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Asymptotic Behavior of Delay Difference Systems

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Abstract—This paper is concerned with a delay difference system

$$\begin{aligned} 2x_n - x_{n-1} &= f\left(y_{n-k}\right), \\ 2y_n - y_{n-1} &= f\left(x_{n-k}\right), \end{aligned} \qquad n \in N, \tag{$*$}$$

where k is a positive integer, and f is a signal transmission function of McCulloch-Pitts type. The difference system $(*)$ can be regarded as the discrete analog of the artifical neural network of two neurons with McCulloch-Pitts nonlinearity. Some interesting results are obtained for the asymptotic behavior of the system $(*)$.

Keywords—Asymptotic behavior, Delay difference system.

1. INTRODUCTION

Recently, there has been increasing interest in the study of the asymptotic behavior of solutions for delay difference equations. See, for example, [1–5]. But there are only a limited number of works concerning the asymptotic behavior of delay difference systems; for some results, we refer to [6,7]. Let Z denote the set of all integers. For any $a, b \in Z$, define $N(a) = \{a, a+1, \dots\}$, $N(a,b) = \{a, a+1, \dots, b\}$ when $a \le b$. $N = N(0)$. In this paper, we consider the delay difference system

$$\begin{aligned} 2x_n - x_{n-1} &= f\left(y_{n-k}\right), \\ 2y_n - y_{n-1} &= f\left(x_{n-k}\right), \end{aligned} \qquad n \in N, \tag{1.1}$$

where $k \in N(1)$, and

$$f(x) = \begin{cases} -1, & \text{if } x > \sigma, \\ 1, & \text{if } x \le \sigma, \end{cases} \tag{1.2}$$

for some constant σ. System (1.1) can be regarded as the discrete analog of the artifical neural network of two neurons

$$\begin{aligned} \dot{x} &= -x + f(y(t-\tau)), \\ \dot{y} &= -y + f(x(t-\tau)), \end{aligned} \tag{1.3}$$

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where $\dot{x}$ and $\dot{y}$ are replaced by the backward difference $x_n - x_{n-1}$ and $y_n - y_{n-1}$, respectively. Many interesting applications have been found in system (1.3), for example, image processing of moving objects which can be implemented by a delayed cloning-template [8,9].

In this paper, we parallel discrete time systems results from continuous time systems [8]. However, discrete time systems demonstrate some surprising differences.

By a *solution* of (1.1), we mean a sequence $\{(x_n, y_n)\}$ of points in R^2 that is defined for all $n \in N(-k)$ and satisfies (1.1) for $n \in N$. Let $X : N(-k,-1) \to R^2$. Clearly, for any $\Phi = (\phi, \psi) \in X$, system (1.1) has a unique solution (x_n^Φ, y_n^Φ) satisfying the initial conditions

$$x_i^\Phi = \phi(i), \quad y_i^\Phi = \psi(i), \qquad \text{for } i \in N(-k,-1). \tag{1.4}$$

Our goal is to determine the limiting behavior of (x_n^Φ, y_n^Φ) as $n \to \infty$ for any $\Phi \in X$. In particular, we concentrate on the case where $\phi - \sigma$ and $\psi - \sigma$ have no sign change on $N(-k,-1)$. Namely, $\Phi \in X_\sigma^{+,+} \cup X_\sigma^{+,-} \cup X_\sigma^{-,+} \cup X_\sigma^{-,-} = X_\sigma$ defined by

$$X_\sigma^{\pm,\pm} = \left\{\Phi \in X;\ \Phi = (\phi,\psi),\ \phi \in R_\sigma^\pm, \text{ and } \psi \in R_\sigma^\pm\right\},$$

with

$$R_\sigma^+ = \{\phi;\ \phi : N(-k,-1) \to R \text{ and } \phi(i) - \sigma > 0 \text{ for } i \in N(-k,-1)\},$$

and

$$R_\sigma^- = \{\phi;\ \phi : N(-k,-1) \to R \text{ and } \phi(i) - \sigma \le 0 \text{ for } i \in N(-k,-1)\}.$$

A *positive semicycle* of a sequence $\{x_n\}_{n=-k}^\infty$ relative to σ consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all greater than σ, with $l \ge -k$ and $m \le \infty$ and such that

$$\text{either } l = -k \text{ or } l > -k \text{ and } x_{l-1} \le \sigma,$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } x_{m+1} \le \sigma.$$

A *negative semicycle* of a sequence $\{x_n\}_{n=-k}^\infty$ relative to σ consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all less than or equal to σ, with $l \ge -k$ and $m \le \infty$ and such that

$$\text{either } l = -k \text{ or } l > -k \text{ and } x_{l-1} > \sigma,$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } x_{m+1} > \sigma.$$

For the general background on difference equations, refer to [10–12].

REMARK. The definition of semicycle in this paper is slightly different from that in [12].

The main results of this paper are as follows.

THEOREM 1.1. *Assume that $|\sigma| < 1$, $\Phi = (\phi, \psi) \in X_\sigma^{+,+} \cup X_\sigma^{-,-}$, and $\phi(-1) = \psi(-1)$. Then the solution $\{(x_n^\Phi, y_n^\Phi)\}_{n=-k}^\infty$ of the difference system (1.1) satisfies $x_n = y_n$ for $n \in N$, and the following statements are true.*

(a) *Every positive semicycle of $\{x_n\}$ relative to σ, except perhaps for the first one, has at least k terms, and terms number is less than $\log_2(2^{k+1} - 1 + \sigma)/(1 + \sigma)$.*
(b) *Every negative semicycle of $\{x_n\}$ relative to σ, except perhaps for the first one, has at least k terms, and terms number is less than $\log_2(2^{k+1} - 1 - \sigma)/(1 - \sigma)$.*

COROLLARY 1.1. *Assume that $\Phi = (\phi, \psi) \in X_\sigma^{+,+} \cup X_\sigma^{-,-}$ and $\phi(-1) = \psi(-1)$. Then we have the following.*

(a) *For $0 \le \sigma < 1$, every positive semicycle of $\{x_n\}$ relative to σ, except perhaps for the first one, has just k terms.*
(b) *For $-1 < \sigma \le 0$, every negative semicycle of $\{x_n\}$ relative to σ, except perhaps for the first one, has just k terms.*
(c) *For $|\sigma| < 1/(2^{k+1} - 1)$, there exists $\Phi_0 = (\phi_0, \psi_0) \in X_\sigma^{+,+}$ with $\phi_0(-1) = \psi_0(-1)$ such that the solution $(x_n^{\Phi_0}, y_n^{\Phi_0})$ of (1.1) with initial value Φ_0 is periodic with the minimal period $2k$. Moreover, there is an integer $m \in N(0, 2k-1)$ such that*

$$\lim_{n\to\infty} \left(x_n^{\Phi} - x_{n-m}^{\Phi_0}\right) = 0.$$

THEOREM 1.2. *Let $|\sigma| < 1$, and $\Phi = (\phi, \psi) \in X_\sigma$. Then $(x_n^\Phi, y_n^\Phi) \to (-1, 1)$ as $n \to \infty$ if $\Phi \in X_\sigma^{-,+}$; and $(x_n^\Phi, y_n^\Phi) \to (1, -1)$ as $n \to \infty$ if $\Phi \in X_\sigma^{+,-}$.*

THEOREM 1.3. *Let $|\sigma| > 1$, for any initial value $\Phi = (\phi, \psi) \in X$. Then $(x_n^\Phi, y_n^\Phi) \to (1, 1)$ as $n \to \infty$ for $\sigma > 1$; $(x_n^\Phi, y_n^\Phi) \to (-1, -1)$ for $\sigma < -1$.*

THEOREM 1.4. *Let $\sigma = 1$. Then we have the following.*

(a) *$(x_n^\Phi, y_n^\Phi) \to (1, 1)$ if either*
 (i) *$\phi(-1) \le 1$, $\psi(-1) \le 1$, or*
 (ii) *$\Phi = (\phi, \psi) \in X_\sigma^{+,+}$ and $2^{1-k} \le (1 + \psi(-1))/(1 + \phi(-1)) \le 2^{k-1}$.*
(b) *$(x_n^\Phi, y_n^\Phi) \to (-1, 1)$ if either $\Phi = (\phi, \psi) \in X_\sigma^{-,+}$ or $\Phi = (\phi, \psi) \in X_\sigma^{+,+}$ and $(1+\psi(-1))/(1+\phi(-1)) \ge 2^k$.*
(c) *$(x_n^\Phi, y_n^\Phi) \to (1, -1)$ if either $\Phi = (\phi, \psi) \in X_\sigma^{+,-}$ or $\Phi = (\phi, \psi) \in X_\sigma^{+,+}$ and $(1+\phi(-1))/(1+\psi(-1)) \ge 2^k$.*

THEOREM 1.5. *Let $\sigma = -1$. Then we have the following.*

(a) *$(x_n^\Phi, y_n^\Phi) \to (-1, -1)$ if either*
 (i) *$\phi(-1) > -1$, $\psi(-1) > -1$, or*
 (ii) *$\Phi = (\phi, \psi) \in X_\sigma^{-,-}$ and $2^{1-k} \le (\psi(-1) - 1)/(\phi(-1) - 1) \le 2^{k-1}$.*
(b) *$(x_n^\Phi, y_n^\Phi) \to (-1, 1)$ if $\Phi = (\phi, \psi) \in X_\sigma^{-,+}$ or $\Phi = (\phi, \psi) \in X_\sigma^{-,-}$ and $(\psi(-1) - 1)/(\phi(-1) - 1) \le 2^{-k}$.*
(c) *$(x_n^\Phi, y_n^\Phi) \to (1, -1)$ if $\Phi = (\phi, \psi) \in X_\sigma^{+,-}$ or $\Phi = (\phi, \psi) \in X_\sigma^{-,-}$ and $(\phi(-1) - 1)/(\psi(-1) - 1) \le 2^{-k}$.*

2. PROOFS OF MAIN RESULTS

Assuming $n_0 \in N$, we first note that the difference equation

$$2x_n - x_{n-1} = -1, \qquad n \in N(n_0), \tag{2.1}$$

with initial condition $x_{n_0-1} = a$ is

$$x_n = (1 + a)\left(\frac{1}{2}\right)^{n+1-n_0} - 1, \qquad n \in N(n_0). \tag{2.2}$$

The solution of the difference equation

$$2x_n - x_{n-1} = 1, \qquad n \in N(n_0), \tag{2.3}$$

with initial condition $x_{n_0-1} = a$ is

$$x_n = (a-1)\left(\frac{1}{2}\right)^{n+1-n_0} + 1, \qquad n \in N(n_0). \tag{2.4}$$

For the sake of convenience, in the sequel, x_n denotes x_n^{Φ}, and y_n denotes y_n^{Φ}.

PROOF OF THEOREM 1.1. We only consider the case where $\Phi = (\phi, \psi) \in X_{\sigma}^{+,+}$. The case where $\Phi = (\phi, \psi) \in X_{\sigma}^{-,-}$ is similar, and the proof is omitted.

Using (1.1), we can easily obtain that $x_n = y_n$ for all $n \in N$. Therefore, it suffices to show that the solution $\{x_n\}$ of the equation

$$2x_n - x_{n-1} = f(x_{n-k}), \tag{2.5}$$

with initial conditions

$$x_i \in R_{\sigma}^{+}, \qquad \text{for } i \in N(-k,-1),$$

satisfy (a) and (b).

Assume that the first semicycle of $\{x_n\}$ relative to σ is $\{x_{-k}, \dots, x_{m_1-1}\}$, which is a positive semicycle. This implies that

$$x_{m_1-1} > \sigma, \qquad x_{m_1} \le \sigma. \tag{2.6}$$

By (2.5), we see that

$$2x_n - x_{n-1} = -1, \qquad n \in N(-k, m_1+k-1).$$

By (2.2), we have

$$x_n = (1 + x_{m_1-1})\left(\frac{1}{2}\right)^{n-m_1+1} - 1, \qquad n \in N(m_1, m_1+k-1). \tag{2.7}$$

Assume that the second semicycle of $\{x_n\}$ relative to σ is $\{x_{m_1}, \dots, x_{m_2-1}\}$. This is a negative semicycle, which implies that

$$x_{m_2-1} \le \sigma, \qquad x_{m_2} > \sigma. \tag{2.8}$$

By (2.7), it is easy to see that $x_{m_1+k-1} \le x_{m_1+k-2} \le \cdots \le x_{m_1} \le \sigma$. Thus, $m_2 \ge m_1 + k$, and

$$2x_n - x_{n-1} = 1, \qquad n \in N(m_1+k, m_2+k-1).$$

By (2.4), we get

$$\begin{aligned} x_n &= (x_{m_1+k-1} - 1)\left(\frac{1}{2}\right)^{n-m_1-k+1} + 1 \\ &= -\left(\frac{1}{2}\right)^{n-m_1-k} + (1 + x_{m_1-1})\left(\frac{1}{2}\right)^{n-m_1+1} + 1. \end{aligned}$$

In particular,

$$x_{m_2-1} = -\left(\frac{1}{2}\right)^{m_2-m_1-k-1} + (1 + x_{m_1-1})\left(\frac{1}{2}\right)^{m_2-m_1} + 1.$$

Noting (2.6) and (2.8), we see that

$$-\left(\frac{1}{2}\right)^{m_2-m_1-k-1} + (1+\sigma)\left(\frac{1}{2}\right)^{m_2-m_1} + 1 < x_{m_2-1} \le \sigma,$$

which implies

$$m_2 - m_1 < \log_2 \frac{2^{k+1} - 1 - \sigma}{1 - \sigma}.$$

Assume that the third semicycle of $\{x_n\}$ relative to σ is $\{x_{m_2}, \dots, x_{m_3-1}\}$ clearly this is a positive semicycle. By a similar way to the second semicycle, we can prove

$$k \le m_3 - m_2 < \log_2 \frac{2^{k+1} - 1 + \sigma}{1 + \sigma}.$$

The proof of Theorem 1.1 is complete by induction.

PROOF OF COROLLARY 1.1. If $0 \le \sigma < 1$, since

$$\log_2 \frac{2^{k+1} - 1 + \sigma}{1 + \sigma} < \log_2 2^{k+1} = k + 1,$$

we see that (a) holds.

Similarly, we know that (b) holds.

If $|\sigma| < 1/(2^{k+1} - 1)$, then

$$\log_2 \frac{2^{k+1} - 1 + \sigma}{1 + \sigma} < k + 1 \qquad \text{and} \qquad \log_2 \frac{2^{k+1} - 1 - \sigma}{1 - \sigma} < k + 1.$$

By Theorem 1.1, we know that every semicycle of $\{x_n\}$ relative to σ, except perhaps for the first one, has k terms. We consider the case where $\Phi = (\phi, \psi) \in X_\sigma^{+,+}$, the case where $\Phi = (\phi, \psi) \in X_\sigma^{-,-}$ is similar and the proof is omitted. Assume that m_1 is the least nonnegative integer such that (2.6) holds. Then the i^{th} semicycle of $\{x_n\}$ is $\{x_{m_1+(i-2)k}, x_{m_1+(i-2)k+1}, \dots, x_{m_1+(i-1)k-1}\}$ for $i \in N(2)$. From (2.5), we have, for $i \in N(1)$,

$$\begin{aligned} 2x_n - x_{n-1} &= -1, \qquad n \in N\left(m_1 + (2i-2)k, m_1 + (2i-1)k - 1\right), \\ 2x_n - x_{n-1} &= 1, \qquad n \in N\left(m_1 + (2i-1)k, m_1 + 2ik - 1\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} x_n &= (1 + x_{m_1-1}) \left(\frac{1}{2}\right)^{n-m_1+1} - 1, && n \in N\left(m_1, m_1 + k - 1\right); \\ x_n &= (x_{m_1+k-1} - 1) \left(\frac{1}{2}\right)^{n-m_1-k+1} + 1 \\ &= \left((1 + x_{m_1-1}) \left(\frac{1}{2}\right)^{k} - 2\right) \left(\frac{1}{2}\right)^{n-m_1-k+1} + 1 \\ &= -\left(\frac{1}{2}\right)^{n-m_1-k} + (1 + x_{m_1-1}) \left(\frac{1}{2}\right)^{n-m_1+1} + 1, && n \in N\left(m_1 + k, m_1 + 2k - 1\right). \end{aligned}$$

In general, for $i \in N(1)$, we have

$$x_n = \begin{cases} \left(\frac{1}{2}\right)^{n-m_1-k} \dfrac{2^{(2i-2)k} - 1}{2^k + 1} + (1 + x_{m_1-1}) \left(\frac{1}{2}\right)^{n-m_1+1} - 1, \\ \qquad \text{for } n \in N\left(m_1 + (2i-2)k, m_1 + (2i-1)k - 1\right); \\ -\left(\frac{1}{2}\right)^{n-m_1-k} \dfrac{2^{(2i-1)k} + 1}{2^k + 1} + (1 + x_{m_1-1}) \left(\frac{1}{2}\right)^{n-m_1+1} + 1, \\ \qquad \text{for } n \in N\left(m_1 + (2i-1)k, m_1 + 2ik - 1\right). \end{cases} \tag{2.9}$$

Let $\Phi_0 = (\phi_0, \psi_0) \in X_\sigma^{+,+}$ with $\phi_0(-1) = \psi_0(-1) = (2^k - 1)/(2^k + 1)$. Then from (2.9), we get, for $i \in N(1)$,

$$x_n^{\Phi_0} = \begin{cases} \dfrac{1}{2^k + 1} \left(\dfrac{1}{2}\right)^{n-(2i-1)k} - 1, & n \in N\left((2i-2)k, (2i-1)k - 1\right), \\ -\dfrac{1}{2^k + 1} \left(\dfrac{1}{2}\right)^{n-2ik} + 1, & n \in N\left((2i-1)k, 2ik - 1\right). \end{cases}$$

Clearly, $\{x_n^{\Phi_0}\}_{n=0}^{\infty}$ is periodic with minimal period $2k$. In view of (2.9), it is easy to see that

$$x_{m_1+j} - x_j^{\Phi_0} = \left(x_{m_1-1} - \frac{2^k-1}{2^k+1}\right)\left(\frac{1}{2}\right)^{j+1} \to 0, \qquad \text{as } j \to \infty. \tag{2.10}$$

Let $m = m_1 - 2k[m_1/2k]$, where $[\cdot]$ denotes the greatest function. Then $m \in N(0, 2k-1)$, and by (2.10),

$$\lim_{n\to\infty}\left(x_n - x_{n-m}^{\Phi_0}\right) = 0.$$

This completes the proof of Corollary 1.1.

PROOF OF THEOREM 1.2. We only prove the case where $\Phi = (\phi, \psi) \in X_\sigma^{-,+}$, the case where $\Phi \in X_\sigma^{+,-}$ is similar and the proof is omitted.

Since $\Phi \in X_\sigma^{-,+}$, from (1.1), we see that

$$\begin{aligned} 2x_n - x_{n-1} &= -1, \\ 2y_n - y_{n-1} &= 1, \end{aligned} \tag{2.11}$$

for $n \in N(0, k-1)$. Therefore,

$$\begin{aligned} x_n &= (1 + \phi(-1))\left(\frac{1}{2}\right)^{n+1} - 1, \\ y_n &= (\psi(-1) - 1)\left(\frac{1}{2}\right)^{n+1} + 1, \end{aligned} \tag{2.12}$$

for $n \in N(0, k-1)$. This implies $x_n < \sigma$, $y_n > \sigma$, for $n \in N(0, k-1)$, and (2.11) is satisfied for $n \in N(k, 2k-1)$. Thus, $x_n < \sigma$, $y_n > \sigma$, for $n \in N(k, 2k-1)$. Repeating this procedure, we can obtain that (x_n, y_n) satisfies (2.12) for all $n \in N$, from which we know that $(x_n, y_n) \to (-1, 1)$ as $n \to \infty$.

PROOF OF THEOREM 1.3. We only prove the case where $\sigma > 1$, the case where $\sigma < -1$ is similar and the proof is omitted.

In view of (1.1), we see that

$$\begin{aligned} 2x_n - x_{n-1} &\le 1, \\ 2y_n - y_{n-1} &\le 1, \end{aligned} \tag{2.13}$$

for $n \in N$. By induction, this implies

$$\begin{aligned} x_n &\le (\phi(-1) - 1)\left(\frac{1}{2}\right)^{n+1} + 1, \\ y_n &\le (\psi(-1) - 1)\left(\frac{1}{2}\right)^{n+1} + 1, \end{aligned} \qquad n \in N. \tag{2.14}$$

From (2.14), we see that there exists a positive integer m_1 such that $x_n < \sigma$, $y_n < \sigma$, for $n \in N(m_1)$. Thus,

$$\begin{aligned} 2x_n - x_{n-1} &= 1, \\ 2y_n - y_{n-1} &= 1, \end{aligned} \qquad n \in N(m_1 + k). \tag{2.15}$$

Therefore,

$$\begin{aligned} x_n &= (x_{m_1+k-1} - 1)\left(\frac{1}{2}\right)^{n-m_1-k+1} + 1, \\ y_n &= (y_{m_1+k-1} - 1)\left(\frac{1}{2}\right)^{n-m_1-k+1} + 1, \end{aligned} \qquad n \in N(m_1 + k), \tag{2.16}$$

which implies that $(x_n, y_n) \to (1, 1)$ as $n \to \infty$.

PROOF OF THEOREM 1.4. We first prove (a). In Case (i), the conclusion can be obtained by a similar argument as that in Case 1 for the proof of Theorem 1.3. Now we consider Case (ii) where $\Phi = (\phi, \psi) \in X_\sigma^{+,+}$ and $2^{1-k} \le (1+\psi(-1))/(1+\phi(-1)) \le 2^{k-1}$. By the symmetry of (1.1), there is no harm in assuming that $(1+\psi(-1))/(1+\phi(-1)) \ge 1$. By using (2.2), we get

$$\begin{aligned} x_n &= (1+\phi(-1))\left(\frac{1}{2}\right)^{n+1} - 1, \\ y_n &= (1+\psi(-1))\left(\frac{1}{2}\right)^{n+1} - 1, \end{aligned} \tag{2.17}$$

for $n \in N(0, k-1)$. Assume that m_1 is the least nonnegative integer such that

$$x_{m_1 - 1} > \sigma, \qquad x_{m_1} \le \sigma, \tag{2.18}$$

then (2.17) holds for $n \in N(0, m_1 + k - 1)$. By (2.17),

$$y_{m_1+k-1} = (1+\psi(-1))\left(\frac{1}{2}\right)^{m_1+k} - 1 \le (1+\phi(-1))\left(\frac{1}{2}\right)^{1-k+m_1+k} - 1 = x_{m_1} \le \sigma,$$

which implies

$$x_n \le \sigma, \quad y_n \le \sigma, \qquad \text{for } n \in N(m_1 + k - 1).$$

Therefore,

$$\begin{aligned} 2x_n - x_{n-1} &= 1, \\ 2y_n - y_{n-1} &= 1, \end{aligned} \qquad n \in N(m_1 + 2k - 1).$$

In view of (2.4), we see that $(x_n, y_n) \to (1,1)$ as $n \to \infty$.

Consider the conclusion (b). If $\Phi = (\phi, \psi) \in X_\sigma^{-,+}$, then the conclusion follows from a similar argument as that in Case 1 for the proof of Theorem 1.2. If $\Phi = (\phi, \psi) \in X_\sigma^{+,+}$ and $(1 + \psi(-1))/(1+\phi(-1)) \ge 2^k$, then (2.17) holds for $n \in N(0, k-1)$. Letting m_1 be as in Case (ii) of (a) such that (2.18) holds, then we can prove

$$\begin{aligned} y_{m_1+i} &= (1+\psi(-1))\left(\frac{1}{2}\right)^{m_1+i+1} - 1 \\ &\ge (1+\phi(-1))\left(\frac{1}{2}\right)^{m_1+i+1-k} - 1 \\ &\ge (1+\phi(-1))\left(\frac{1}{2}\right)^{m_1} - 1 \\ &= x_{m_1-1} > \sigma, \qquad i \in N(0, k-1). \end{aligned}$$

Thus,

$$x_{m_1+i} \le \sigma, \quad y_{m_1+i} > \sigma, \qquad \text{for } i \in N(0, k-1).$$

Similar to the previous case, we know the conclusion holds.

By the symmetry of (1.1), we see that conclusion of (c) holds. This completes the proof of Theorem 1.4.

Theorem 1.5 can be proved in a similar fashion to Theorem 1.4, we omitted the proof.

REFERENCES

1. J.S. Yu and S.S. Cheng, A stability criterion for neutral difference equation with delay, *Appl. Math. Lett.* **7** (6), 75–80, (1994).
2. Z. Zhou and Q.Q. Zhang, Uniform stability of nonlinear difference systems, *J. Math. Anal. Appl.* **225**, 486–500, (1998).

3. Z. Zhou, J.S. Yu and Z.C. Wang, Global attractivity of neutral difference equations, *Computers Math. Applic.* **36** (6), 1–10, (1998).
4. V.L.J. Kocic and G. Ladas, Global attractivity in nonlinear delay difference equations, *Proc. Amer. Math. Soc.* **115**, 1083–1088, (1992).
5. J.S. Yu, Asymptotic stability for a linear difference equation with variable delay, *Computers Math. Applic.* **36** (10–12), 203–210, (1998).
6. R.D. DeGroat, L.R. Hunt, D.A. Linebarger and M. Verma, Discrete-time nonlinear system stability, *IEEE Trans. Circuits Syst. I* **39**, 834–840, (1992).
7. T. Ushio, Limitation of delay feedback control in nonlinear discrete-time systems, *IEEE Trans. Circuits Syst. I* **43**, 815–816, (1996).
8. L. Huang and J. Wu, Joint effects of threshold and synaptic delay on dynamics of artificial neural networks with McCulloch-Pitts nonlinearity (to appear).
9. T. Roska and L.O. Chua, Cellular neural networks with nonlinear and delay-type template elements, *Int. J. Circuit Theory Appl.* **20**, 469–481, (1992).
10. R.P. Agarwal, *Difference Equations and Inequalities: Theory, Method and Applications*, Marcel Dekker, New York, (1992).
11. R.P. Agarwal and P.J.Y. Wong, *Advanced Topics in Difference Equations*, Kluwer, Dordrecht, (1997).
12. V.L.J. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic, (1993).