The structure of shift invariant spaces on a locally compact abelian group

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Abstract

We investigate shift invariant subspaces of $L^2(G)$, where $G$ is a locally compact abelian group. We show, among other things, that every shift invariant space can be decomposed as an orthogonal sum of spaces each of which is generated by a single function whose shifts form a Parseval frame.

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1. Introduction

In the last decade, shift invariant subspaces of $L^2(\mathbb{R}^n)$ have been studied from different aspects, by many authors such as: Aldroubi, Benedetto, Bownik, de Boor, DeVore, Li, Ron, Rzeszotnik, Shen, Weiss and Wilson, cf. [1,3,4,6,7,12,16]. This theory plays an important role in many areas, specially in the theory of wavelets, and multiresolution analysis. It has been used to show a new characterization of orthonormal wavelets conjectured by Weiss [14], a result originally proved in [5] by applying the techniques of [11,12].

In this paper we investigate the structure of shift invariant subspaces of $L^2(G)$, where $G$ is a locally compact abelian group. Our results generalize some of the results appearing in the literature on shift invariant spaces. Such a unified approach seems to be useful, since it describes the basic features of shift invariant spaces, and includes most of the special cases.

This paper is organized as follows: In Section 2, we state some preliminaries and notations related to locally compact abelian groups and shift invariant spaces. Moreover we give a characterization of elements in a principle shift invariant subspace of $L^2(G)$. In Section 3, we prove a necessary and sufficient condition for shifts of a function $\varphi \in L^2(G)$ to be an orthonormal system. In particular, we investigate a necessary and sufficient condition for shifts of a function $\varphi \in L^2(G)$ to be a Parseval frame. We also construct a function in a principle shift invariant space whose shifts form a Parseval frame. We show a decomposition of a shift invariant subspace of $L^2(G)$ into an orthogonal sum

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of spaces each of which is generated by a single function whose shifts form a Parseval frame. Moreover we find a Parseval frame for every shift invariant space.

2. Notations and preliminary results

Throughout this paper we assume that $G$ is a locally compact abelian group. It is well known that such a group possesses a Haar measure $\mu$ that is unique up to a multiplication by constants. We refer to the usual text books about locally compact groups [8,9]. We shall denote the measure of a measurable set $A$ by $|A|$. Let $\hat{G}$ denote the dual group of $G$ equipped with the compact convergence topology. The elements of $\hat{G}$ which we usually denote by $\xi$, are characters on $G$, but one can also regard elements of $G$ as characters on $\hat{G}$. More precisely $\hat{G} = G$ [8, Pontrjagin duality theorem].

Let the Fourier transform $\hat{\cdot} : L^1(G) \to C_0(\hat{G})$, $f \to \hat{f}$, be defined by $\hat{f}(\xi) = \int_G f(x)\hat{\xi}(x)\,dx$. The Fourier transform can be extended to a unitary isomorphism from $L^2(G)$ to $L^2(\hat{G})$ known as the Plancherel transform [8, Plancherel theorem].

Suppose $G$ is a locally compact abelian group and $H$ is a closed subgroup of $G$. Let $G/H$ be the quotient group whose Haar measure is $\mu$ (which is unique up to a constant factor). If this factor is suitably chosen we have

$$\int_G f(x)\,dx = \int_{G/H} \int_H f(xy)\,dy\,d\mu(xH), \quad f \in L^1(G).$$

This identity is known as Weil’s formula [8].

A subgroup $L$ of $G$ is called a uniform lattice if it is discrete and co-compact (i.e. $G/L$ is compact). The subgroup $L^\perp = \{\xi \in \hat{G} ; \xi(L) = \{1\}\}$ is called the annihilator of $L$ in $\hat{G}$.

Let $L$ be a uniform lattice in $G$. Then the identities $L^\perp = \hat{G}/L$ and $\hat{G}/L^\perp = \hat{L}$, together with the fact that a locally compact abelian group is compact if and only if its dual group is discrete [8], imply that the subgroup $L^\perp$ is a uniform lattice in $\hat{G}$ (see also [10,15]).

We now define a shift invariant space in $L^2(G)$.

**Definition 2.1.** Let $G$ be a locally compact abelian group and $L$ be a uniform lattice in $G$. A closed subspace $V \subseteq L^2(G)$ is called a shift invariant space (with respect to $L$) if $f \in V$ implies $T_k f \in V$, for any $k \in L$, where $T_k$ is the translation operator defined by $T_k f(x) = f(k^{-1}x)$ for all $x \in G$. For $\varphi \in L^2(G)$, $\text{span}(T_k \varphi; \ k \in L)$ is called the principle shift invariant space generated by $\varphi$ and will be denoted by $V_\varphi$.

Let $\varphi \in L^2(G)$. We denote by $L^2(\hat{L}, w_\varphi)$ the space of all functions $r : \hat{L} \to \mathbb{C}$, which satisfy

$$\int_{\hat{L}} |r(\xi)|^2 w_\varphi(\xi)\,d\xi < \infty,$$

where

$$w_\varphi(\xi) = \sum_{\eta \in L^\perp} |\hat{\varphi}(\xi \eta)|^2. \quad (2.1)$$

Note that $w_\varphi \in L^1(\hat{L})$. Indeed, by Weil’s formula and the Plancherel theorem $\int_{\hat{L}} \sum_{\eta \in L^\perp} |\hat{\varphi}(\xi \eta)|^2\,d\xi = \int_{\hat{G}} |\hat{\varphi}(\xi)|^2\,d\xi = \|\varphi\|^2$. In this case $\|r\|^2_{L^2(\hat{L}, w)} = \int_{\hat{L}} |r(\xi)|^2 w_\varphi(\xi)\,d\xi$ is a norm in $L^2(\hat{L}, w)$.

The following proposition gives a characterization of elements in a principle shift invariant subspace of $L^2(G)$ in terms of their Fourier transforms.

**Proposition 2.2.** Let $\varphi \in L^2(G)$. Then $f \in V_\varphi$ if and only if $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$, for some $r \in L^2(\hat{L}, w_\varphi)$.

**Proof.** Consider $A_\varphi = \text{span}(T_k \varphi; \ k \in L)$, then $V_\varphi = \overline{A_\varphi}$. For $f \in A_\varphi$ let $f(x) = \sum_{i=1}^n a_i \varphi(k_i^{-1}x)$, $a_i \in \mathbb{C}$, $k_i \in L$, $1 \leq i \leq n$, for some $n \in \mathbb{N}$. Then we have
\[ \hat{f}(\xi) = \sum_{i=1}^{n} a_i \hat{\xi}(k_i) \hat{\phi}(\xi) = r(\xi) \hat{\phi}(\xi) \]  

(2.2)

where \( r(\xi) = \sum_{i=1}^{n} a_i \hat{\xi}(k_i) \). Conversely every trigonometric polynomial will give us a function \( f \in A_\phi \), via formula (2.2). So \( f \in A_\phi \) if and only if \( \hat{f}(\xi) = r(\xi) \hat{\phi}(\xi) \) where \( r \) is a trigonometric polynomial. Denote the set of all trigonometric polynomials by \( P \). The operator \( U: A_\phi \to P \) given by \( U(f) = r \) is an isometry which is onto. In fact by using the Plancherel theorem and Weil’s formula we have

\[ \|f\|_2^2 = \int_\hat{G} |\hat{f}(\xi)|^2 \, d\xi \]
\[ = \int_\hat{L} \sum_{\eta \in L^\perp} |r(\xi\eta)|^2 |\hat{\phi}(\xi\eta)|^2 \, d\xi \]
\[ = \int_\hat{L} w_\phi(\xi) |r(\xi)|^2 \, d\xi \]
\[ = \|r\|_{L^2(\hat{L}, w_\phi)}^2. \]  

(3.2)

Therefore there is a unique isometry \( \tilde{U}: \overline{A_\phi} \to \overline{P} \), which extends \( U \) from \( V_\phi \) onto \( \overline{P} = L^2(\hat{L}, w_\phi) \). (Note that for a compact abelian group \( G \) the set of all trigonometric polynomials is dense in \( L^2(G) \) [13].) \( \square \)

Note that in the case \( G = \mathbb{R}, \mathbb{Z} \) is a uniform lattice. In this case we have the following corollary which is also proved in [14, Theorem 1.2.4].

**Corollary 2.3.** Let \( V_\phi \) be a principle shift invariant subspace of \( L^2(\mathbb{R}) \). Then \( f \in V_\phi \) if and only if \( \hat{f}(\xi) = r(\xi) \hat{\phi}(\xi) \), for some \( r \in L^2(\mathbb{T}, w_\phi) \), where \( w_\phi(\xi) = \sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi + k)|^2 \).

### 3. Main results

It is natural to ask if for any given principle shift invariant space we can find a function \( \phi \) in \( L^2(G) \) whose shifts are orthonormal. In general the answer is not affirmative. (However, we will find another kind of generator for every principle shift invariant space; see Corollary 3.8 below.) In the following theorem we state a necessary and sufficient condition for shifts of a function \( \phi \) in \( L^2(\hat{G}) \) to be an orthonormal system. Throughout this section the notations are as in Section 2.

**Proposition 3.1.** Suppose that \( \phi \in L^2(G) \), then \( \{T_k \phi; \ k \in L\} \) is an orthonormal system in \( L^2(G) \) if and only if

\[ w_\phi = 1 \quad a.e. \text{ on } \hat{G}. \]

(3.1)

**Proof.** We have to show that for \( k_1, k_2 \in L \), \( \langle T_{k_1} \phi, T_{k_2} \phi \rangle = \delta_{k_1,k_2} \) if and only if \( w_\phi(\xi) = 1 \ a.e. \ \xi \in \hat{G} \). Using the Plancherel theorem and Weil’s formula, we have

\[ \langle T_{k_1} \phi, T_{k_2} \phi \rangle = \langle \phi, T_{k_2} T_{k_1}^{-1} \phi \rangle \]
\[ = \langle \hat{\phi}, T_{k_2} T_{k_1}^{-1} \hat{\phi} \rangle \]
\[ = \int_{\hat{G}} |\hat{\phi}(\xi)|^2 \xi \xi(k_2 k_1^{-1}) \, d\xi \]
\[ = \int_{\hat{L}} \sum_{\eta \in L^\perp} |\hat{\phi}(\xi\eta)|^2 \xi \xi(k_2 k_1^{-1}) \, d\xi \]
\[ = \int_{\hat{L}} w_\phi(\xi) \xi \xi(k_2 k_1^{-1}) \, d\xi. \]

(3.2)
Pontrjagin duality theorem and [8, Proposition 4.3] imply that $L$ is an orthonormal basis for $L^2(\mathcal{L})$, i.e. \( \{ e_k: k \in L \} \) is an orthonormal basis for the Hilbert space $L^2(\mathcal{L})$, where $e_k(\xi) = \xi(k)$ for $\xi \in \hat{G}$. Now (3.2) completes the proof. \qed

If $V_\varphi$ is a principle shift invariant space and $w_\varphi$ is given by (2.1), then the set supp $w_\varphi$ is called the spectrum of $V_\varphi$ and is denoted by $\Omega_\varphi$. (Note that by supp $w_\varphi$ we mean the set of all $\xi$ such that $w_\varphi(\xi) \neq 0$. Also our convention is that all measurable sets are determined up to a null set.) In the case of Proposition 3.1, $\Omega_\varphi$ is equal to $\hat{G}$. The following example shows the existence of principle shift invariant spaces which do not satisfy this property.

Example 3.2. Let $G = (\mathbb{R}^2, +)$, $L = \mathbb{Z}^2$ (so $L^1 = \mathbb{Z}^2$), $E = [0, 1/2] \times [1/2, 3/2]$, and $\varphi$ be given by $\hat{\varphi} = \chi_E$. Then $w_\varphi(\xi) = \sum_{k \in \mathbb{Z}^2} \chi_{E}(\xi + k)$. So $\Omega_\varphi = \bigcup_{k \in \mathbb{Z}^2} (E + k) \neq \mathbb{R}^2$.

Now we shall determine how the information about orthogonality of $V_{\varphi_1}$ and $V_{\varphi_2}$ can be transferred into some other information about the generators $\varphi_1$ and $\varphi_2$ in $L^2(G)$.

Proposition 3.3. The spaces $V_{\varphi_1}$ and $V_{\varphi_2}$ are orthogonal if and only if

$$\sum_{\eta \in L^1} \hat{\varphi}_1(\xi \eta) \hat{\varphi}_2(\xi \eta) = 0 \quad \text{a.e. } \xi \in \hat{G}.$$  

Proof. Observe that by the Plancherel theorem $\langle T_{k_1} \varphi_1, T_{k_2} \varphi_2 \rangle = 0$ for $k_1, k_2 \in L$, if and only if

$$\int_L \sum_{\eta \in L^1} \hat{\varphi}_1(\xi \eta) \hat{\varphi}_2(\xi \eta) \xi(k_2 k_1^{-1}) d\xi = 0.$$ 

By an argument similar to the proof of Proposition 3.1 the desired result follows. \qed

As a consequence of Propositions 3.1 and 3.3, we have the following corollary (see also [16]).

Corollary 3.4.

(i) Suppose $\psi \in L^2(\mathbb{R})$. Then \{ $\psi(-k); k \in \mathbb{Z}$ \} is an orthonormal system if and only if $\sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + k)|^2 = 1$, for a.e. $\xi \in \mathbb{R}$.

(ii) For any two functions $\varphi, \psi \in L^2(\mathbb{R})$ the sets \{ $\varphi(-k); k \in \mathbb{Z}$ \} and \{ $\psi(-k); k \in \mathbb{Z}$ \} are biorthogonal, if and only if $\sum_{k \in \mathbb{Z}} \hat{\varphi}(\xi + k) \hat{\psi}(\xi + k) = 0$, for a.e. $\xi \in \mathbb{R}$.

Definition 3.5. Let $\mathcal{H}$ be a Hilbert space. A subset $X \subseteq \mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist two numbers $0 < A \leq B < \infty$ so that

$$A \|h\|^2 \leq \sum_{\eta \in X} |\langle h, \eta \rangle|^2 \leq B \|h\|^2 \quad \text{for } h \in \mathcal{H}. \quad (3.3)$$

If $A = B = 1$, $X$ is called a Parseval frame.

Now we prove that every principle shift invariant space has generators whose shifts form a Parseval frame. The key is the following theorem.

Theorem 3.6. Let $\varphi \in L^2(G)$. The shifts of $\varphi$ (with respect to $L$) form a Parseval frame for the space $V_\varphi$, if and only if

$$w_\varphi = \chi_{\Omega_\varphi} \quad \text{a.e. on } \hat{G}. \quad (3.4)$$

Proof. Let $\varphi \in L^2(G)$. By Proposition 2.2, for every $f \in V_\varphi$ we have $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$, for some $r \in L^2(\mathcal{L}, w_\varphi)$. So by using the Plancherel theorem and Weil’s formula,
\[ \langle f, T_k \varphi \rangle = \langle \hat{f}, \hat{T}_k \varphi \rangle = \int_{\widehat{G}} r(\xi) |\hat{\varphi}(\xi)|^2 \xi(k) \, d\xi \]
\[ = \int_{\widehat{L}} \sum_{\eta \in L^\perp} |\hat{\varphi}(\eta \xi)|^2 r(\xi) \xi(k) \, d\xi \]
\[ = \int_{\widehat{L}} w_\varphi(\xi) \xi(k) r(\xi) \, d\xi. \]

Consequently,
\[
\sum_{k \in L} |\langle f, T_k \varphi \rangle|^2 = \sum_{k \in L} \int_{\widehat{L}} w_\varphi(\xi) \xi(k) r(\xi) \, d\xi \int_{\widehat{L}} r(\eta) w_\varphi(\eta) \eta(k) \, d\eta
\]
\[ = \sum_{k \in L} \int_{\widehat{L}} \overline{F}(\xi) \overline{k}(\xi) \, d\xi \int_{\widehat{L}} F(\eta) k(\eta) \, d\eta
\]
\[ = \sum_{k \in L} F(k) \overline{F}(k)
\]
\[ = \langle F, F \rangle_{L^2(\widehat{L})}
\]
\[ = \int_{\widehat{L}} |r(\xi)|^2 |w_\varphi(\xi)|^2 \, d\xi. \tag{3.5} \]

(where \( F(\xi) = r(\xi)w_\varphi(\xi) \)). But by definition, \( \{T_k \varphi; \ k \in L\} \) is a Parseval frame for \( V_\varphi \) if and only if
\[
\sum_{k \in L} |\langle f, T_k \varphi \rangle|^2 = \|f\|^2 \text{ for every } f \in V_\varphi. \tag{3.6} \]

By (2.3) and (3.5), this is equivalent to \( \int_{\widehat{L}} |r(\xi)|^2 |w_\varphi(\xi)|^2 \, d\xi = \int_{\widehat{L}} |r(\xi)|^2 w_\varphi(\xi) \, d\xi \), or \( \int_{\widehat{L}} |r(\xi)|^2 w_\varphi(\xi) \chi_{\Omega_\varphi}(\xi) - w_\varphi(\xi) \, d\xi = 0 \), for all \( r \in L^2(\widehat{L}, w_\varphi) \). Put \( r = \chi_M \), where \( M = \{\xi \in \Omega; \ w_\varphi(\xi) > 1\} \) or \( M = \{\xi \in \Omega; \ w_\varphi(\xi) < 1\} \). We see that \( \chi_{\Omega_\varphi}(\xi) = w_\varphi(\xi) \), for a.e. \( \xi \in \widehat{G} \) if and only if (3.4) holds. \( \square \)

Remark 3.7. Equality (3.4) is obviously a more general version of equality (3.1) that characterizes the orthonormality of the system \( \{T_k \varphi; k \in L\} \).

Corollary 3.8. If \( V_\varphi \) is a principle shift invariant space and \( \psi \) is given by
\[
\hat{\psi}(\xi) = \begin{cases} \hat{\varphi}(\xi) w_\varphi(\xi)^{-1/2}, & \xi \in \Omega, \\ 0, & \text{otherwise}, \end{cases}
\]
then \( \{T_k \psi, \ k \in L\} \) is a Parseval frame for \( V_\varphi \).

Proof. First by Proposition 2.2, we have \( \psi \in V_\varphi \), since \( s \in L^2(\widehat{L}, w_\varphi) \) where \( s(\xi) = w_\varphi(\xi)^{-1/2} \), in fact
\[
\|s\|^2 = \int_{\widehat{L}} |s(\xi)|^2 w_\varphi(\xi) \, d\xi = \int_{\widehat{L}} \chi_{\Omega}(\xi) \, d\xi = |\widehat{L} \cap \Omega| \leq |\widehat{L}| < \infty \quad \text{(since } \widehat{L} \text{ is compact}).
\]
Also we have \( \sum_{\eta \in L^\perp} |\hat{\psi}(\xi \eta)|^2 = \chi_{\Omega}(\xi) \). By Theorem 3.6 the proof is complete. \( \square \)

Definition 3.9. If \( V_\varphi \) is a principle shift invariant space and the system \( \{T_k \varphi, k \in L\} \) is a Parseval frame for \( V_\varphi \), the function \( \varphi \) is called a Parseval frame generator of \( V_\varphi \).
Corollary 3.8 shows that every principle shift invariant space has a Parseval frame generator.

Now we show the existence of a decomposition of a shift invariant subspace of $L^2(G)$ into an orthogonal sum of spaces each of which is generated by a single function whose shifts form a Parseval frame.

**Theorem 3.10.** Let $G$ be a locally compact abelian group and let $L$ be a uniform lattice in $G$. If $V$ is a shift invariant space in $L^2(G)$, then there exists a family of functions $\{\varphi_\alpha\}_{\alpha \in I}$ in $L^2(G)$ (where $I$ is an index set), such that

$$V = \bigoplus_{\alpha \in I} V_{\varphi_\alpha},$$

(3.7)

and $\varphi_\alpha$ is a Parseval frame generator of the space $V_{\varphi_\alpha}$. Moreover, $f \in V$ if and only if

$$\hat{f}(\xi) = \sum_{\alpha \in I} r_\alpha(\xi) \hat{\varphi}_\alpha(\xi),$$

(3.8)

and $\|f\|^2 = \sum_{\alpha \in I} \|r_\alpha\|^2_{L^2(\hat{\mathbb{L}} \cap \Omega_{\varphi_\alpha}, w_{\varphi_\alpha})}$, where $r_\alpha \in L^2(\hat{\mathbb{L}} \cap \Omega_{\varphi_\alpha}, w_{\varphi_\alpha})$ and $\Omega_{\varphi_\alpha}$ is the spectrum of $V_{\varphi_\alpha}$, for every $\alpha \in I$.

**Proof.** Choose a non-zero $\varphi \in V$ and apply Corollary 3.8 to obtain $\psi \in V_\varphi$ satisfying (3.4). By Zorn’s lemma there is a maximal collection $\{V_\varphi\}_{\alpha \in I}$ of mutually orthogonal principle shift invariant subspaces of $V$ so that $\varphi_\alpha$ is a Parseval frame generator of $V_{\varphi_\alpha}$, for every $\alpha \in I$. If there was a non-zero $g \in V$, orthogonal to all the $V_{\varphi_\alpha}$’s, then the principle shift invariant space generated by $g$ would be orthogonal to the $V_{\varphi_\alpha}$’s, contradicting maximality. Hence (3.7) holds. If $\hat{f}$ is given by (3.8), then clearly $f \in V$ by Proposition 2.2. For the converse, let $P_\alpha$ be the orthogonal projection onto the space $V_{\varphi_\alpha}$. For every $f \in V$ we have $f = \sum_{\alpha \in I} P_\alpha f$, so $\hat{f} = \sum_{\alpha \in I} (P_\alpha \hat{f})$. By Proposition 2.2 we obtain $P_\alpha \hat{f}(\xi) = r_\alpha(\xi) \hat{\varphi}_\alpha(\xi)$, where $r_\alpha \in L^2(\hat{\mathbb{L}} \cap \Omega_{\varphi_\alpha}, w_{\varphi_\alpha})$, for every $\alpha$. Since $\varphi_\alpha$ is a generator for $V_{\varphi_\alpha}$, it follows $\|P_\alpha f\|^2 = \|r_\alpha\|^2_{L^2(\hat{\mathbb{L}} \cap \Omega_{\varphi_\alpha}, w_{\varphi_\alpha})}$, for all $\alpha \in I$. Thus

$$\|f\|^2 = \sum_{\alpha \in I} \|P_\alpha f\|^2 = \sum_{\alpha \in I} \|P_\alpha f\|^2 = \sum_{\alpha \in I} \|r_\alpha\|^2_{L^2(\hat{\mathbb{L}} \cap \Omega_{\varphi_\alpha}, w_{\varphi_\alpha})}. \quad \square$$

**Remark 3.11.** Using the above theorem we can find a Parseval frame for every shift invariant subspace of $L^2(G)$: If $\{T_k \varphi_\alpha\}_{k \in \mathbb{L}}$ is a Parseval frame for $V_{\varphi_\alpha}$, for every $\alpha \in I$, then $\{T_k \varphi_\alpha\}_{k \in \mathbb{L}, \alpha \in I}$ is a Parseval frame for the orthogonal sum $\bigoplus_{\alpha \in I} V_{\varphi_\alpha}$. Indeed, for every $f = \sum_{\alpha \in I} P_\beta f \in \bigoplus_{\alpha \in I} V_{\varphi_\alpha}$, where $P_\beta$ is the orthogonal projection onto $V_{\varphi_\beta}$, we have

$$\sum_{\alpha \in I} \sum_{k \in \mathbb{L}} |\langle T_k \varphi_\alpha, f \rangle|^2 = \sum_{\alpha \in I} \sum_{\beta \in I} \sum_{k \in \mathbb{L}} |\langle T_k \varphi_\alpha, P_\beta f \rangle|^2$$

$$= \sum_{\alpha \in I} \sum_{k \in \mathbb{L}} |\langle T_k \varphi_\alpha, P_\alpha f \rangle|^2$$

$$= \sum_{\alpha \in I} \|P_\alpha f\|^2$$

$$= \sum_{\alpha \in I} \|P_\alpha f\|^2$$

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$$= \|f\|^2.$$

**Example 3.12.** Let $\psi \in L^2(\mathbb{R})$. For $j, k \in \mathbb{Z}$, define $\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k)$. Obviously, $W_j := \text{span} \{\psi_{j,k}; k \in \mathbb{Z}\}$ is a shift invariant space, for $j \geq 0$. So by Theorem 3.10, there exists a sequence of functions $\{\varphi_n\}_{n=1}^\infty$ that are mutually orthogonal and for each $n$, $\{\varphi_n(., -k); k \in \mathbb{Z}\}$ is a Parseval frame for $V_{\varphi_n}$ and $V = \bigoplus_{n=1}^\infty V_{\varphi_n}$ (since $\mathbb{R}$ is second countable, each $W_j$ is separable and so the decomposition is countable).
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