(k) \mid k \in K \text{ and } N(k) = 1\}$. Bruck was further able to show that if σ is any field automorphism which generates the automorphism group of K over F, then the set $\{J^{\sigma}(k) \mid k \in K$ and $N(k) = 1\}$, where $J^{\sigma}(k)$ is defined via

$$J^{\sigma}(k) = \{(x, kx^{\sigma}) : x \in K\}, \tag{3}$$

is a set of $\frac{q^{n+1}-1}{q-1}$ pairwise disjoint n-dimensional spaces which cover the points of $\mathcal{N}$ as well. We call any of these sets of n-spaces contained in a norm-surface a *hyperregulus.* (We note that Ostrom [11] proved essentially the same result from an algebraic point of view.)

The term hyperregulus was first used by Ostrom [12] in a slightly more general context. The importance of these hyperreguli is that they serve the same "reversal" role as reguli do in $\mathcal{PG}(3, q)$. In particular, if a spread $\mathcal{S}$ contains a hyperregulus $\mathcal{Q}$, then we can obtain a new spread from $\mathcal{S}$ by removing $\mathcal{Q}$ and replacing it with any of the hyperreguli which cover the points of the norm-surface generated by $\mathcal{Q}$. The hyperreguli we consider here are referred to by Ostrom as André hyperreguli, as they were first explored by André [1]. However, as we will show, these André hyperreguli can be used to construct planes which are not André planes.

We can now define a *subregular spread* of $\mathcal{PG}(2n+1, q)$ to be any spread which can be obtained from a regular spread of $\mathcal{PG}(2n+1, q)$ by a sequence of hyperregulus reversals. Trivially, the regular spread of $\mathcal{PG}(2n+1, q)$ is a subregular spread. The André spreads are also subregular. Indeed, André's construction essentially consists of partitioning the spaces of the regular spread $\mathcal{S}$ of Eq. (2) into the two spaces $J(\infty)$ and $J(0)$ and the $q-1$ hyperreguli $\{J(k) \mid k \in K \text{ and } N(k) = f\}$ for each $f \in F^*$, where F^* is the set of nonzero elements of F, each of which covers a norm-surface $\mathcal{N}_f$. To construct an André spread, one merely picks one hyperregulus from each $\mathcal{N}_f$.

Our goal now is to construct subregular spreads of these higher-dimensional spaces, which give rise to planes which are not André planes. Following Bruck's theory for $\mathcal{PG}(3, q)$, we proceed by considering circle geometries, a generalization of inversive planes.

3. A CONNECTION WITH CIRCLE GEOMETRIES

The abstract definition of a circle geometry is given by Bruck in the papers [4, 5]. For our purposes, however, we can merely deal with the classical examples. Let q be a prime power and d a positive integer. The *d-dimensional circle geometry* over $\mathcal{GF}(q)$, denoted $\mathcal{CG}(d, q)$, is the incidence system whose points are the points of the projective line $\mathscr{PG}(1,q^d)$, and whose blocks (called *circles*) are the order q sublines of this projective line. We note that the $\mathscr{CG}(d,q)$'s are $3-(q^d+1,q+1,1)$ designs, which were studied by Witt under the name *spherical designs*.

We model the projective line $\mathscr{PG}(1,q^d)$ using a two-dimensional vector space over $\mathscr{GF}(q^d)$ with homogeneous coordinates. So, the points of $\mathscr{PG}(1,q^d)$ are the one-dimensional subspaces $\{\langle(1,x)\rangle \mid x \in \mathscr{GF}(q^d)\}$ together with $\langle(0,1)\rangle$. For short, we identify the subspace $\langle(1,x)\rangle$ with the field element x, and the subspace $\langle(0,1)\rangle$ with the symbol ∞. We also use this convention in labeling the points of $\mathscr{CG}(d,q)$.

The following theorem motivates the study of circle geometries. The proof of this theorem is quite tedious, but simply involves a comparison of how one constructs reguli in $\mathscr{PG}(2n+1,q)$ and how one constructs sublines in $\mathscr{PG}(1,q^{n+1})$.

THEOREM 3.1. *Let q be a prime power and n a positive integer. Let $\mathscr{S}$ be the regular spread of $\mathscr{PG}(2n+1,q)$ defined in Eq.* (2). *Then if we take the spaces in $\mathscr{S}$ as points, and the reguli contained in $\mathscr{S}$ as circles, the resulting structure is isomorphic to the circle geometry $\mathscr{CG}(n+1,q)$. Further, there exists an isomorphism α which maps the space $J(k)$ onto the field element k for all $k \in \mathscr{GF}(q^{n+1})$, and maps $J(\infty)$ onto ∞.*

Since α is an isomorphism from $\mathscr{S}$ onto $\mathscr{CG}(\mathrm{n}+1,q)$, we can consider the natural mapping

$$\theta : \mathrm{Aut}(\mathscr{S}) \to \mathrm{Aut}(\mathscr{CG}(n+1,q)) \tag{4}$$

which is a surjective group homomorphism (see Bruck [5]). In particular, any collineation of $\mathscr{PG}(2n+1,q)$ which leaves $\mathscr{S}$ invariant can be considered as a permutation of the spaces of $\mathscr{S}$. Since collineations of $\mathscr{PG}(2n+1,q)$ map reguli onto reguli, this action induces an automorphism of the circle geometry under α. One can compute the kernel of this homomorphism, which is the group of mappings in $\mathrm{Aut}(\mathscr{PG}(2n+1,q))$ which fix each space of $\mathscr{S}$ setwise. This is the content of the following proposition, which can easily be proved (see Dover [8] for details in the case $n=2$.)

PROPOSITION 3.2. *Let q be a prime power and n a positive integer. Let $\mathscr{S}$ be the regular spread of $\mathscr{PG}(2n+1,q)$ given by Eq.* (2) *and let K be the field $\mathscr{GF}(q^{n+1})$. If θ is the homomorphism given in Eq.* (4), *then* $\mathrm{Ker}(\theta)$ *is a cyclic group of order $\frac{q^{n+1}-1}{q-1}$ which consists of the collineations induced by the matrices $\{\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right) : k \in K^*\}$. Further, this group cyclically permutes the points of each space in $\mathscr{S}$.*

For the construction of subregular spreads of $\mathscr{PG}(2n+1,q)$, we need to focus on the image of a hyperregulus under the mapping α. This was done by

Bruck in [5], and the resulting object in $\mathscr{CG}(n+1,q)$ is called a *cover*. While Bruck gave an abstract development of covers using the groups associated with $\mathscr{CG}(n+1,q)$, we wish to give a slightly different development, still based on Bruck's work, with which it will be easier to work.

Let $F = \mathscr{GF}(q)$ and $K = \mathscr{GF}(q^{n+1})$. Let $\mathscr{S}$ be the regular spread in Eq. (2) and let α be the isomorphism given in Theorem 3.1. Since the set of spaces $\mathscr{Q} = \{J(k) \mid k \in K \text{ and } N(k) = 1\}$ forms a hyperregulus in $\mathscr{S}$, the set $N_1 = \{k \in K \mid N(k) = 1\}$, which is the image of $\mathscr{Q}$ under α, is a cover $\mathscr{CG}(n+1,q)$.

In [5], Bruck computes the automorphism group of $\mathscr{CG}(n+1,q)$. It turns out that it is 3-transitive and consists of all mappings of the form

$$x\phi = \frac{ax^\sigma + b}{cx^\sigma + d}, \qquad a, b, c, d \in K, \tag{5}$$

with the restrictions that $ad - bc \neq 0$ and the map $\sigma \in \mathrm{Aut}(K)$ is a field automorphism of K. We take the usual conventions on ∞, namely $\infty\phi = a/c$ if $c \neq 0$ and $\infty\phi = \infty$ if $c = 0$.

It is easy to see that the image of N_1 under any automorphism of $\mathscr{CG}(n+1,q)$ is again a cover, and so the preimage of this new cover under α is a new hyperregulus in $\mathscr{S}$. With this in mind, our construction method will be the following. We would like to find a set of pairwise disjoint covers in $\mathscr{CG}(n+1,q)$. From this set of covers, we can use α^{-1} to map back to a set of pairwise disjoint hyperreguli in $\mathscr{S}$. Then, by doing a replacement of each of these hyperreguli, we will obtain a subregular spread.

Note that there are some unanswered questions here. First, we have no result analogous to Orr's result [10] that any subregular spread can be obtained by reversing a pairwise disjoint set of hyperreguli. Further, once a set of pairwise disjoint hyperreguli is chosen, it is unknown how choosing from among the various replacements for a given hyperregulus will affect the type of spread obtained. These are interesting questions which we do not tackle here.

4. EXAMPLES OF SUBREGULAR SPREADS OF $\mathscr{PG}(5,q)$

We would now like to focus on the following setting. Let q be an *odd* prime power, and define $F = \mathscr{GF}(q)$ and $K = \mathscr{GF}(q^3)$.

The basic tool we need for this entire section is a straightforward lemma about finite fields. Let K be a field and F a subfield of K. The *trace function* of K over F is the function which maps an element of K to the sum of all of its algebraic conjugates. If $K = \mathscr{GF}(q^3)$ and $F = \mathscr{GF}(q)$, then $\mathrm{Tr}(x) = x + x^q + x^{q^2}$ for all $x \in K$.

LEMMA 4.1. *Let $F = \mathcal{GF}(q)$ and $K = \mathcal{GF}(q^3)$. Then, for all $a, b \in K^*$*

$$N(a+b) = N(a) + N(b) + \mathrm{Tr}\left(N(b)\frac{a}{b} + N(a)\frac{b}{a}\right).$$

where N is the norm function of K over F, and Tr *is the trace function of K over F.*

Proof.

$$\begin{aligned} N(a+b) &= (a+b)(a+b)^q(a+b)^{q^2} \\ &= a^{q^2+q+1} + ab^{q^2+q} + a^q b^{q^2+1} + a^{q^2} b^{q+1} \\ &\quad + ba^{q^2+q} + b^q a^{q^2+1} + b^{q^2} a^{q+1} + b^{q^2+q+1} \\ &= N(a) + N(b) + \mathrm{Tr}(ab^{q^2+q} + ba^{q^2+q}) \\ &= N(a) + N(b) + \mathrm{Tr}\left(N(b)\frac{a}{b} + N(a)\frac{b}{a}\right). \quad \blacksquare \end{aligned}$$

As mentioned before, our method for constructing subregular spreads of $\mathcal{PG}(5, q)$ is to find a set of pairwise disjoint covers in $\mathcal{CG}(3, q)$. We can then map these covers to a set of pairwise disjoint hyperreguli in $\mathcal{S}$, the regular spread given in Eq. (2) of $\mathcal{PG}(5, q)$.

We begin by giving a method for constructing a pair of disjoint covers. The construction given here was originally obtained by extensive computer experimentation using the package MAGMA [7]. The proof is quite technical, as it consists of some involved arithmetic in finite fields.

THEOREM 4.2. *Let $d \in K = \mathcal{GF}(q^3)$, q odd, and $q \geq 5$, be such that $N(d) \neq 0, 1$. Let $m \in \mathcal{GF}(q) \backslash \{0, 1\}$ be such that $N(d)m + \frac{1}{4}(N(d)-1)^2$ is a square (possibly zero) in $F = \mathcal{GF}(q)$. Let $b \in K$ be an element of norm $\frac{1}{2}(N(d)-1)m + m[N(d)m + \frac{1}{4}(N(d)-1)^2]^{1/2}$. If $N_1 = \{x \in K \mid N(x) = 1\}$, and $\phi \in \mathrm{Aut}(\mathcal{CG}(3, q))$ defined via*

$$x\phi = \frac{x+b}{(md/b)x + d},$$

we have $N_1 \cap N_1\phi = \emptyset$.

Proof. We first note that ϕ is indeed an automorphism of $\mathcal{CG}(3, q)$, since $1(d) - b\frac{md}{b} = d - md \neq 0$, as $m \neq 1$. Further, we note $N(b) \neq 0$, since this would force either $m = 0$ or $\frac{1}{2}(N(d)-1) = -[N(d)m + \frac{1}{4}(N(d)-1)^2]^{1/2}$. Squaring both sides and cancelling yields $N(d)m = 0$, which is not possible as neither $N(d)$ nor m is 0.

Now, by way of contradiction, suppose there exists $u \in N_1 \cap N_1\phi$.Then, for some $x \in N_1$, $u = x\phi$, which implies that there exists $x \in K$ such that $N(x) = 1$ and $N(x\phi) = 1$.

If $N(x\phi) = 1$, we have the string of implications

$$N(x\phi) = 1 \Rightarrow N\left(\frac{x+b}{(md/b)x+d}\right) = 1 \Rightarrow N(x+b) = N(d)N\left(\frac{m}{b}x+1\right).$$

As noted above, neither m nor b is 0, and $x \neq 0$ since $N(x) = 1$. So, we can use Lemma 4.1 to expand the above norms as

$$N(x) + \mathrm{Tr}\left[\frac{N(x)b}{x} + \frac{N(b)x}{b}\right] + N(b)$$
$$= N(d)\left[N\left(\frac{mx}{b}\right) + \mathrm{Tr}\left[\frac{N(mx/b)}{mx/b} + \frac{mx}{b}\right] + N(1)\right].$$

Noting that $N(x) = 1$ and $N(m) = m^3$, since $m \in F$, we can simplify and collect like terms to obtain

$$\mathrm{Tr}\left[\frac{b(1 - N(d)m^2/N(b))}{x} + \frac{(N(b) - N(d)m)x}{b}\right] = -1 - N(b)$$
$$+ N(d) + \frac{N(d)m^3}{N(b)}. \qquad (6)$$

By our hypothesis, $N(b) = \frac{1}{2}(N(d)-1)m + m[N(d)m + \frac{1}{4}(N(d)-1)^2]^{1/2}$, so we can use this to compute the quantity

$$\frac{N(d)m^2}{N(b)} = \frac{N(d)m^2}{(1/2)(N(d)-1)m + m[N(d)m + (1/4)(N(d)-1)^2]^{1/2}}.$$

Rationalizing the denominator and simplifying in this expression yields

$$\frac{N(d)m^2}{N(b)} = -\tfrac{1}{2}(N(d)-1) + [N(d)m + \tfrac{1}{4}(N(d)-1)^2]^{1/2}. \qquad (7)$$

We can now plug these two facts into Eq. (6) to find that

$$\mathrm{Tr}\left[\frac{b}{x}(1 + \tfrac{1}{2}(N(d)-1) - [N(d)m + \tfrac{1}{4}(N(d)-1)^2]^{1/2}) \right.$$
$$\left. + \frac{mx}{b}(\tfrac{1}{2}(N(d)-1) + [N(d)m + \tfrac{1}{4}(N(d)-1)^2]^{1/2} - N(d))\right]$$

$$= -1 - \tfrac{1}{2}(N(d)-1)m - m[N(d)m + \tfrac{1}{4}(N(d)-1)^2]^{1/2}$$
$$+ N(d) - \tfrac{1}{2}(N(d)-1)m + m[N(d)m + \tfrac{1}{4}(N(d)-1)^2]^{1/2},$$

which with some simplification gives

$$\mathrm{Tr}\left[\left(\frac{b}{x} - \frac{mx}{b}\right)\left(\frac{N(d)+1}{2} - \left[N(d)m + \frac{(N(d)-1)^2}{4}\right]^{1/2}\right)\right]$$
$$= (N(d)-1)(1-m). \tag{8}$$

We note that $\frac{1}{2}(N(d)+1) - [N(d)m + \frac{1}{4}(N(d)-1)^2]^{1/2} \in F^*$. It is clearly in F, and it is not 0, since if it were we would have the implications

$$\tfrac{1}{2}(N(d)+1) = [N(d)m + \tfrac{1}{4}(N(d)-1)^2]^{1/2}$$
$$\Rightarrow \tfrac{1}{4}(N(d)+1)^2 = N(d)(m) + \tfrac{1}{4}(N(d)-1)^2$$
$$\Rightarrow \tfrac{1}{2}N(d) = N(d)m - \tfrac{1}{2}N(d)$$

which forces m to be 1, which is a contradiction. The above expression is thus not zero, so we can divide by it in Eq. (8) to get

$$\mathrm{Tr}\left[\frac{b}{x} - \frac{mx}{b}\right] = \frac{(N(d)-1)(1-m)}{(1/2)(N(d)+1) - [N(d)m + (1/4)(N(d)-1)^2]^{1/2}}.$$

Again rationalizing and simplifying yields

$$\mathrm{Tr}\left[\frac{b}{x} - \frac{mx}{b}\right] = \left(\frac{N(d)-1}{N(d)}\right)\left(\frac{N(d)+1}{2} + \left[N(d)m + \frac{(N(d)-1)^2}{4}\right]^{1/2}\right). \tag{9}$$

Now, let $z \in K$ be any element such that $N(z) = \frac{N(b)}{m}$. Note that such a z exists and is not 0 since $N(b) \neq 0$ and $m \neq 0$. Define $\alpha = -zx$ and $\beta = \frac{zb}{N(z)}$. Again, we note that since $z \neq 0$, neither α nor β is 0. We can then compute the quantities

$$N(\alpha) = -N(z) = -(\tfrac{1}{2}(N(d)-1) + [N(d)m + \tfrac{1}{4}(N(d)-1)^2]^{1/2})$$

and

$$N(\beta) = \frac{N(z)N(b)}{N(z)^3} = \frac{m^2}{N(b)}.$$

Using Eq. (7), we know this latter quantity is

$$N(\beta) = \frac{1}{N(d)} \left(\tfrac{-1}{2} (N(d) - 1) + [N(d)m + \tfrac{1}{4} (N(d) - 1)^2]^{1/2} \right).$$

Two final computations yield

$$\frac{N(\alpha)\beta}{\alpha} = \frac{-N(z)bz}{-zN(z)x} = \frac{b}{x}$$

and

$$\frac{N(\beta)\alpha}{\beta} = \frac{-N(b)x}{N(z)b} = \frac{-mx}{b}.$$

With some simplification, one can use these expressions to compute that

$$N(\alpha) + N(\beta) = \left(\frac{1 - N(d)}{N(d)}\right)\left(\frac{N(d) + 1}{2} + \left[N(d)m + \frac{(N(d) - 1)^2}{4}\right]^{1/2}\right).$$

We can now simplify Eq. (9) as

$$\mathrm{Tr}\left[\frac{N(\alpha)\beta}{\alpha} + \frac{N(\beta)\alpha}{\beta}\right] = -N(\alpha) - N(\beta),$$

which by Lemma 4.1 gives the implication

$$N(\alpha + \beta) = 0 \Rightarrow \alpha + \beta = 0.$$

Substituting the definitions of α and β gives

$$-zx + \frac{z}{N(z)} b = 0 \Rightarrow x = \frac{b}{N(z)},$$

in which we can take norms on both sides to obtain

$$1 = \frac{N(b)}{N(z)^3} = \frac{m^3 N(b)}{N(b)^3} = \frac{m^3}{N(b)^2},$$

which forces $N(b)^2 = m^3$. Finally, we calculate

$$N(b)^2 = \tfrac{1}{4}(N(d)-1)^2 m^2 + (N(d)-1)m^2\,[N(d)m + \tfrac{1}{4}(N(d)-1)^2]^{1/2} + N(d)m^3 + \tfrac{1}{4}(N(d)-1)^2 m^2$$
$$= m^2\left(\frac{(N(d)-1)^2}{2} + (N(d)-1)\left[N(d)m + \frac{(N(d)-1)^2}{4}\right]^{1/2} + N(d)m\right).$$

Setting this equal to m^3 and dividing both sides by m^2 yields the equation

$$\tfrac{1}{2}(N(d)-1)^2 + (N(d)-1)[N(d)m + \tfrac{1}{4}(N(d)-1)^2]^{1/2} + (N(d)-1)m = 0.$$

Since $N(d) \neq 1$, $N(d)-1 \neq 0$, so we can divide both sides by $N(d)-1$ to get

$$\tfrac{1}{2}(N(d)-1) + [N(d)m + \tfrac{1}{4}(N(d)-1)^2]^{1/2} + m = 0.$$

Moving the square root to the right hand side and squaring both sides, we find that

$$m^2 + m(N(d)-1) + \tfrac{1}{4}(N(d)-1)^2 = N(d)m + \tfrac{1}{4}(N(d)-1)^2,$$

which after simplification yields

$$m^2 - m = 0.$$

This implies that m is 0 or 1. However, these were the two specifically excluded values for m; this is a final contradiction. ■

For the construction of disjoint pairs of covers, we now need only pick appropriate values for m, b, and d in this theorem. There are many ways in which this can be done, and an exploration of the distinct possibilities would be of interest. However, we would like to focus on a specific example here.

Let $q = 1 \pmod 4$ be a prime power. Then, -1 is a square in $\mathcal{GF}(q)$; let i be one fixed square root of -1 in this field. Let $m = 2$, $d = -1$, and $b = i - 1$. We then check that $N(d) = -1$; $N(d)m + \frac{1}{4}(N(d)-1)^2 = -1$, which is a square; and $N(b) = 2i - 2 = \frac{1}{2}(N(d)-1)m + m[N(d)m + \frac{1}{4}(N(d)-1)^2]^{1/2}$, where we take i to be the square root of -1 in this expression. So these values satisfy the hypotheses of Theorem 4.2, and we have a pair of disjoint covers in $\mathcal{CG}(3, q)$. By applying the map α^{-1} to each of these covers, we obtain a pair of pairwise disjoint hyperreguli in the spread $\mathcal{S}$. Finally, we can replace these hyperreguli with the appropriate images of the hyperreguli in Eq. (3) in various ways to obtain several (possibly equivalent) subregular spreads. We note that there may in principle be other hyperreguli which cover the same set of points as the given replacements, but Pomareda [13] has shown that this is not possible in $\mathscr{PG}(5,q)$.

A natural question is whether one can extend this set of disjoint covers to a larger set. We wish to give two answers to this question; in both cases we provide a third cover disjoint from both of our previous two. The first method works only when q is an even power of 3, while the second works for all prime powers $q > 5$ which are not divisible by 5. Note that these two cases do overlap; it is not clear whether or not the first method is simply a special case of the second.

We first assume that q is an even power of 3. If so, then $q \equiv 1 \pmod 4$ and we already have a pair of disjoint covers in $\mathscr{CG}(3,q)$, namely $N_1 = \{x \in \mathscr{GF}(q) \mid N(x) = 1\}$ and $N_1 f$, where $xf = \frac{x+(i+1)}{(i-1)x-1}$.

THEOREM 4.3. *Let q be a power of 3 such that $q \equiv 1 \pmod 4$, i.e., $q = 3^{2h}$, for some $h \geq 1$. Let i be a square root of -1 in $F = \mathscr{GF}(q)$, and let $K = \mathscr{GF}(q^3)$. Let f be defined via $xf = \frac{x+(i+1)}{(i-1)x-1}$ and let g be defined via $xg = \frac{x+(1-i)}{(-1-i)x-1}$. If $N_1 = \{x \in K \mid N(x) = 1\}$, then $N_1 \cap N_1 f = N_1 \cap N_1 g = N_1 f \cap N_1 g = \emptyset$.*

Proof. f was shown to satisfy the conditions of Theorem 4.2 above, so $N_1 \cap N_1 f = \emptyset$. To look at g in terms of this theorem, let $m = 2$, $d = -1$, and $b = 1 - i$. As above, $N(d) = -1$, $N(d)m + \frac{1}{4}(N(d) - 1)^2 = -1$ is a square, and $N(b) = -2i - 2 = \frac{1}{2}(N(d) - 1)m + m[N(d)m + \frac{1}{4}(N(d) - 1)^2]^{1/2}$, where the square root is now taken to be $-i$. Noting that $xg = \frac{x+b}{(md/b)+d}$ with these choices, we have that N_1 and $N_1 g$ are also disjoint covers from Theorem 4.2.

Finally, we need to show that $N_1 f$ and $N_1 g$ are pairwise disjoint. The key to this computation is to note that f is an involution, so that $N_1 f$ and $N_1 g$ are pairwise disjoint if and only if N_1 and $N_1(gf)$ are pairwise disjoint. We compute an expression for $x(gf)$ as

$$\begin{aligned} x(gf) &= \left(\frac{x+(1-i)}{(-1-i)x-1}\right)f \\ &= \frac{(x+(1-i))/((-1-i)x-1)+(i+1)}{(i-1)((x+(1-i))/((-1-i)x-1))-1} \\ &= \frac{x+(1-i)+(i+1)((-1-i)x-1)}{(i-1)(x+(1-i))-((-1-i)x-1)} \\ &= \frac{(1-2i)x-2i}{2ix+(1+2i)} \\ &= \frac{x+(-1-i)}{(1+i)x-i}. \end{aligned}$$

Let us again apply Theorem 4.2. Let $m = 2$, $d = -i$, and $b = -1 - i$. Then, $N(d) = -i^3 = i$; $N(d)m + \frac{1}{4}(N(d) - 1)^2 = 0$, which is a square; and $N(b) = i - 1 = \frac{1}{2}(N(d) - 1)m + m[N(d)m + \frac{1}{4}(N(d) - 1)^2]^{1/2}$. Since $x(gf) = \frac{x+b}{(md/b)x+d}$ for these choices, we find that N_1 and $N_1(gf)$ are two disjoint covers in $\mathscr{CG}(3, q)$, which implies $N_1 f$ and $N_1 g$ are also disjoint. ■

By this theorem, we are able to construct new subregular spreads by mapping the three covers N_1, $N_1 f$, and $N_1 g$ back to $\mathscr{S}$ and reversing the resulting hyperreguli. However, this only works in a very special case, when q is a even power of 3. We would now like to give another result of this type, which works for most other cases.

THEOREM 4.4. *Let $q \equiv 1 \pmod 4$ and $q \not\equiv 0 \pmod 5$. Let $F = \mathscr{GF}(q)$ and $K = \mathscr{GF}(q^3)$. Let $N_1 = \{x \in K \mid N(x) = 1\}$ and define the mapping f as above via $xf = \frac{x-(i+1)}{(i-1)x-1}$, where $i^2 = -1$. Define $h \in \mathrm{Aut}(\mathscr{CG}(3, q))$ via $xh = \frac{x+\delta}{-ix+\delta}$, where $\delta \in K$ and $N(\delta) = 1 + i$. Then, $N_1 \cap N_1 f = N_1 \cap N_1 h = N_1 f \cap N_1 h = \emptyset$.*

Proof. We first note that $N_1 \cap N_1 f = \emptyset$ by Theorem 4.2. To show $N_1 \cap N_1 h = \emptyset$, we show that h satisfies the conditions of Theorem 4.2. For this, we write h as

$$xh = \frac{x + \delta}{(-i\delta/\delta)x + \delta}.$$

In terms of the statement of Theorem 4.2, we have $b = \delta$, $m = -i$, and $d = \delta$. Checking the conditions of this theorem, we have

$$\begin{aligned} N(d)m + \tfrac{1}{4}(N(d) - 1)^2 &= (1 + i)(-i) + \tfrac{1}{4}((i + 1) - 1)^2 \\ &= \tfrac{3}{4} - i = (-1 + \tfrac{1}{2} i)^2 \end{aligned}$$

and

$$\begin{aligned} \frac{m(N(d) - 1)}{2} + m\left[N(d)m + \frac{(N(d) - 1)^2}{4}\right]^{1/2} &= \frac{-i^2}{2} - i\left(-1 + \frac{i}{2}\right) \\ &= 1 + i = N(b). \end{aligned}$$

Thus h satisfies the conditions of Theorem 4.2, and therefore we can conclude that $N_1 \cap N_1 h = \emptyset$.

To show $N_1 f \cap N_1 h = \emptyset$, since f is an involution, we can show the equivalent statement that $N_1 \cap N_1(hf) = \emptyset$. We can now apply the same ideas as used above. We first compute $x(hf)$ as

$$x(hf) = \left(\frac{x+\delta}{-ix+\delta}\right)f = \frac{(x+\delta)/(-ix+\delta)+(i+1)}{(x+\delta)/(-ix+\delta)-1}$$
$$= \frac{(x+\delta)+(1+i)(-ix+\delta)}{(i-1)(x+\delta)-(-ix+\delta)},$$

in which we can collect like terms to yield

$$x(hf) = \frac{(2-i)x+(2+i)\delta}{(2i-1)x+(i-2)\delta}.$$

Since $q \not\equiv 0 \pmod 5$, 2 is not a square root of -1, and thus $2-i$ is not 0. So we can normalize the above to get

$$x(hf) = \; = \frac{x+((3+4i)/5)\delta}{((3i-4)/5)x-\delta}.$$

To cast this into the form from Theorem 4.2, we let $d' = -\delta$,and $b' = \frac{3+4i}{5}\delta$. To find m', we note that $\frac{m'd'}{b'} = \frac{3i-4}{5}$. Solving this for m' yields $m' = \frac{7i+24}{25}$. With these definitions, we can write

$$x(hf) = \frac{x+b'}{(m'd'/b')x+d'},$$

and show that our conditions hold.

First, we have

$$N(d')m' + \tfrac{1}{4}(N(d')-1)^2 = -(1+i)\frac{7i+24}{25} + \tfrac{1}{4}(i-2)^2 = \left(\frac{-4+3i}{10}\right)^2.$$

To show $N(b') = \frac{1}{2}(N(d')-1)m' + m'[N(d')m' + \frac{1}{4}(N(d')-1)^2]^{1/2}$, we compute both sides of the expression. For the left-hand side, we find

$$N(b') = \left(\frac{3+4i}{5}\right)^3 N(\delta) = \frac{-161-73i}{125},$$

while for the right-hand side we obtain

$$\tfrac{1}{2}(N(d')-1)m' + m'[N(d')m' + \tfrac{1}{4}(N(d')-1)^2]^{1/2}$$
$$= \frac{7i+24}{25}\left(\frac{-1-i-1}{2} + \frac{-4+3i}{10}\right)$$

$$= \frac{7i+24}{25}\left(\frac{-7-i}{5}\right)$$

$$= \frac{-161-73i}{125}.$$

Thus, hf satisfies the conditions of Theorem 4.2, which implies $N_1 f \cap N_1 h = \emptyset$ as claimed. ■

As before, we can again use this set of pairwise disjoint covers to construct new subregular spreads of $\mathscr{PG}(5, q)$.

5. THE ANDRÉ SPREADS

In the previous section, we have given three constructions of subregular spreads of $\mathscr{PG}(5, q)$ for various values of q, with $q \equiv 1 \pmod 4$ (though our principal result (Theorem 4.2) works for all odd prime powers greater than 3). We have not yet shown however that these spreads are not already known. In particular, we must show that the spreads we have obtained are not André spreads. In order to do this, we must study the André spreads in more depth.

As in the two-dimensional case, the key to defining an André spread is the concept of linearity. Let $\mathscr{S}$ be a regular spread of $\mathscr{PG}(2n+1, q)$ with q a prime power and n a positive integer as given in Eq. (2). In the standard construction of an André spread, one partitions the spaces of $\mathscr{S}$ into the $q-1$ hyperreguli $\{J(k) | k \in \mathscr{GF}(q^3)$ and $N(k) = f\}$, for each nonzero $f \in \mathscr{GF}(q)$, and the spaces $J(0)$ and $J(\infty)$, and then (possibly) reverses each of the hyperreguli. We would like to investigate this phenomenon with respect to the circle geometries involved.

If we map these hyperreguli to $\mathscr{CG}(n+1, q)$ using the isomorphism α from Theorem 3.1, we obtain a set of $q-1$ pairwise disjoint covers N_f for each nonzero $f \in \mathscr{GF}(q)$, where $N_f = \{k \in \mathscr{GF}(q^3) | N(k) = f\}$, which cover the points of $\mathscr{CG}(3, q) \setminus \{0, \infty\}$. It turns out (see Bruck [5]) that the points 0 and ∞ are the *carriers* of each of these $q-1$ covers, in a well-defined sense (how one defines the carriers of a cover requires one to delve into the group theory of the circle geometries in much more detail than is necessary for our purposes, so we refer the interested reader to Bruck). In general, we call a set of covers which share a pair of carriers a *linear set* of covers, and we call the corresponding set of hyperreguli a *linear set* as well. We further exploit this correspondence by referring to the spaces in the spread $\mathscr{S}$ which map to the carriers of this linear set *carriers* too.

The critical point here is that one obtains an André spread by reversing a linear set of hyperreguli. So it will be important for us to show that a given pair of hyperreguli form a nonlinear set. This is impossible if $n = 1$, for every pair of disjoint reguli in a regular spread of $\mathscr{PG}(3, q)$ forms a linear set. However, Bruck [5] shows that a cover of $\mathscr{CG}(n+1, q)$ has a unique pair of carriers, at least if $n+1$ is prime. So to show that a set of covers is nonlinear, it suffices to show that the covers therein do not share carriers.

However, showing that we reversed a nonlinear set of hyperreguli is not sufficient to show that the spread we obtain is not André. It is conceivable that the spread we obtain could also be obtained by starting with a different regular spread and reversing a linear set of hyperreguli. We wish to show that this usually does not happen, at least in $\mathscr{PG}(5, q)$. We proceed with a series of lemmas.

LEMMA 5.1. *Let q be a prime power, and let $\mathscr{A}$ be an André spread in $\mathscr{PG}(5, q)$. Then there exist three regular spreads $\mathscr{T}$, $\mathscr{T}'$, and $\mathscr{T}''$ such that $\mathscr{A}$ is a subset of the union of these three spreads. Further, $\mathscr{A}$ can be obtained from any one of these spreads by reversing a linear set of pairwise disjoint hyperreguli.*

Proof. Since $\mathscr{A}$ is an André spread, there exists a regular spread $\mathscr{T}$ such that $\mathscr{A}$ is obtained from $\mathscr{T}$ by reversing a linear set $\mathscr{Q}$ of hyperreguli in $\mathscr{T}$. Without loss of generality, we can coordinatize the spread $\mathscr{T}$ as in Eq. (2) so that the carriers of the hyperreguli in $\mathscr{Q}$ are $J(\infty)$ and $J(0)$. (See Bose and Bruck [6] for details on such coordinatization.) Thus, the hyperreguli that are being reversed all have the form $N[f] = \{J(k) \mid k \in \mathscr{GF}(q^3) \text{ and } N(k) = f\}$ for some nonzero $f \in \mathscr{GF}(q)$.

By considering $N[f]$ as the image of $N[1]$ under the collineation induced by the matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & f\end{smallmatrix}\right)$, Pomareda [13] has shown that there are exactly two hyperreguli which can replace $N[f]$, namely $N'[f] = \{J^q(k) \mid N(k) = f\}$ and $N''[f] = \{J^{q^2}(k) \mid N(k) = f\}$. These planes come from Eq. (3), using the field automorphisms $x \to x^q$ and $x \to x^{q^2}$, respectively.

Define $\mathscr{T}' = \bigcup_{f \in \mathscr{GF}(q)^*} N'[f] \cup \{J(0), J(\infty)\}$ and $\mathscr{T}'' = \bigcup_{f \in \mathscr{GF}(q)^*} N''[f] \cup \{J(0), J(\infty)\}$, where $\mathscr{GF}(q)^*$ denotes the set of nonzero field elements.

Consider the following mapping $\phi: V \to V$, where $V = \{(x, y) \mid x, y \in \mathscr{GF}(q^3)\}$, defined via $(x, y)\phi = (x^q, y)$. It is straightforward to check that ϕ is a linear mapping over V, and thus induces a collineation of $\mathscr{PG}(5, q)$, which we will also call ϕ. Clearly $J(\infty)\phi = J(\infty)$ and one can compute that $J(k)\phi = J^{q^2}(k)$. Thus, $\mathscr{T}\phi = \mathscr{T}''$ which implies $\mathscr{T}''$ is a regular spread. Applying ϕ to $\mathscr{T}''$ shows that $\mathscr{T}'$ is also a regular spread. Since $J(0)$ and $J(\infty)$ are in all three of $\mathscr{T}$, $\mathscr{T}'$, and $\mathscr{T}''$, and every other plane of $\mathscr{A}$ lies in either $N[f]$, $N'[f]$, or $N''[f]$ for some nonzero $f \in \mathscr{GF}(q)$, it follows immediately that $\mathscr{A}$ is contained in the union of these three regular spreads. The final assertion is now straightforward. ∎

The following lemma can be proved by extending the methods of Bruck [2] in proving a similar result about regular spreads of $\mathscr{PG}(3, q)$. The details are tedious and thus we omit them.

LEMMA 5.2. *Let $\mathscr{T}$ and $\mathscr{T}'$ be two distinct regular spreads of $\mathscr{PG}(5, q)$, with q a prime power. Then $\mathscr{T}$ and $\mathscr{T}'$ meet in at most $q + 1$ planes.*

We can now prove:

THEOREM 5.3. *Let q be a prime power greater than 2, and let $\mathscr{S}$ be a regular spread of $\mathscr{PG}(5, q)$. If $\mathscr{P}$ is any spread obtained from $\mathscr{S}$ by reversing a nonlinear, nonempty set of at most $q - 2$ pairwise disjoint hyperreguli, then $\mathscr{P}$ is not an André spread.*

Proof. By way of contradiction, suppose $\mathscr{P}$ is an André spread. Then by Lemma 5.1, there exist three regular spreads $\mathscr{T}$, $\mathscr{T}'$, and $\mathscr{T}''$ whose union contains all planes of $\mathscr{P}$ and such that $\mathscr{P}$ can be obtained from any of these three regular spreads by reversing a linear set of pairwise disjoint hyperreguli. Since $\mathscr{P}$ was obtained from $\mathscr{S}$ by reversing a nonlinear set, none of $\mathscr{T}$, $\mathscr{T}'$, or $\mathscr{T}''$ can be $\mathscr{S}$.

Since we reversed at most $q - 2$ hyperreguli in $\mathscr{S}$ to obtain $\mathscr{P}$, $\mathscr{P}$ must have at least $q^2 + q + 3$ planes in common with $\mathscr{S}$. These planes must lie in the union of the three regular spreads $\mathscr{T}$, $\mathscr{T}'$, and $\mathscr{T}''$. So, by the pigeonhole principle, at least one of these three spreads must have at least $\frac{1}{3}(q^2 + q + 3)$ planes in common with $\mathscr{S}$. However, this contradicts Lemma 5.2 since distinct regular spreads of $\mathscr{PG}(5, q)$ share at most $q + 1$ planes, and $\frac{1}{3}(q^2 + q + 3) > q + 1$ for all $q > 2$. Thus $\mathscr{P}$ cannot be an André spread. ■

We note briefly that this theorem is trivially true if $q = 2$, as $q - 2 = 0$ and the hypotheses cannot be satisfied. Of course, there is only one spread of $\mathscr{PG}(5, 2)$ up to projective equivalence.

At this point, it is simple to verify that the three types of spreads generated in the previous section are not André spreads. For example, we have noted that the carriers of N_1 are 0 and ∞. If f is the mapping $xf = \frac{x + (i+1)}{(i-1)x - 1}$, then the carriers of $N_1 f$ are $0f = -(i + 1)$ and $\infty f = \frac{1}{i-1}$. Since these covers do not share carriers, they form a nonlinear set. So if we pull back to the regular spread $\mathscr{S}$ of $\mathscr{PG}(5, q)$, we obtain a pair of disjoint, nonlinear hyperreguli. Assuming we actually do a reversal, we obtain a subregular spread which does not give rise to an André plane by the above theorem.

6. CONCLUSION

There are many avenues yet to be explored in this area. The most important next step is to give examples of subregular spreads of $\mathscr{PG}(2n + 1, q)$, where $n > 2$, which are not André. In addition, we have also not given any examples of non-André subregular spreads of $\mathscr{PG}(5, q)$ with q even; technically, we have also not done so if $q \equiv 3 \pmod 4$, but such examples are easy to generate using Theorem 4.2.

Many of the classical questions regarding subregular spreads also remain. For example, how large can a nonlinear set of pairwise disjoint hyperreguli be? Is there an analog of the flock conjecture for this higher-dimensional setting? One can also ask how many projectively distinct subregular spreads there are. Even a count of the number of projectively distinct André spreads in higher dimensions seems unknown.

We have also left unanswered many of the normal questions regarding the collineation groups of the planes we have constructed. For each set of pairwise disjoint hyperreguli we have obtained, there are several different, but possibly equivalent, ways to replace each one. By brute force computation, it is often possible to compute the inherited automorphism groups of these planes (i.e., the group of collineations which leave the ambient Desarguesian plane invariant as well); the details are quite messy in general, and we refer the interested reader to Dover [8] for an example of such a computaton.

Of course, one might ask if this inherited automorphism group is the full collineation group of the plane. The analogous statement for planes obtained from subregular spreads of $\mathscr{PG}(3, q)$ was proven by Walker [14], where he showed that for any spread obtained by reversing a set of k pairwise disjoint reguli, the collineation group of the resulting plane is inherited unless $k = q - 1$ or $k = \frac{q-1}{2}$. It would be interesting to see if an analogous result holds in the higher-dimensional cases; Ostrom [11] has obtained some results in this direction, but does not seem to have solved the whole problem.

A final issue of interest is the characterization problem; in particular, one may ask if it is possible to determine that a plane arises from reversing a set of pairwise disjoint hyperreguli by looking at some properties of the collineation group of the plane. Indeed, it is not difficult to see that each hyperregulus replacement is given by images of a suitable subspace under the kernel homology group given in Proposition 3.2; Ostrom [11] has given a general group-theoretic characterization of planes obtained by this kernel homology action. Much recent work has been devoted to this and related problems; for example, we refer the reader to Jha and Johnson [9].

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